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# REGULAR $P$-ALMOST COTANGENT STRUCTURES

MANUEL DE LEÓN*, ISABEL MÉNDEZ** AND MODESTO SALGADO**

## 1. Introduction.

Bruckheimer [Br] introduced the concept of an almost cotangent structure. An almost cotangent structure on a $2n$-dimensional manifold $M$ consists of an almost symplectic form $\omega$, together with an $n$-dimensional distribution $V$ which is a Lagrangian distribution for $\omega$. An almost cotangent structure is obtained by abstracting two of the cotangent bundle's most important geometric ingredients, namely its canonical symplectic form and vertical distribution (hence the name). Clark and Goel interpreted an almost cotangent structure as a certain type of $G$-structure and proved that this $G$-structure is integrable if and only if each of its constituent $G$-structures is integrable, that is, $\omega$ is symplectic and $V$ is involutive.

Obviously, the integrability of an almost cotangent structure implies that it is *locally* equivalent to the cotangent bundle $T^*N$ of an $n$-dimensional manifold $N$. Recently, Thompson demonstrated that a regular almost cotangent structure (that is, an integrable almost cotangent structure verifying some global hypotheses) determines an element of the second de Rham cohomology group $H^2(N, R)$ of the base manifold $N$. In fact, there is a one-to-one correspondence between equivalence classes of regular almost cotangent manifolds and elements of $H^2(N, R)$. The vanishing of this cohomology class characterizes cotangent bundles in the class of regular almost cotangent manifolds. The situation may be contrasted with the case of almost tangent manifolds [CTh].

In a previous paper [LMS3] we have introduced a natural generalization of almost cotangent structures. This new class of geometric structures (called *p-almost cotangent structures*) consists in a family of almost presymplectic forms $\omega_a$ of rank $2n$ together with a family of

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$n$-dimensional distributions $V_a$, $1 \leq a \leq p$ verifying some compatibility conditions on a $(p+1)n$-dimensional manifold $M$. In particular, we suppose that $V_a \cap (\underset{b \neq a}{+} V_b) = 0$. Then $V = V_1 \oplus \ldots \oplus V_p$ is a $pn$-dimensional distribution. Again, a $p$-almost cotangent structure can be interpreted as a certain type of $G$-structure and its integrability is proved to be equivalent to the integrablity of its constituent $G$-structures (that is, $\omega_a$ is a presymplectic form and $V$, $V_a$ are involutive, for any $a$).

In this paper, we establish some global results, similar to the ones obtained by Thompson. In the first theorem (Theorem 3. 2) we prove that, under some global hypotheses, an integrable $p$-almost cotangent manifold is diffeomorphic to the cotangent bundle $T_{p^1}{}^*N$ of $p^1$-covelocities of the base manifold $N$. Moreover, in the second theorem (Theorem 4. 1) we prove that every regular $p$-almost cotangent structure determines an element of $H^2(N, R) \times \overset{p}{\cdots} \times H^2(N, R)$ in such a way that the vanishing of this $p$-tupla of cohomology classes characterizes $T_{p^1}{}^*N$ in the class of regular $p$-almost cotangent structures. This last result may be contrasted with the corresponding one for integrable $p$-almost tangent structures ([LMS1], [LMS2]).

## 2. The cotangent bundle of $p^1$-covelocities.

Let $N$ be an $n$-dimensional manifold. By $T_{p^1}{}^*N$ we denote the *cotangent bundle of $p^1$-covelocities* of $N$, that is, the manifold of all 1-jets of mappings from $N$ to $R^p$ with target $0 \in R^p$. The manifold $T_{p^1}{}^*N$ is locally characterized as follows: if $(x^i)$ is a coordinate system on $N$ then the coordinates $(x^i, x^1{}_i, \ldots, x^p{}_i)$ on $T_{p^1}{}^*N$ are defined by

$$x^i(j_{x,0}{}^1 f) = x^i(x),$$
$$x^a{}_i(j_{x,0}{}^1 f) = (\partial f^a / \partial x^i)_{/x}, \ 1 \leq i \leq n, \ 1 \leq a \leq p,$$

where $j_{x,0}{}^1 f$ is the 1-jet at $x \in N$ of the map $f : N \to R^p$, $f(x) = 0$ and $f^a = \tau^a \circ f$, $\tau^a : R^p \to R$ being the canonical projection $\tau^a(t^1, \ldots, t^p) = t^a$. Clearly, $T_{p^1}{}^*N$ is a manifold of dimension $(p+1)n$. Two such coordinate systems $(x^i, x^a{}_i)$, $(\bar{x}^i, \bar{x}^a{}_i)$ with intersecting domains are related by a change of coordinates whose Jacobian matrix has the form

$$(2.1)\qquad \begin{pmatrix} A & 0 & 0 & 0 \\ B_1 & C & 0 & 0 \\ B_2 & 0 & C & 0 \\ \cdots & \cdots & \cdots & \cdots \\ B_p & 0 & 0 & C \end{pmatrix}$$

where

$A=[\partial x^i/\partial \bar{x}^j]$, $C=[\partial \bar{x}^j/\partial x^i]$, $B_a=[(\partial^2\bar{x}^k/\partial x^i\partial x^l)(\partial x^l/\partial \bar{x}^j)x^a{}_i]$,

where $1\leq i, j, k, l\leq n$, $1\leq a\leq p$.

Next, we shall prove that $T_{p^1}{}^*N$ has the structure of vector bundle over $N$ with standard fibre the vector space $R^{pn}$. To do this, we proceed as follows. We denote by $\pi_0 : T_{p^1}{}^*N\rightarrow N$ the canonical projection defined by

$$\pi_0(j_{x,0}{}^1f)=x.$$

Now, we have a canonical diffeomorphism

$$\Lambda : T_{p^1}{}^*N\rightarrow T^*N\overset{p}{\oplus\cdots\oplus}T^*N$$

of $T_{p^1}{}^*N$ with the Whitney sum of $T^*N$ with itself $p$ times; $\Lambda$ is given by

$$\Lambda(j_{x,0}{}^1f)=(j_{x,0}{}^1f^1, \cdots, j_{x,0}{}^1f^p).$$

Then each element $\theta\in(T_{p^1}{}^*N)_x=\pi^{-1}(x)$, $x\in N$, may be identified, via $\Lambda$, with a $p$-tupla $(\theta^1, \cdots, \theta^p)$ of 1-forms $\theta^a\in T_x{}^*N$, $1\leq a\leq p$. If we now define

$$\lambda\theta+\nu\phi=(\lambda\theta^1+\nu\phi^1, \cdots, \lambda\theta^p+\nu\phi^p)$$

where $\theta=(\theta^a)$, $\phi=(\phi^a)\in(T_{p^1}{}^*N)_x$, $\lambda, \nu\in R$, then it is easy to prove that $\pi_0 : T_{p^1}{}^*N\rightarrow N$ is a vector bundle over $N$ isomorphic, as vector bundles, with the Whitney sum of $T^*N$ with itself $p$ times. Obviously, when $p=1$, then $T_{p^1}{}^*N=T^*N$. Moreover, for each $a$, $1\leq a\leq p$, we have two canonical projections

$$(\rho^a)_0 : T_{p^1}{}^*N\rightarrow T^*N$$

and

$$(\pi_a)_0 : T_{p^1}{}^*N\rightarrow T_{(p-1)^1}N$$

defined by

$$(\rho^a)_0(\theta^1, \ldots, \theta^p)=\theta^a$$

and

$$(\pi_a)_0(\theta^1, \ldots, \theta^p)=(\theta^1, \ldots, \check{\theta}^a, \ldots, \theta^p),$$

respectively (here the circumflex over a term means that it is to be omitted). Then we locally have

$$(\rho^a)_0(x^i, x^b_i) = (x^i, x^a_i).$$

Thus, we have $p$ canonical vertical distributions $(V_a)_0 = \text{Ker } T(\pi_a)_0$ such that $(V_a)_0 \cap (\underset{b\neq a}{+}(V_b)_0) = 0$. Furthermore, we have

$$V_0 = \bigoplus_{a=1}^{p}(V_a)_0 = \text{Ker } T\pi_0.$$

We also may define a canonical injection

$$j_a : T^*N \to T_{p^1}{}^*N$$

given by

$$j_a(\alpha) = (0, \ldots, \overset{a}{\alpha}, \ldots, 0).$$

So, we locally have

$$j_a(x^i, y_i) = (x^i, 0, \ldots, y^i, \ldots, 0).$$

Now, we may define $p$ presymplectic forms $(\omega_a)_0$ of rank $2n$, $1 \leq a \leq p$ as follows. First, we define $p$ canonical 1-forms $(\lambda_a)_0$ by setting

$$(\lambda_a)_0(\theta)(X) = \theta^a(x)(T\pi_0(X))$$

$X \in T_\theta(T_{p^1}{}^*N)$, $\pi_0(\theta) = x$. Locally, $(\lambda_a)_0$ is given by

$$(\lambda_a)_0 = x^a{}_i dx^i.$$

Then $(\omega_a)_0 = -d(\lambda_a)_0$ is a canonical presymplectic form on $T_{p^1}{}^*N$ locally given by

$$(\omega_a)_0 = dx^i \wedge dx^a{}_i$$

If we denote by $(s_a)_0 : T(T_{p^1}{}^*N) \to T^*(T_{p^1}{}^*N)$ the vector bundle homomorphism given by $(s_a)_0(X) = i(X)(\omega_a)_0$, then

$$(K_a)_0 = \text{Ker}(s_a)_0 = \bigoplus_{\substack{b=1\\ b\neq a}}^{p}(V_b)_0.$$

When $p=1$, the canonical 1-form $\lambda_0 = (\lambda_1)_0$ is the Liouville form on $T^*N$ (see [Go], [LR1], [LR2]).

We next show that there is a construction on $T_{p^1}{}^*N$ which generalizes the vertical lift construction on the cotangent bundle $T^*N$ (see [YI]). Suppose that $y \in T_{p^1}{}^*N$ and that $\pi(y) = x$. Define a map

$$T_x{}^*M \to T_y(T_{p^1}{}^*M)$$

by

$$\alpha \to \alpha^{(a)} = Tj_a(\alpha)^v,$$

where $\alpha^v \in T_{y_a}(T^*N)$ is the vertical lift of $\alpha$ to $T^*N$. If $\alpha = \alpha_i dx^i$, then we have

$$\alpha^{(a)} = -\alpha_i(\partial/\partial x^a{}_i).$$

Thus, $\alpha^{(a)} \in (V_a)_y$, $1 \leq a \leq p$ and, then $\alpha^{(a)}$ is called the *(a)-vertical* *lift* of $\alpha$ to $T_{p^1}{}^*N$. Furthermore, a simple computation on local coordinates shows that

$$[\alpha^{(a)}, \beta^{(b)}]=0,$$

where $\alpha, \beta$ are 1-forms on $N$ and $1\leq a, b\leq p$.

## 3. Integrable $p$-almost cotangent structures

In this section, we briefly recall some definitions and results about $p$-almost cotangent structures. As the name suggest, a $p$-almost cotangent structure is obtained by abstracting the most important geometric ingredients of the cotangent bundle of $p^1$-covelocities.

DEFINITION 3.1. A *$p$-almost cotangent structure* on a $(p+1)n$-dimensional manifold $M$ consists of a family $(\omega_a, V_a; 1\leq a\leq p)$ of almost presymplectic forms $\omega_a$ of rank $2n$, together with $pn$-dimensional distributions $V_a$ such that

(1) $V_a\cap(\underset{b\neq a}{+}V_b)=0$,

(2) $K_a=\text{Ker}\ s_{\omega_a}=\bigoplus_{\substack{b=1\\ b\neq a}}^{p} V_b$, where $s_{\omega_a}: TM\to T^*M$ is defined by

$X\to s_{\omega_a}(X)=i_X\omega_a$.

Such a manifold $M$ is called a *$p$-almost cotangent manifold*.

If we put $V=\bigoplus_{a=1}^{p} V_a$, then $V$ is a $pn$-dimensional distribution on $M$. In [LMS3], we proved that giving a $p$-almost cotangent structure is the same as giving a $G$-structure on $M$, $G$ being the subgroup of $Gl((p+1)n, R)$ which consists of all matrices of the form (2.1) where $A\in Gl(n, R)$ and $C=(A^{-1})^t$, $A^tB_a=B_a{}^tA$, $1\leq a\leq p$. Furthermore, the following integrability theorem has been proved.

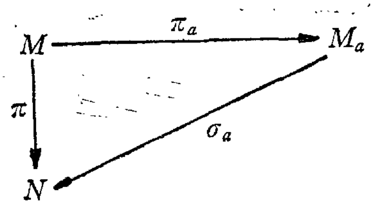
THEOREM 3.1. *A $p$-almost cotangent structure $(\omega_a, V_a)$ on $M$ is integrable iff the distributions $V$, $V_a$ are involutive and the almost presymplectic forms $\omega_a$ are closed, namely $d\omega_a=0$ (then $\omega_a$ are presymplectic forms).*

Now, let $M$ be an integrable $p$-almost cotangent manifold with $p$-almost cotangent structure $(\omega_a, V_a)$ and suppose that $(\omega_a, V_a)$ defines a fibration. That is, we suppose:

(1) the space of leaves $M_a=M/K_a$ defined by the involutive distribut-

ion $K_a$ is a quotient manifold of $M$; then $\rho_a : M \to M_a$ is a fibration whose fibres are precisely the leaves of $K_a$, and
(2) the space of leaves $N=M/V$ defined by the involutive distribution $V$ is a quotient manifold of $M$; then $\pi : M \to N$ is a fibration whose fibres are precisely the leaves of $V$.

Consequently, we have a canonical projection $\sigma_a : M_a \to N$ such that the following diagram




is commutative. Therefore, we have:
(1) Since $K_a = \mathrm{Ker}\, s_{\omega_a}$, then there exists a unique symplectic form $\bar{\omega}_a$ on $M_a$ such that

$$(\pi_a)^* \bar{\omega}_a = \omega_a .$$

(2) Since $K_a \cap V_a = 0$, then $V_a$ induces an involutive distribution $\bar{V}_a = T\pi_a V_a$ on $M_a$. The leaves of $V_a$ projects, via $\pi_a$, onto the leaves of $\bar{V}_a$. So, $\sigma_a$ is a fibration whose fibres are precisely the leaves of $\bar{V}_a$.

Now, let $(x^i, x^{i+cn}, 1 \leq c \leq p)$ be a coordinate system adapted for the integrable $p$-almost cotangent structure. Then we have

$$V = \langle \partial/\partial x^{i+cn};\ 1 \leq c \leq p \rangle, \quad V_a = \langle \partial/\partial x^{i+an} \rangle,$$
$$K_a = \langle \partial/\partial x^{i+bn};\ 1 \leq b \leq p, b \neq a \rangle$$
$$\omega_a = dx^i \wedge dx^{i+an}.$$

Thus, $N$ and $M_a$ have induced coordinates $(x^i)$ and $(x^i, x^{i+an})$, respectively. Therefore, we have

$$\pi(x^i, x^{i+cn}) = (x^i),\ \pi_a(x^i, x^{i+cn}) = (x^i, x^{i+an}),\ \sigma_a(x^i, x^{i+an}) = (x^i),$$
$$\bar{\omega}_a = dx^i \wedge dx^{i+an}.$$

We now show that there is a construction on an integrable $p$-almost cotangent manifold which defines a fibration, which generalizes the vertical lift construction on a cotangent bundle of $p^1$-covelocities.

Suppose that $y \in M$, $\pi(y) = x$, $\pi_a(y) = y_a$ and that $\alpha \in T_x^* N$. We define the $(a)$-*vertical lift* of $\alpha$ to $M$ as the unique tangent vector $\alpha^{(a)} \in (V_a)_y$ such that

$$i((T\pi_a\alpha^{(a)}))\bar{\omega}_a=(\sigma_a)^*\alpha.$$

If $\alpha=\alpha_i dx^i$, then we have

$$\alpha^{(a)}=-\alpha_i(\partial/\partial x^{i+an}).$$

Therefore, if $\alpha$ is a 1-form on $N$ there are $p$ vertical vector fields $\alpha^{(a)}$, $1\leq a\leq p$, such that $\alpha^{(a)}\in V_a$.

PROPOSITION 3.1. *For all 1-forms $\alpha, \beta$ on $N$, we have*

$$[\alpha^{(a)}, \beta^{(b)}]=0.$$

*Proof.* It follows directly from the local expressions of $\alpha^{(a)}$ and $\beta^{(b)}$.

As we have said, a $p$-almost cotangent structure defines a $G$-structure. One of the most important results in the theory of $G$-structures states that if a $G$-structure is integrable, then there is a symmetric $G$-connection (see [Fu]). The connection no need to be unique. In the present context of our integrable $p$-almost cotangent structure we can affirm the existence of a symmetric connection $\nabla$ which verifies

$$\nabla\omega_a=0,\ \ \nabla V\subset V,\ \ \nabla V_a\subset V_a,\ \ 1\leq a\leq p.$$

Such a connection is said to be *adapted* for the $p$-almost cotangent structure.

PROPOSITION 3.2. *Let $\nabla$ be a symmetric connection which is adapted for an integrable $p$-almost cotangent structure which defines a fibration $\pi: M\rightarrow N$. Then $\nabla$ induces, by restriction, a flat connection on each leaf of the foliations $V$ and $V_a$, $1\leq a\leq p$.*

*Proof.* Firstly, note that it is sufficient to show that for all 1-forms $\alpha$ and $\beta$ on $N$

$$\nabla_{\beta^{(b)}}\alpha^{(a)}=0. \tag{3.1}$$

To show (3.1), it is enough to prove that for all vector fields $Z$ on $M$

$$\omega_c(\nabla_{\beta^{(b)}}\alpha^{(a)}, Z)=0,\ \ 1\leq c\leq p. \tag{3.2}$$

(In fact, if $\omega_c(X, Z)=0$ for all vector fields $Z$, then $X\in\bigcap_{c=1}^{p}K_c=0$).

Now, note that we only have to prove (3.2) when $Z$ is itself a vertical vector field or $Z$ is complementary to $V$. In the first case, we only need prove (3.2) when $Z=\gamma^{(d)}$, $\gamma$ being a 1-form on $N$. Then we have

$$\omega_c(\nabla_{\beta^{(b)}}\alpha^{(a)}, \gamma^{(d)})=0,$$

since $\nabla V_a \subset V_a$ and $\omega_{c/V\times V}=0$. Next, suppose that $Z$ is complementary to $V$. We only need prove (3.2) when $Z$ is $\pi$-related to a vector field $\bar{Z}$ on $N$. If $\alpha=\alpha_i dx^i$, $\beta=\beta_i dx^i$, $\bar{Z}=Z^i(\partial/\partial x^i)$, then we have

$$\omega_c(\nabla_{\beta^{(b)}}\alpha^{(a)}, Z)=\beta^{(b)}(\omega_c(\alpha^{(a)}, Z))-\omega_c(\alpha^{(a)}, \nabla_{\beta^{(b)}}Z)$$

(since $\nabla\omega_c=0$)

$$=\beta^{(b)}(\omega_c(\alpha^{(a)}, Z))-\omega_c(\alpha^{(a)}, \nabla_Z\beta^{(b)})-\omega_c(\alpha^{(a)}, [\beta^{(b)}, Z]).$$

(since $\nabla$ is symmetric)

But $\beta^{(b)}(\omega_c(\alpha^{(a)}, Z))=\beta^{(b)}(\alpha_i Z^i)=0$, since $\alpha_i Z^i$ is a function on $N$. Likewise,

$$\omega_c(\alpha^{(a)}, [\beta^{(b)}, Z])=0,$$

since $[\beta^{(b)}, Z]\in V$. Finally, $\nabla_Z\beta^{(b)}\in V_b$; then the second term also vanishes.

We now state and prove the main result of this section.

THEOREM 3.2. *Let $(M, \omega_a, V_a)$ be an integrable $p$-almost cotangent structure which defines a fibration $\pi: M\to N$. Suppose that $\nabla$ is a symmetric connection adapted for the $p$-almost cotangent structure such that the flat connection induced by it on each leaf of $V$ and $V_a$ is geodesically complete. Suppose further that each fibre of $\pi$ is connected and simply connected and that each leaf of $V_a$ is connected, for all $a$. Then $M$ is diffeomorphic to $T_{p}{}^{1*}N$. Moreover, the diffeomorphism, $F$ say, can be chosen such that $\omega_a=F^*((\omega_a)_0+\pi_0{}^*\phi_a)$, where $(\omega_a)_0$ are the canonical presymplectic forms on $T_{p}{}^{1*}N$, $\pi_0: T_{p}{}^{1*}N\to N$ the canonical projection and $\phi_a$ are $p$ closed 2-forms on $N$.*

*Proof*. Firstly, we prove that $M$ is an affine bundle modelled on $T_{p}{}^{1*}N$. Thus, we shall construct a map $\rho: M\times_N T_{p}{}^{1*}N\to M$ such that

$$\rho_x: \pi^{-1}(x)\times(T_{p}{}^{1*}N)_x\to\pi^{-1}(x)$$

is a free transitive action of the vector space $(T_{p}{}^{1*}N)_x$ on $\pi^{-1}(x)$, for all $x\in N$.

For any $\alpha=(\alpha_1, \dots, \alpha_p)\in(T_{p}{}^{1*}N)_x$, where $\alpha_a\in T_x{}^*N$, $1\le a\le p$, we define a vertical vector field $A_a$ on $\pi^{-1}(x)$ by

$$(A_a)_y = ((\alpha_a)^{(a)})_y, \; y \in \pi^{-1}(x).$$

Then $A_a \in V_a$. Consequently,

$$\nabla_{A_a} A_a = 0,$$

and so $A_a$ is a geodesic vector field. Hence by our completeness assumption, $A_a$ generates a global one-parameter group of transformations of $\pi^{-1}(x)$:

$$\Phi_{A_a} : R \times \pi^{-1}(x) \to \pi^{-1}(x).$$

Let $t \to \Phi_{A_a}(t, y)$ be the integral curve of $A_a$ such that $\Phi_{A_a}(0, y) = y$, $y \in \pi^{-1}(x)$. We define $\rho_x$ by

(3.3) $\quad \rho_x(y, (\alpha_1, \ldots, \alpha_p)) = \Phi_{A_p}(1, \ldots, \Phi_{A_2}(1, \Phi_{A_1}(1, y)), \ldots)$

We shall show that (3.3) defines a free transitive action. First, for any $\alpha = (\alpha_a), \beta = (\beta_a) \in (T_{p^1}{}^*N)_x$, the corresponding vector fields $A_a$, $B_b$, $1 \leq a, b \leq p$ on $\pi^1(x)$ verifying $[A_a, B_b] = 0$ by Proposition 3.1. Their one-parameter groups therefore commute and the composition of their flows yields another one-parameter group whose generator is $A_a + B_b$, that is

(3.4) $\quad \Phi_{A_a}(t, \Phi_{B_b}(t, y)) = \Phi_{B_b}(t, \Phi_{A_a}(t, y)) = \Phi_{A_a + B_b}(t, y)$

and hence

$$\rho_x(\rho_x(y, \alpha), \beta) = \rho_x(\rho_x(y, \beta), \alpha) = \rho_x(y, \alpha + \beta).$$

Then $\rho_x$ defines an action of $(T_{p^1}{}^*N)_x$ on $\pi^{-1}(x)$.

Secondly, we show that $\rho_x$ is transitive. Let ( , ) be any scalar product on $(T_{p^1}{}^*N)_x$ and define a Riemannian metric $g_a$ on each leaf of $V_a$ by setting

$$g_a(A_a, B_a) = (\alpha_a, \beta_a).$$

Since $\nabla A_a = 0$ and $g_a(A_a, B_a)$ is constant, it follows that $\nabla$ is the Riemannian connection of $g_a$. Now, let $y, z$ be any two points of $\pi^{-1}(x)$. Since each leaf of $V$ is a local product of leaves of $V_1, \ldots, V_p$, we must consider two cases. If $y, z$ belong to the same local slice, then, from the Hopf-Rinow theorem, $y$ and $z$ may be joined by a piecewise differentiable curve $\gamma$ which consists of $p$ geodesic arcs $\{\gamma_1, \ldots, \gamma_p\}$ in such a way that $\gamma_a$ is a geodesic arc on a leaf of the foliation $V_a$. We may suppose that $\gamma_1(0) = y, \gamma_1(1) = \gamma_2(0), \ldots, \gamma_{p-1}(1) = \gamma_p(0)$, $\gamma_p(1) = z$ and

$$\dot{\gamma}_1(0) = (\alpha_1)^{(1)}, \ldots, \dot{\gamma}_p(0) = (\alpha_p)^{(p)}.$$

Therefore, we have

$$\rho_x(y, (\alpha_1, \ldots, \alpha_p)) = \Phi_{A_p}(1, \ldots, \Phi_{A_2}(1, \Phi_{A_1}(1, y)), \ldots) = z.$$

If $y, z$ are not in the same local slice, they can also be joined by a piecewise differentiable curve $\gamma=\{\gamma_1, \ldots, \gamma_q\}$ which consists of geodesic arcs in a such a way that $\gamma_r$, $1 \leq r \leq q$ is a geodesic arc on a leaf of the foliation $V_a$, for some $a$. Using (3.4) one can finds an element $\alpha=(\alpha_a) \in (T_{p^1}{}^*N)_x$ such that

$$\gamma_1(0)=(\alpha_1)^{(1)},$$

and

$$z=\Phi_{A_p}(1, \ldots, \Phi_{A_2}(1, \Phi_{A_1}(1, y)), \ldots).$$

Consequently, we have

$$z=\rho_x(y, (\alpha_a)).$$

Thirdly, we prove that the action is free. Let $\Gamma(y)$ be the isotropy group of $y \in \pi^{-1}(x)$ under the action of $(T_{p^1}{}^*N)_x$, that is

$$\Gamma(y)=\{\alpha=(\alpha_a) \in (T_{p^1}{}^*N)_x / \rho_x(y, \alpha)=y\}.$$

From the definition of $\rho_x$, one can easily prove that the following diagram

$$\begin{array}{ccc} (T_{p^1}{}^*N)_x = T_x{}^*N \oplus \overset{p}{\ldots} \oplus T_x{}^*N & \xrightarrow{\bar{\rho}_x} & \pi^{-1}(x) \\ \Big\downarrow \Psi & \nearrow \exp_y & \\ T_y(\pi^{-1}(x)) & & \end{array}$$

is commutative, where $\exp_y$ denotes the exponential map of $\nabla$ restricted to $\pi^{-1}(x)$, $\Psi$ is the linear isomorphism given by

$$\Psi(\alpha)=\Psi(\alpha_1, \ldots, \alpha_p)=(\alpha_1)^{(1)}+\ldots+(\alpha_p)^{(p)}$$

and $\bar{\rho}_x$ is defined by $\bar{\rho}_x(\alpha)=\rho_x(y, \alpha)$. Since $\exp_y$ is a local diffeo morphism, then $\bar{\rho}_x$ is so also. Therefore

$$\Gamma(y)=(\bar{\rho}_x)^{-1}(y)$$

must be a discrete (additive) subgroup of $(T_{p^1}{}^*N)_x$. Then the elements of $\Gamma(y)$ are integral linear combinations of some $k$ linearly independent vectors $\xi_1, \ldots, \xi_k$, where $1 \leq k \leq pn$. So we have

$$(T_{p^1}{}^*N)_x / \Gamma(y) \cong (R^k \times R^{pn-k}) / Z^k \cong T^k \times R^{pn-k},$$

where $T^k$ is a $k$-torus. But, since $(T_{p^1}{}^*N)_x$ acts transitively on $\pi^{-1}(x)$, then $\pi^{-1}(x)$ is diffeomorphic to the coset space $(T_{p^1}{}^*N)_x / \Gamma(y)$; if $\Gamma(y)$ is non-trivial, then $\pi^{-1}(x)$ is diffeomorphic to $T^k \times R^{pn-k}$, which is not simply connected. Thus, $\Gamma(y)$ must be trivial and the action is free.

So far we have shown that $M$ is an affine bundle modelled on $T_{p^1}{}^*N$. Now we choose a (global) section $s$ of $M$ over $N$ (Which is possible if we suppose that $M$ is a paracompact manifold). Then $M$ may be identified with $T_{p}{}^{1*}N$, with $s$ playing the role of zero section. We call the resulting diffeomorphism $F$ and consider the 2-forms

$$\Omega_a=\omega_a-F^*(\omega_a)_0,\quad 1\leq a\leq p.$$

They are closed and verify $i(X)\Omega_a=0$, for any vertical vector field $X \in V$. Then there exist $p$ closed 2-forms $\phi_a$ on $N$ such that

$$\Omega_a=\pi^*\phi_a,\quad 1\leq a\leq p.$$

Since $F$ is fibred over the identity on $N$, we have

$$\omega_a=F^*((\omega_a)_0+\pi_0{}^*\phi_a),\quad 1\leq a\leq p$$

and the result follows.

Corollary 3.1. *Suppose that $(M, \omega_a, V_a)$ verifies all the hypotheses of the Theorem 3.2 except that the leaves of $V$ are simply connected. Then if the leaves of $V$ are assumed to be mutually homeomorphic, $T_{p}{}^{1*}N$ is a covering space of $M$ and the leaves of $V$ are of the form $T^k\times R^{pn-k}$, where $T^k$ is a $k$-dimensional torus. Moreover, if it is assumed that the leaves of $V$ are compact, then $T_{p}{}^{1*}N$ is a covering space of $M$ and this leaves are diffeomorphic to $T^{pn}$.*

## 4. Regular $p$-almost cotangent structures.

We shall define a $p$-almost cotangent structure $(M, \omega_a, V_a)$ to be *regular* if it verifies all the hypotheses of the Theorem 3.2 and say in that case that $(M, \pi, N, \omega_a)$ is a *regular $p$-almost cotangent structure.* If $(M, \pi, N, \omega_a)$, $(\bar{M}, \bar{\pi}, N, \bar{\omega}_a)$ are two regular $p$-almost cotangent structures, they will be said to be *equivalent* if there exists a bundle morphism $F: M\to\bar{M}$ fibred over the identity on $N$ such that

$$F^*\bar{\omega}_a-\omega_a=\pi^*(d\alpha_a),$$

where $\alpha_a$, $1\leq a\leq p$ is a 1-form on $N$, that is, $F^*\bar{\omega}_a-\omega_a$ is cohomologous to zero for any $a$.

Proposition 4.1. *There is a one-to-one correspondence between the set of equivalence classes of regular $p$-almost cotangent structures for a fixed $N$ and elements of $H^2(N, R)\times\overset{p}{\cdots}\times H^2(N, R)$, where $H^2(N, R)$ is the second de Rham cohomology group of $N$.*

*Proof.* Let $(M, \pi, N, \omega_a)$ be a regular $p$-almost cotangent structure.

By Theorem 3.2, there is a diffeomorphism $F_s : M \to T_p{}^{1*}N$ fibred over the identity on $N$ such that

$$\omega_a = F_s{}^*(\omega_a)_0 - \pi^*(\phi_a)_s,$$

where $\phi_a$, $1 \le a \le p$ is a closed 2-form on $N$ and $s$ is the section of $M$ over $N$ which is used to define $F_s$ and upon which it and $\phi_a$ depend. Then $F_s \circ s = s_0$ (zero section of $T_p{}^{1*}N$) and, consequently

$$s^*\omega_a = (\phi_a)_s, \quad 1 \le a \le p.$$

Let $s$ be another section of $M$ over $N$ and $F_s$ the corresponding diffeomorphism of $M$ with $T_p{}^{1*}N$ such that

$$\omega_a = F_{\bar{s}}{}^*(\omega_a)_0 - \pi^*(\phi_a)_{\bar{s}},$$

where $(\phi_a)_{\bar{s}}$ are closed 2-forms on $N$. Then there is a section $\sigma$ of $T_p{}^{1*}N$ over $N$ such that $\bar{s} = s + \sigma$ (that is, $\sigma(x)$ is the vector of $(T_p{}^{1*}N)_x$ such that $\bar{s}(x) = s(x) + \sigma(x)$, for any point $x \in N$). A simple computation shows that

$$F_s \circ \bar{s} = \sigma.$$

Then

$$\begin{aligned}(\phi_a)_{\bar{s}} &= \bar{s}^*\omega_a = \bar{s}^*(F_s{}^*((\omega_a)_0 + \pi^*(\phi_a)_s) \\ &= (F_s \circ \bar{s})^*(\omega_a)_0 + (\pi \circ \bar{s})^*(\phi_a)_s = \sigma^*(\omega_a)_0 + (\phi_a)_s \\ &= (\phi_a)_s + d\sigma_a,\end{aligned}$$

where $\sigma = (\sigma_1, \ldots, \sigma_p)$ is identified with a $p$ tupla of 1-forms on $N$. Then $(\phi_a)_{\bar{s}} - (\phi_a)_s$ is cohomologous to zero for any $a$, and therefore $(M, \pi, N, \omega_a)$ determines an element

$$([(\phi_1)_s], \ldots, [(\phi_p)_s]) \in H^2(N, R) \times \overset{p}{\ldots} \times H^2(N, R).$$

Now, we prove that the mapping defined above is surjective. Let $([\phi_a])$ be a $p$-tupla of cohomology classes of degree 2 on $N$. Then the corresponding regular $p$-almost cotangent structure is given by

$$(T_p{}^{1*}N, \pi_0, N, (\omega_a)_0 + \pi_0{}^*\phi_a).$$

Finally, to prove that the mapping is injective, suppose that $(M, \pi, N, \omega_a)$, $(\bar{M}, \bar{\pi}, N, \bar{\omega}_a)$ are regular $p$-almost cotangent structures and that $F$ and $\bar{F}$ are the respective diffeomorphisms corresponding to the sections $s$ and $\bar{s}$. Then we have

$$\bar{s}^*\omega_a - s^*\omega_a = \bar{\phi}_a - \phi_a = d\alpha_a, \text{ for some 1-forms } \alpha_a,\ 1 \le a \le p.$$

Therefore

$$\begin{aligned}(\bar{F}^{-1} \circ F)^*\omega_a &= F^*((\bar{F}^{-1})^*\omega_a) = F^*((\omega_a)_0 + (\bar{F}^{-1})^*\bar{\pi}^*\bar{\phi}_a) \\ &= F^*(\omega_a)_0 + (\bar{\pi} \circ \bar{F}^{-1} \circ F)^*\bar{\phi}_a = F^*(\omega_a)_0 + \pi^*\bar{\phi}_a \\ &\text{(since } \bar{\pi} \circ \bar{F} = \pi_0 \text{ and } \pi_0 \circ F = \pi)\end{aligned}$$

$$=F^*(\omega_a)_0+\pi^*\bar{s}^*\bar{\omega}_a=\omega_a-\pi^*\phi_a+\pi^*\bar{s}^*\bar{\omega}_a$$
$$=\omega_a-\pi^*(\bar{\phi}_a-\phi_a)=\omega_a-\pi^*(d\alpha_a).$$

Then $F^{-1}\circ F$ is an equivalence of regular $p$-almost cotangent structures. This ends the proof.

The last result of this section shows that the vanishing of the element of $H^2(N, R)\times\overset{p}{\dots}\times H^2(N, R)$ characterizes $T_{p^1}{}^*N$ as a regular $p$-almost cotangent structure up to equivalence.

THEOREM 4. 1. *Suppose that* $(M, \pi, N, \omega_a)$ *is a regular $p$-almost cotangent structure. Then* $(M, \pi, N, \omega_a)$ *is equivalent to* $(T_{p^1}{}^*N, \pi_0, N, (\omega_a)_0)$ *iff the element of* $H^2(N, R)\times\overset{p}{\dots}\times H^2(N, R)$ *it determines is zero. In such a case $\omega_a$ is exact for any $a$,* $1\leq a\leq p$, *say* $\omega_a=-d\lambda_a$, *and the equivalence $F$ verifies* $F^*(\lambda_a)_0=\lambda_a$.

*Proof*. Clearly, the element of $H^2(N, R)\times\dots\times H^2(N, R)$ determined by $(T_{p^1}{}^*N, \pi_0, N, (\omega_a)_0)$ is zero. Furthermore, the element of $H^2(N, R)\times\dots\times H^2(N, R)$ determined by a regular $p$-almost cotangent structure $(M, \pi, N, \omega_a)$ is zero iff there is a section $s$ such that $s^*\omega_a$ is an exact 1-form on $N$. Now, suppose that there is a section $s$ of $M$ over $N$ such that $s^*\omega_a=d\alpha_a$, $1\leq a\leq p$. If $\bar{s}=s-\alpha$, where $\alpha$ is the section of $T_{p^1}{}^*N$ determined by the $\alpha_a$'s, then we have

$$\bar{s}^*\omega_a=s^*\omega_a-d\alpha_a=0.$$

Thus, if $F_{\bar{s}}$ is the diffeomorphism defined by $\bar{s}$, we obtain

$$\omega_a=F_{\bar{s}}^*(\omega_a)_0=F_{\bar{s}}^*(-(d(\lambda_a)_0)=-d(F_{\bar{s}}^*(\lambda_a)_0).$$

Consequently, $(M, \pi, N, \omega_a)$ is equivalent to $(T_{p^1}{}^*N, \pi_0, N, (\omega_a)_0)$ and $\omega_a=-d\lambda_a$, where

$$\lambda_a=F_{\bar{s}}^*(\lambda_a)_0.$$

Conversely, let us suppose that $(M, \pi, N, \omega_a)$ is equivalent to $(T_{p^1}{}^*N, \pi_0, N, (\omega_a)_0)$. Then there is a bundle isomorphism $F: M\to T_{p^1}{}^*N$ such that

$$F^*(\omega_a)_0-\omega_a=\pi^*(d\alpha_a),\quad 1\leq a\leq p.$$

Consider the section of $M$ over $N$ defined by $s=F^{-1}\circ s_0$. Then the corresponding diffeomorphism $F_s$ verifies

$$F_s^*(\omega_a)_0-\omega_a=\pi^*(\phi_a)_s$$

and

$$(\phi_a)_s = s^*\omega_a = s_0^*(F^{-1})^*\omega_a = s_0^*((\omega_a)_0 - (F^{-1})^*\pi^*(d\alpha_a))$$
$$= -(\pi \circ F^{-1} \circ s_0)^*(d\alpha_a) = d\alpha_a$$
$$(\text{since } s_0^*(\omega_a)_0 = 0 \text{ and } \pi_0 \circ F = \pi)$$

Therefore, if we take $\bar{s} = s - \alpha$, where $\alpha$ is the section of $T_{p^1}{}^*N$ defined by $(\alpha_a)$, then we have

$$\bar{s}^*\omega_a = s^*\omega_a - d\alpha_a = 0.$$

Now, we proceed as above.

REMARK. From Theorem 4.1, we deduce that in a geometrical sense, relative to the choice of the section $s$, a regular $p$-almost cotangent structure $(M, \pi, N, \omega_a)$ is completely equivalent to $(T_{p^1}{}^*N, \pi_0, N, (\omega_a)_0)$ (even as vector bundles!). Moreover, the choice of $s$ is not arbitrary; in fact, one must choose $s$ so that $s^*\omega_a = 0$.

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*CECIME
Consejo Superior de Investigaciones Científicas
Serrano, 123, 28006
Madrid, Spain
**Departamento de Geometría y Topología
Facultad de Matemáticas
Universidad de Santiago de Compostela,
Spain