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REDUCED GAUGE GROUP AND THE MODULI SPACE OF STABLE BUNDLES

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0. When one considers the equivalence relation among holomorphic structures on a smooth complex vector bundle E over a complex manifold M, the natural "gauge group" is the group $GL(E)$ of all smooth bundle automorphisms of E. This group can be reduced to the "unitary group" $U(E)$ (see (5.2)) through Hermite-Einstein connections [2]. When Yang-Mills equation is concerned on a bundle E with $c_1(E)=0$, the equivalence relation is provided by the special unitary group $SU(E)$. In the paper, we show that moduli spaces using $U(E)$ are the same as the moduli spaces using $SU(E)$, if the first Betti number of M vanishes (Theorems (3.3), (4.2), (5.4)).

1. Let E be a smooth complex vector bundle over a compact oriented Riemannian manifold M. We fix a unitary structure h on E. The set of all connections D on E compatible with h is denoted by $Con(E)$. It is an affine space parallel to the space $A^1(u(E))$ of 1-forms on M with values in the real vector bundle $u(E)$ of skew-hermitian endomorphisms of E. The group $U(E)$ of all isometries g of the bundle E acts on the space of connections;

(1.1) $$g(D)=D-\tilde{D}(g)\cdot g^{-1},$$

where $\tilde{D}$ is the covariant derivative on the bundle $End\ E$ induced from the connection D on E. The kernel of the group homomorphism

(1.2) $$det : U(E)\to C^\infty(M, S^1)$$

is the group $SU(E)$ of the special unitary automorphisms of E. Now we have an obvious surjection

(1.3) $$Con(E)/SU(E)\to Con(E)/U(E)$$

2. Let $L=det\ E:=\bigwedge^r E$, $r=rank(E)$, be the line bundle obtained from E. We equip L with the unitary structure $k=det\ h$ induced from

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h. Then the natural map

(2.1) $$det : Con(E) \to Con(L)$$

is a surjection. If $D \in Con(E)$ and $\alpha \in A^1(u(E))$, then

(2.2) $$det(D+\alpha) = det(D) + tr(\alpha).$$

If $\nabla \in Con(L)$, then fiber over ∇ is denoted by $Con(E, \nabla)$ so that

(2.3) $$Con(E) = \cup_{\nabla} Con(E, \nabla).$$

Then the spaces $Con(E, \nabla)$ are all isomorphic affine spaces parallel to $A^1(su(E))$, where $su(E)$ is the real vector bundle of trace-free skew-hermitian endomorphisms of E. The decomposition (2.3) corresponds to the decomposition of the Lie algebra

(2.4) $$u(r) = center \oplus su(r).$$

Since $Con(E, \nabla)$ is invariant under the action of $SU(E)$. we have a commutative diagram of surjections;

(2.5) $$\begin{array}{ccc} Con(E)/SU(E) & \to & Con(E)/U(E) \\ l \downarrow & & \downarrow \\ Con(L) & \to & Con(L)/U(L) \end{array}$$

Here $U(L)$ is equal to $C^\infty(M, S^1)$ and the left vertical arrow l is a trivial fibration;

(2.6) $$Con(E)/SU(E) \simeq Con(L) \times (Con(E, \nabla)/SU(E)).$$

for any $\nabla \in Con(L)$.

(2.7) Remark. It is easy to see that the upper arrow in (2.5) is an injection on each fiber of l. For, if D_1 and D_2 are elements of $Con(E, \nabla)$ such that $D_1 = g(D_2) = D_2 - \tilde{D}_2(g)g^{-1}$ for some $g \in U(E)$, then by taking determinant, we get

$$\nabla = \nabla - tr\, \tilde{D}_2(g)g^{-1}.$$

Thus the following 1-form on M vanishes identically;

(2.8) $$tr\, \tilde{D}_2(g)g^{-1} = d(det\, g) \cdot det\, g^{-1}.$$

This shows that $det\, g$ is locally constant and hence

$$D_1 = \tilde{g}(D_2), \quad \tilde{g} \in SU(E, h),$$

where $\tilde{g} = g / \sqrt[r]{det\, g}$.

3. The Yang-Mills functional $YM : Con(E) \to \boldsymbol{R}$ is defined by

$$YM(D) = \int_M |R|^2 vol = \int_M tr(R \wedge {}^*R),$$

where vol is the volume element of M and $*$ is the Hodge star. The Yang-Mills functional is invariant under the gauge group $U(E)$. The critical points of YM are called *Yang–Mills connections* and the set of all Yang–Mills connections is denoted by $YMC(E)$. We will assume that $YMC(E)$ is nonempty. Then the diagram (2.5) induces a commutative diagram of surjections;

$$(3.1)\qquad \begin{array}{ccc} YMC(E)/SU(E) & \to & YMC(E)/U(E) \\ \downarrow & & \downarrow \\ Z^1(M)\simeq YMC(L) & \to & YMC(L)/U(L)\simeq H^1(M;\boldsymbol{R})/H^1(M;\boldsymbol{Z}), \end{array}$$

where $Z^1(M)$ is the space of closed 1–forms on M. Here the left vertical arrow is again a trivial fibration;

$$(3.2)\qquad YMC(E)/SU(E)\simeq YMC(L)\times(YMC(E,\nabla)/SU(E)),$$

where $YMC(E,\nabla)=YMC(E)\cap Con(E,\nabla)$. Note that when $c_1(E)=0$, $YMC(E,\nabla)/SU(E)$ is the moduli space of Yang–Mills connections for the $SU(r)$–bundle E, where ∇ is the trivial connection for a trivialization of $det\,E$ ([3]).

(3.3) THEOREM. *If the first Betti number $b_1(M)=0$, then*

$$YMC(E,\nabla)/SU(E)\simeq YMC(E)/U(E)$$

for any $\nabla\in YMC(L)$.

Proof. The map $YMC(E,\nabla)/SU(E)\to YMC(E)/U(E)$ is injective as seen in the remark (2.7). Let $D\in YMC(E)$ be given. Then

$$det\,D=\nabla+\alpha$$

for some pure imaginary closed 1–form α on M. Since $b_1(M)=0$,

$$\alpha=\sqrt{-1}\,df$$

for some $f\in C^\infty(M,\boldsymbol{R})$. Now let $g=\exp(\sqrt{-1}f/r)1_E\in U(E)$. Then $g(D)\in YMC(E,\nabla)$, since

$$\begin{aligned} det\ g(D)&=(det\ g)(det\ D)=det\ D-(d\ det(g))/det(g)\\ &=det\ D-\sqrt{-1}\,df=\nabla. \end{aligned}$$

Thus the equivalence class in $YMC(E,\nabla)/SU(E)$ represented by $g(D)$ is mapped to the equivalence class in $YMC(E)/U(E)$ represented by D. Q. E. D.

4. If we consider self–dual connections on a 4–manifold M, then as in (3.1) we also have a commutative diagram with surjective arrows;

$$(4.1)\qquad \begin{array}{ccc} SDC(E)/SU(E) & \to & SDC(E)/U(E) \\ \downarrow & & \downarrow \\ Z^1(M)\simeq SDC(L) & \to & SDC(L)/U(L)\simeq H^1(M;\boldsymbol{R})/H^1(M;\boldsymbol{Z}), \end{array}$$

where $SDC(E)$ denotes the set of all self–dual connections on the hermitian vector bundle (E, h). We will assume that $SDC(E)$ is nonempty. Let

$$SDC(E, \nabla) = SDC(E) \cap Con(E, \nabla)$$

for any $\nabla \in SDC(L)$. Then theorem (3.3) is true with YMC replaced by SDC, and the proof is the same.

(4.2) THEOREM. *If* $\dim M = 4$ *and* $b_1(M) = 0$, *then*

$$SDC(E, \nabla)/SU(E) \simeq SDC(E)/U(E)$$

for any $\nabla \in SDC(L)$.

5. In this section, we assume that M is a compact Kähler manifold with a Kähler (1, 1)-form Φ. As before, a C^∞ hermitian vector bundle E over M is fixed. For a unitary connection D on E, let K be the contraction of the curvature tensor R with respect to the Kähler from Φ,

$$K = \sqrt{-1} \wedge R,$$

where $\wedge$ is the adjoint of the exterior mutiplication by Φ.

(5.1) DEFINITION. A connection $D \in Con(E)$ is called a *Hermite-Einstein connection* if R is of type (1, 1) and $K = c \cdot 1_E$ for some constant c.

The constant c in the definition is uniquely determined [6] in terms of a topological invariant of E and the cohomology class $[\Phi] \in H^2(M, \mathbf{Z})$. The set of all Hermite-Einstein connections on E is denoted by $HEC(E)$. We assume that $HEC(E)$ is nonempty. A *holomorphic structure* on E is a semi-connection ([1], [5], [6])

$$D'' : A^{0,0}(E) \to A^{0,1}(E)$$

such that $D'' \circ D'' = 0$. The set of all holomorphic structures on E is denoted by $Hol(E)$. Then the group $GL(E)$ of the bundle automorphisms of E acts naturally on $Hol(E)$;

$$g(D'') = g \circ D'' \circ g^{-1}.$$

It is known that a Hermite-Einstein connection determines a holomorphic structure on E and there is a natural embedding ([2], [4])

$$HEC(E)/U(E) \to Hol(E)/GL(E). \tag{5.2}$$

As results of Kobayashi-Lübke [6] and Uhlenbeck-Yau [7], the image of this map is exactly the isomorphism classes of quasi-stable holomorphic structures (a holomorphic structure is *quasi-stable* if and only if it is a direct sum of stable holomorphic structures with the same constant c as in (5.1)). Now as in section 1, we consider the natural projection

$$HEC(E) \to HEC(L) \simeq H^1(M, O),$$

where $L = det\, E$ is equipped with the metric $k = det\, h$, as before. Then we have a commutative diagram of surjections

$$\begin{array}{ccc} HEC(E)/SU(E) & \to & HEC(E)/U(E) \\ \downarrow & & \downarrow \\ HEC(L) & \to & HEC(L)/U(L) \simeq Pic^0(M) \end{array} \tag{5.3}$$

Now the following result is easily obtained.

(5.4) THEOREM. *If M is a compact Kähler manifold with $b_1(M) = 0$, then*

$$HEC(E, \nabla)/SU(E) \simeq HEC(E)/U(E)$$

for any $\nabla \in HEC(L)$.

When M is a Kähler surface and $c_1(E) = 0$, then Hermite–Einstein connections are exactly anti-self-dual connections. Thus we obtain;

(5.5) COROLLARY. *If M is a Kähler surface with $b_1(M) = 0$ and E is a smooth vector bundle with $c_1(E) = 0$, then the moduli space of quai-stable holomorphic structures on E is the same as the moduli space of anti-self-dual $SU(r)$-connections on E.*

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