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A NOTE ON THE QUOTIENT RING R((X)) OF THE POWER SERIES RING $R[[X]]$

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In this paper all rings are assumed to be commutative and contain identities. Let $R[X]$ be the polynomial ring in an indeterminate X over a ring R. Let S be the set of all elements
$a_0+a_1x+\ldots+a_nx^n$ of $R[X]$ such that $R=(a_0, a_1, \cdots, a_n)$. Denote the quotient ring $S^{-1}R[X]$ by $R(X)$. Then we have the following well known results, (see [3][4]).

(a) If $\{M_\beta\}_{\beta\in B}$ is the set of maximal ideals of R then $S=R[X]-\bigcup_{\beta\in B} M_\beta[X]$.

(b) S is a multiplicative system in $R[X]$ consisting entirely of regular elements in $R[X]$.

(c) If Q is a P-primary ideal of R then $QR(X)$ is a $PR(X)$-primary ideal of $R(X)$ and $QR(X)\cap R=Q$.

(d) If $\{M_\beta\}_{\beta\in B}$ is the set of maximal ideals of R then $\{M_\beta R(X)\}_{\beta\in B}$ is the set of maximal ideals of $R(X)$.

To get the analogous results in the formal power series ring $R[[X]]$, we choose T be the set of all $f=\sum_{i=0}^{\infty}a_iX^i\in R[[X]]$ such that $A_f=R$ where A_f is the ideal of R generated by the coeffiients of f : $A_f=(a_0, a_1, a_2, \cdots)$. Then T is a multiplicative system in $R[[X]]$. Denote the quotient ring $T^{-1}R[[X]]$ by $R((X))$. But the following example in [2] indicates that there may exist an element f of $R[[X]]$ such that f has a unit coefficient but f is a zero divisor in $R[[X]]$.

EXAMPLE. Let A be a commutative ring with identity; let $\{Y, X_0, X_1, X_2, \cdots, X_i, \cdots\}$ be a set of indeterminates over A; and let $R=A[Y, \{X_i\}_{i=0}^{\infty}]/(X_0Y, \{X_i-X_{i+1}Y\}_{i=0}^{\infty})$.
Let $y=\bar{Y}$ and $f=y-X\in R[[X]]$. Then f has a unit coefficient

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so $f \in T$. However, letting $x_i = \overline{X}_i$ and $g = \sum_{i=0}^{\infty} x_i X^i$, we see that $f \cdot g = 0$ while $g \neq 0$.

In the following theorem we will show that if R is a Noetherian ring then T consists entirely of regular elements in $R[[X]]$ and the properties (a)-(d) mentioned above can be extended to $R((X))$.

THEOREM 1. *Let R be a Noetherian ring with an identity and $R[[X]]$ the power series ring in an indeterminate X over R and T the set of all $f \in R[[X]]$ such that $A_f = R$, then*

(a) *T is a multiplicative system in $R[[X]]$ consisting entirely of regular elements in $R[[X]]$ and $T = R[[X]] - \bigcup_{\beta \in B} M_\beta[[X]]$ where $\{M_\beta\}_{\beta \in B}$ is the set of maximal ideals of R.*

(b) *If Q is a P-primary ideal of R then $QR((X))$ is a $PR((X))$-primary ideal of $R((X))$ and $QR((X)) \cap R = Q$.*

(c) *If $\{M_\beta\}_{\beta \in B}$ is the set of maximal ideals of R then $\{M_\beta R((X))\}_{\beta \in B}$ is the set of maximal ideals of $R((X))$.*

Proof. (a) If $f \in R[[X]]$, then $A_f = R$ if and only if $A_f \not\subseteq M_\beta$ for each $\beta \in B$. Consequently, $T = R[[X]] - (\bigcup_{\beta \in B} M_\beta[[X]])$. Since R is a Noetherian ring, an element f of $R[[X]]$ is a zero divisor if and only if there is a nonzero element c of R such that $c \cdot f = 0$, [2]. Therefore, no element of T is a zero divisor in $R[[X]]$. Clearly T is a multiplicative system in $R[[X]]$ since $T = R[[X]] - \bigcup_{\beta \in B} M_\beta[[X]]$.

In order to prove (b) we shall need the following lemma.

LEMMA. *If R is a Noetherian ring and Q is a P-primary ideal of R, then $QR[[X]]$ is a $PR[[X]]$-primary ideal of $R[[X]]$.*

Proof. To prove that $QR[[X]]$ is $PR[[X]]$-primary, it suffices, by passage to $R[[X]]/Q[[X]]$, to prove for the case where $Q = (0)$. So suppose that (0) is P-primary. Let $f, g \in R[[X]]$ such that $f \cdot g = 0$ and $f \neq 0$. Then there exists a nonzero element c in R such that $cg = 0$. Then $cA_g = (0)$. Therefore, since (0) is P-primary, it follows that $A_g \subseteq P$; that is, $g \in P[[X]]$. Note that $P[[X]] = PR[[X]]$ since R is a Noetherian ring. Clearly $P[[X]]$ is a prime ideal of $R[[X]]$.

Let $\sum_{i=0}^{\infty} a_i X^i \in P[[X]]$, then a_i is nilpotent for each $i=0, 1, 2, \cdots$, since (0) is P-primary. Since R is Noetherian, $\sum_{i=0}^{\infty} a_i X^i$ is a nilpotent element in $R[[X]]$; therefore, Rad $(R[[X]]=P[[X]]$.
Thus (0) is a $P[[X]]$-primary ideal of $R[[X]]$.

Proof of (*b*). Suppose that Q is P-primary, then $QR[[X]]$ is $PR[[X]]$-Primary by Lemma. Then $QR((X))$ is $PR((X))$-primary since $R((X))$ is a quotient ring of $R[[X]]$.
Clearly, $QR[[X]] \cap R=Q$ and $QR[[X]] \subseteq QR((X)) \cap R[[X]]$. Let $f \in QR((X)) \cap R[[X]]$, then there exists g in T such that $fg \in QR[[X]]$. Then $A_g=R$ and A_g is a cancillation ideal of R.
Therefore, $A_f=A_f\ A_g=A_{fg} \subseteq Q$; that is, $f \in QR[[X]]$ and $QR((X)) \cap R[[X]] \subseteq QR[[X]]$. It follows that $QR((X)) \cap R[[X]]=QR[[X]]$ and $QR((X)) \cap R=QR[[X]] \cap R=Q$.

Proof of (*c*). Let C be an ideal of $R[[X]]$ which is contained in $\bigcup_{\beta \in B} M_\beta[[X]]$. Let E be the set of all elements of R which appear as coefficients of some member of C. It is easy to check that E is an ideal of R. Since $C \subseteq \bigcup_{\beta \in B} M_\beta[[X]]$, we see that $1 \notin E$ and so E is a proper ideal of R; therefore, $E \subseteq M_\beta$ for some β. Then it turned out that $C \subseteq M_\beta[[X]]$ and $\{M_\beta R((X))\}_{\beta \in B}$ is the set of maximal ideals of $R((X))$ (By proposition (4. 8) in [3]).

Recently Al-Ezeh [1] discovered the following theorem.

THEOREM 2. *Let $R[X]$ be the polynomial ring in X over a ring R. Then $R[X]$ is a P. F. ring* (*a P. P. ring*) *if and only if R is a P. F ring* (*a. P. P. ring*).

DEFINITION. A ring R is called a P. F. ring if every principal ideal of R is a flat R-module. A ring R is called a P. P. ring if every principal ideal of R is a projective R-module.

DEFINITION. An ideal I of a ring R is called pure if for each element x in I, there exists an element y in I such that $xy=x$.

To prove Theorem 2, Al-Ezeh introduced the following several propositions as lemmas and theorems in [1].

Proposition 1. *A ring R is a P. F. ring if and only if for each $a \in R$, $ann(a)$ is a pure ideal.*

Proposition 2. *If $I_1, I_2, \ldots, I_n$ are pure ideals in R, then $\bigcap_{j=1}^{n} I_j$ is a pure ideal of R.*

Proposition 3. *If R is a P. F. ring, then R has no nonzero nilpotent element.*

Proposition 4. *Let R be a ring without nilpotent elements and let $h(X) = \sum_{i=0}^{n} h_i X^i \in R[X]$. If $\sum_{i=0}^{m} a_i X^i \in \underset{R[X]}{ann}(h(X))$, then $a_i h_j = 0$ for each $i = 1, \cdots, m$ and $j = 1, \cdots, n$.*

Proposition 5. *Let R be a ring and $a \in R$. Then aR is a projective R-module if and only if the annihilator, $\underset{R}{ann}(a)$, is generated by an idempotent element.*

Using proposition 1 and 5, we can easily show that if R is a P. F. (P. P.) ring then $R(X)$ is a P. F. (P. P.) ring. We can extend this result to the ring $R((X))$ if R is a Noetherian ring.

Theorem 3. *Let R be a Noetherian ring. Then R is a P. F. ring if and only if $R[[X]]$ is a P. F. ring.*

Proof. Suppose R is a P. F. ring. Let $h = \sum_{i=0}^{\infty} h_i X^i \in R[[X]]$ and let $f = \sum_{i=0}^{\infty} a_i X^i \in \underset{R}{\text{ann}}(h)$. Since R has no nonzero nilpotent element, $a_i h_j = 0$ for each $i = 0, 1, 2, \cdots$ and $j = 0, 1, 2, \cdots$. Note that proposition 4 holds when $R[X]$ is replaced by $R[[X]]$. Consider the ideals A_f and A_h of R. Since R is Noetherian, A_f and A_h are finitely generated, say $A_f = (a_0, a_1, \cdots, a_m)$ and $A_h = (h_0, h_1, \cdots, h_n)$. Then $a_i \in J = \bigcap_{j=0}^{n} \underset{R}{\text{ann}}(h_j)$ for each $i = 0, 1, \cdots, m$. Since R is a P. F. ring, $\underset{R}{\text{ann}}(h_j)$ is a pure ideal of R; therefore, by Proposition 2, J is a pure ideal of R. Hence there exist $b_0, b_1, \cdots, b_m$ in J such that $a_i b_i = a_i$ for each $i = 0, 1, \cdots, m$. Now, we find an element c in J such that $c\left(\sum_{i=0}^{m} a_i X^i\right) = \sum_{i=0}^{m} a_i X^i$. Al-Ezeh constructed this element c in the proof of theorem 2.

First, $a_0b_0=a_0$. Consider $(a_0+a_1X)(b_0+b_1-b_1b_0)$

$$=a_0b_0+a_0b_1-a_0b_0b_1+a_1b_0X+a_1b_1X-a_1b_0b_1X$$
$$=a_0+a_0b_1-a_0b_1+a_1b_0X+a_1X-a_1b_0X=a_0+a_1X.$$

Let $c_1=b_0+b_1-b_0b_1$. Then $(a_0+a_1X+a_2X^2)(c_1+b_2-b_2c_1)$

$$=(a_0+a_1X)c_1+(a_0+a_1X)(1-c_1)b_2+a_2X^2(c_1+b_2-b_2c_1)$$
$$=a_0+a_1X+a_2c_1X^2+a_2b_2X^2-a_2b_2c_1X^2$$
$$=a_0+a_1X+a_2c_1X^2+a_2X^2-a_2c_1X^2=a_0+a_1X+a_2X^2.$$

Let $c_2=c_1+b_2-b_2c_1, \cdots, c_m=c_{m-1}+b_m-c_{m-1}b_m$.

Suppose $(a_0+a_1X+\cdots+a_kX^k)c_k=(a_0+a_1X+\cdots+a_kX^k)$. Then

$$(a_0+a_1X+\cdots+a_kX^k+a_{k+1}X^{k+1})c_{k+1}$$
$$=(a_0+a_1X+\cdots+a_kX^k+a_{k+1}X^{k+1})(c_k+b_{k+1}-c_kb_{k+1})$$
$$=(a_0+a_1X+\cdots+a_kX^k)c_k+(a_0+a_1X+\cdots+a_kX^k)$$
$$\cdot(b_{k+1}-c_kb_{k+1})+a_{k+1}X^{k+1}(c_k+b_{k+1}-c_kb_{k+1})$$
$$=(a_0+a_1X+\cdots+a_kX^k)+a_{k+1}c_kX^{k+1}+a_{k+1}b_{k+1}X^{k+1}$$
$$-a_{k+1}b_{k+1}c_kX^{k+1}=a_0+a_1X+\cdots+a_kX^k+a_{k+1}X^{k+1}$$

Therefore, $(a_0+a_1X+\cdots+a_mX^t)c_t=a_0+a_1X+\cdots+a_mX^t$ for each $t=0, 1, \cdots, m$. *Let* $c=c_m$, then clearly $c\in J=\bigcap_{j=0}^{n}\underset{R}{\text{ann}}(h_j)$ and $a_0c=a_0$, $a_1c=a_1c=a_1, \cdots, a_mc=a_m$. Since $A_f=(a_0, a_1, \cdots, a_m)$, it follows that $ac=a$ for any $a\in A_f$. Hence $cf=f$; therefore, $\underset{R[[X]]}{\text{ann}}(h)$ is a pure ideal of $R[[X]]$ and $R[[X]]$ is a P. F. ring.

Conversely, assume $R[[X]]$ is a P. F. ring. Let $a\in R$ and $b\in\underset{R}{\text{ann}}(a)$. Then $b\in\underset{R[[X]]}{\text{ann}}(a)$. Since $R[[X]]$ is a P. F. ring, there exists $g=\sum_{i=0}^{\infty}c_ix^i$ in $\underset{R[[X]]}{\text{ann}}(a)$ such that $bg=b$. Then $bc_0=b$ and $c_0\in\underset{R}{\text{ann}}(a)$. Hence $\underset{R}{\text{ann}}(a)$ is a pure ideal of R and R is a P. F. ring.

THEOREM 4. *Let R be a Noetherian ring then R is a P. P. ring if and only if $R[[X]]$ is a. P. P. ring.*

Proof. Suppose that R is a P. P. ring. Let $h=\sum_{i=0}^{\infty}h_ix^i\in R[[X]]$ and $f=\sum_{i=0}^{\infty}a_ix^i\in\underset{R[[X]]}{\text{ann}}(h)$. Since R has no nonzero nilpotent element, $a_ih_j=0$ for each $i=0, 1, 2, \cdots$ and $j=0, 1, 2, \cdots$. Since R is Noetherian, A_h is finitely generated, say $A_h=(h_0, h_1, \cdots, h_n)$. Let $N=\underset{R}{\mathrm{ann}}(h_0, h_1, \cdots, h_n)$, then $a_i\in N$ for each $i=0, 1, 2, \cdots$. Therefore, $f\in N[[X]]$ and $\underset{R[[X]]}{\mathrm{ann}}(h)\subseteq N[[X]]$. If $g=\sum_{i=0}^{\infty}b_ix^i\in N[[X]]$ then $A_g\subseteq N=\mathrm{ann}(A_h)$; therefore, $g\in\underset{R[[X]]}{\mathrm{ann}}(h)$. Hence $\underset{R[[X]]}{\mathrm{ann}}(h)=N[[X]]$ and $N=\underset{R}{\mathrm{ann}}(h_0, h_1, \cdots, h_n)=\bigcap_{j=0}^{n}\underset{R}{\mathrm{ann}}(h_j)$. Since R is a P.P. ring, $\underset{R}{\mathrm{ann}}(h_j)=e_jR$ for each $j=0, 1, \cdots, n$ where e_j is an idempotent element. Then $N=\bigcap_{j=0}^{n}e_jR=(e_1e_2\cdots e_n)R=eR$ where $e=e_1e_2\cdots e_n$ is an idempotent element. Therefore, $\underset{R[[X]]}{\mathrm{ann}}(h)=eR[[X]]$, i.e. $\underset{R[[X]]}{\mathrm{ann}}(h)$ is generated by an idempotent element e, hence $R[[X]]$ is a P.P. ring.

Conversely, suppose that $R[[X]]$ is a P.P. ring. Let $a\in R$. Then $\underset{R[[X]]}{\mathrm{ann}}(a)=gR[[X]]$ for some $g\in R[[X]]$ such that $g^2=g$. If $g=\sum_{i=0}^{\infty}b_iX^i$ then $b_0{}^2=b_0$ and $ab_0=0$. Let $b\in\underset{R}{\mathrm{ann}}(a)$, then $ba=0$. Then, $b\in\underset{R[[X]]}{\mathrm{ann}}(a)$ so $b\in gR[[X]]$ and $b=b_0c$ for some $c\in R$. Hence $\underset{R}{\mathrm{ann}}(a)\subseteq b_0R$. To show the opposite inclusion, let $d\in b_0R$. Then $d=b_0d_0$ for some $d_0\in R$. Since $b_0\in\underset{R}{\mathrm{ann}}(a)$, $d\in\underset{R}{\mathrm{ann}}(a)$; therefore, $\underset{R}{\mathrm{ann}}(a)\supseteq b_0R$ so $\mathrm{ann}(a)=b_0R$. Thus R is a P.P. ring.

COROLLARY 5. *Let R be a Noetherian ring. Then R is a P.F. ring (a P.P. ring) if and only if $R((X))$ is a P.F. ring (a P.P. ring).*

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