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ON GENERATORS AND NONLINEAR SEMIGROUPS IN BANACH SPACES*

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1. Introduction

Let E be a real Banach space and let C be a subset of E. Let $S(t)$: $C \to C$ be a continuous semigroup of type ω, which we denote by $S \in Q_\omega(C)$. The following is the well-known problem: Does there exist an operator A which generates S via the exponential formula:

$$S(t)x = \lim_{n\to\infty} (I + \frac{t}{n}A)^{-n}x$$

for every $x \in C$? For the Hilbert space H and $S \in Q_0(H)$, the problem was completely solved positively by Komura [9, 10]. In case that C is a closed convex subset of H and $S \in Q_0(C)$, the proof was simplified by Kato [9]. We also find the proof in [3]. On the other hand, in [5], Crandall and Liggett proved that if C is a closed convex subset of a finite dimensional Banach space and $S \in Q_0(C)$, then there exists an operator A which generates S via an exponential formula. Baillon [1] established an existence theorem of an operator A, which was conjectured by Komura, under the assumption that C is a closed convex subset of a uniformly smooth Banach space and $S \in Q_\omega(C)$. In particular, Reich [14] showed that if C is a closed convex subset of a reflexive Banach space with a uniformly Gâteaux differentiable norm and $S \in Q_0(C)$, then there is a unique operator A which generates S via the exponential formula.

In this paper, we prove the existence of an operator A under assumption that C is a closed convex subset of a Banach space with a uniformly Gâteaux differentiable norm and $S \in Q_\omega(C)$. The idea of our proof is essentially due to the work [1] of Baillon. However our proof is of interest in view of use of Banach limit.

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2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. Let $U=\{x\in E : \|x\|=1\}$ be its unit sphere. The norm of E is said to be Gâteaux differentiable (and E is said to be smooth) if

$$\lim_{t\to 0}(\|x+ty\|-\|x\|)/t \tag{2.1}$$

exists for each $x, y\in U$. It is said to be Fréchet differentiable if for each $x\in U$, this limit is attained uniformly for $y\in U$. The norm is said to be uniformly Gâteaux differentiable if for each $y\in U$, the limit (2.1) exists uniformly for $x\in U$. The space E is said to have a uniformly Fréchet differentiable norm (and E is said to be uniformly smooth) if the limit is attained uniformly for $(x, y)\in U\times U$. Every Banach space with a uniformly convex dual has a uniformly Gâteaux differentiable norm, but there are reflexive Banach spaces with a uniformly Gâteaux differentiable norm that are not even isomorphic to a uniformly smooth space (cf. [12, p. 149]).

Recall that the duality map J from E into the family of nonempty (by the Hahn–Banach theorem) weak–star compact subsets of E^* is defined by

$$J(x)=\{x^*\in E^* : (x, x^*)=\|x\|^2=\|x^*\|^2\}$$

for each $x\in E$. If $\phi(x)=\frac{1}{2}\|x\|^2$, J coincides with the subdifferential $\partial\phi$ of ϕ:

$$J(x)=\{x^*\in E^* : (z, x^*)\leq\frac{1}{2}\|x+z\|^2-\frac{1}{2}\|x\|^2 \text{ for every } z\in E\}$$

(cf. [2]). It is well–known that if E is smooth, then J is single valued. It is also known that if E has a uniformly Gâteaux differentiable norm, J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak–star topology of E^* (cf. [6, 7]).

Let S be a directed set, and let LIM be a Banach generalized limit. Then, if $\{x_\alpha : \alpha\in S\}$ is a bounded subset of E, we can define the real valued continuous convex and coercive function ϕ on E by

$$\phi(z)=\mathrm{LIM}\|x_\alpha-z\|^2.$$

By the methods of [13], we obtain the following.

LEMMA 2.1. *Let C be a nonempty closed convex subset of a real Banach space E with a uniformly Gâteaux differentiable norm, and let S be a* *directed set, let* $\{x_\alpha : \alpha\in S\}$ *be a bounded set of* E, *and let LIM be a Banach generalized limit. Let* $u\in C$. *Then* ϕ *attains its minimum over* C *at* u *if and only if*

$$LIM(z-u, J(x_\alpha-u))\leq 0 \qquad (2.2)$$

for all $z\in C$, *where* J *is the duality map of* E.

Proof. For $z\in C$ and $t\in(0,1)$, we have

$$\|x_\alpha-u\|^2\geq\|x_\alpha-u-t(z-u)\|^2+2t(z-u, J(x_\alpha-u-t(z-u))).$$

Let $\varepsilon>0$ be given. Since the norm of E is uniformly Gâteaux differentiable, the duality map J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak-star topology of E^*. Therefore,

$$|(z-u, J(x_\alpha-u-t(z-u))-J(x_\alpha-u))|<\varepsilon$$

if t is close enough to 0. Consequenty, we have

$$\begin{aligned}(z-u, J(x_\alpha-u))&<\varepsilon+(z-u, J(x_\alpha-u-t(z-u)))\\ &\leq\varepsilon+\frac{1}{2t}\{\|x_\alpha-u\|^2-\|x_\alpha-u-t(z-u)\|^2\}\end{aligned}$$

and hence

$$\mathrm{LIM}(z-u, J(x_\alpha-u))\leq\varepsilon+\frac{1}{2t}\{\mathrm{LIM}\|x_\alpha-u\|^2-\mathrm{LIM}\|x_\alpha-u-t(z-u)\|^2\}<\varepsilon.$$

Therefore we have $\mathrm{LIM}(z-u, J(x_\alpha-u))\leq 0$. Conversely, suppose (2.2) holds. Let $z, u\in C$. Then, since

$$\|x_\alpha-z\|^2-\|x_\alpha-u\|^2\geq 2(u-z, J(x_\alpha-u))$$

for all $\alpha\in S$, we have

$$\phi(u)=\mathrm{LIM}\|x_\alpha-u\|^2=\min_{z\in C}\{\mathrm{LIM}\|x_\alpha-z\|^2\}$$

and completes the proof.

If A is a subset of $E\times E$ and $x\in C$, we let $Ax=\{y\in E : [x, y]\in A\}$, $D(A)=\{x\in E : Ax\neq\phi\}$, and $R(A)=\cup\{Ax : x\in D(A)\}$. Its inverse is defined by $A^{-1}y=\{x\in E : y\in Ax\}$. A is called accretive if for all $x_i\in D(A)$, $y_i\in Ax$, $i=1,2$ and $r>0$, $\|x_1-x_2\|\leq\|x_1-x_2+r(y_1-y_2)\|$. A is accretive if and only if for each $x_i\in D(A)$ and $y_i\in Ax_i$, $i=1,2$, there exists $j\in J(x_1-x_2)$ such that $(y_1-y_2, j)\geq 0$. Let I denote the identity operator. If A is accretive, we can define, for $r>0$, the resolvent of A, $J_r : R(I+rA)\to D(A)$ by $J_r=(I+rA)^{-1}$.

A continuous semigroup of type ω ($\omega\in R$) on a subset $C\subset E$ is a function $S(t) : C\to C$ satisfying the following conditions:

$S(t_1+t_2)x=S(t_1)S(t_2)x$ for $t_1, t_2\geq 0$ and $x\in C$,
$\|S(t)x-S(t)y\|\leq e^{\omega t}\|x-y\|$ for $t\geq 0$ and $x, y\in C$,
$\lim_{t\downarrow 0} S(t)=S(0)x=x$ for $x\in C$.

We denote by $S\in Q_\omega(C)$. If $\omega=0$, $S(t)$ is said to be a continuous semigroup of nonlinear contractions. The strong (negative) generator A_0 of $S(t)$ on C is defined by

$$A_0x=\lim_{t\downarrow 0}(x-S(t)x)/t \text{ for } x\in D(A_0),$$

where $D(A_0)=\{x\in C: \lim_{t\downarrow 0}(x-S(t)x)/t \text{ exists}\}$.

Let B be a subset of E. Then we denote by $\mathrm{cl}(B)$ the closure of B and $\overline{\mathrm{co}}(B)$ its closed convex hull.

We conclude this section with the following well-known results.

THEOREM 2.1 [5]. *Let A be an operator in a real Banach space E. Let $\omega\in R$ such that $(A+\omega I)$ is accretive. If $R(I+\lambda A)\supset cl(D(A))$ for small enough $\lambda>0$, then*

$$S(t)x=\lim_{n\to\infty}\left(I+\frac{t}{n}A\right)^{-n}x \tag{2.3}$$

exists for all $x\in cl(D(A))$ and $t>0$. Moreover, $S\in Q_\omega(cl(D(A)))$.

THEOREM 2.2 [4, 5, 12]. *In addition to the conditions of Theorem 2.1, suppose that*

(a) *$R(I+\lambda A)\supset\overline{co}(D(A))$ for all small enough $\lambda>0$,*

(b) *A is a cloesd subset of $E\times E$.*

Then for all $x\in D(A)$ and $T\in[0,\infty)$, the following are equivalent:

(i) *$u(t)$ is a strong solution of*

$$\frac{du(t)}{dt}+Au(t)\ni 0 \qquad a.e.\ t\in[0,T),$$
$$u(0)=x.$$

(ii) $u(t)=\lim_{n\to\infty}\left(I+\frac{t}{n}A\right)^{-n}x \qquad t\in[0,T),$

and $u(t)$ is strongly differentiable a.e. on $[0,T)$.

3. Resolvent consistency

In this Section, we prove the following result.

THEOREM 3.1. *Let E be a Banach space with a uniformly Gâteaux differentiable norm. Let C be a closed convex subset of E and $S\in Q_\omega(C)$.*

If $S(t)x$ is uniformly continuous on bounded (t, x) sets, then for each $x\in C$ and $\lambda>0$ such that $\lambda\omega<1$,

$$\lim_{t\downarrow 0}(I+\frac{\lambda}{t}(I-S(t)))^{-1}x$$

exists.

Proof. We first note that $(I+(\lambda/t)(I-S(t)))^{-1}x$ does exist for all $x\in C$ and positive λ and t. To see this, let $x\in C$ and define $T: C\to C$ by

$$Ty=\frac{t}{t+\lambda}x+\frac{\lambda}{t+\lambda}S(t)y$$

for all $y\in C$. Then T is a strict contraction for $t>0$ if $\omega\leq 0$ and $t\in (0, (1-\lambda\omega)/\omega]$ if $\omega>0$. Therefore, for small enough λ and t, T has a unique fixed point $J_{\lambda,t}x=(I+(\lambda/t)(I-S(t)))^{-1}x$. For a fixed λ, $J_{\lambda,t}x$ will be denoted by y_t.

For $t>0$, we define

$$\alpha(t, x)=\sup\{\|S(\tau)x-x\| : 0\leq\tau\leq t\}.$$

We note that since $S(t)$ is continuous, $\lim_{t\downarrow 0}\alpha(t, x)=0$.

Our next step is to prove that $\{y_t\}$ remains bounded as $t\downarrow 0$. Since $S(t)y_t=(I+t/\lambda)y_t-(t/\lambda)x$, we have for $k\in N$

$$\begin{aligned} e^{\omega t}\|y_t-S(kt)x\| &\geq \|S(t)y_t-S((k+1)t)x\| \\ &=\left\|\left(1+\frac{t}{\lambda}\right)y_t-S((k+1)t)x-\frac{t}{\lambda}x\right\| \\ &\geq\left(1+\frac{t}{\lambda}\right)\|y_t-S((k+1)t)x\|-\frac{t}{\lambda}\|S((k+1)t)x-x\| \\ &=\|y_t-S((k+1)t)x\|+\frac{t}{\lambda}[\|y_t-S((k+1)t)x\| \\ &\quad -\|S((k+1)t)x-x\|]. \end{aligned}$$

Summing from $k=0$ to $n-1$, we obtain

$$\begin{aligned} \|y_t-x\| &\geq e^{-n\omega t}\|y_t-S(nt)x\| \\ &\quad +\frac{t}{\lambda}\sum_{k=1}^{n}e^{-k\omega t}[\|y_t-S(kt)x\|-\|S(kt)x-x\|]. \end{aligned}$$

If $\omega=0$, we have

$$\begin{aligned} \|y_t-x\| &\geq\|y_t-x\|-\|S(nt)x-x\| \\ &\quad +\frac{t}{\lambda}\sum_{k=1}^{n}[\|y_t-x\|-2\|S(kt)x-x\|] \end{aligned}$$

$$\geq \|y_t - x\| - \alpha(nt, x) + \frac{nt}{\lambda}\|y_t - x\| - \frac{2nt}{\lambda}\alpha(nt, x).$$

Thus $\|y_t - x\| \leq \left(2 + \frac{\lambda}{nt}\right)\alpha(nt, x)$.

Fix a positive T, define an integer $n = n_t$ and α_t by $T = nt + \alpha_t$, $0 \leq \alpha_t < t$. Then we have

$$\limsup_{t \downarrow 0} \|y_t - x\| \leq \left(2 + \frac{\lambda}{T}\right)\alpha(T, x). \tag{3.1}$$

If $\omega \neq 0$, we have by the same method,

$$\begin{aligned} \|y_t - x\| &\geq e^{-n\omega t}(\|y_t - x\| - \|S(nt)x - x\|) \\ &\quad + \frac{t}{\lambda}\sum_{k=1}^{n} e^{-k\omega t}[\|y_t - x\| - 2\|S(kt)x - x\|] \\ &\geq \|y_t - x\|\left[e^{-n\omega t} + \frac{t}{\lambda}\sum_{k=1}^{n} e^{-k\omega t}\right] \\ &\quad - \alpha(nt, x)\left[e^{-n\omega t} + \frac{2t}{\lambda}\sum_{k=1}^{n} e^{-k\omega t}\right], \end{aligned}$$

so that

$$\|y_t - x\| \leq \left[\left(e^{-n\omega t} + \frac{2t}{\lambda}\sum_{k=1}^{n} e^{-k\omega t}\right) \Big/ \left(e^{-n\omega t} + \frac{t}{\lambda}\sum_{k=1}^{n} e^{-k\omega t} - 1\right)\right]\alpha(nt, x)$$

Thus we obtain for $\lambda\omega < 1$,

$$\limsup_{t \downarrow 0} |y_t - x\| \leq \left[2 + \frac{\lambda\omega e^{-\omega T}}{1 - e^{-\omega T}}\right]\frac{\alpha(T, x)}{1 - \lambda\omega}. \tag{3.2}$$

Therefore $\{y_t\}$ is bounded as $t \downarrow 0$.

Let $z \in C$. For small enough $t > 0$, we have

$$\begin{aligned} e^{2\omega t}\|y_t - S(kt)z\|^2 &\geq \|S(t)y_t - S((k+1)t)z\|^2 \\ &\geq \|y_t - S((k+1)t)z + \frac{t}{\lambda}(y_t - x)\|^2 \\ &\geq \|y_t - S((k+1)t)z\|^2 \\ &\quad + \frac{2t}{\lambda}(y_t - x, J(y_t - S((k+1)t)z)). \end{aligned}$$

Summing we have

$$\begin{aligned} &\|y_t - z\|^2 \\ &\geq e^{-2n\omega t}\|y_t - S(nt)z\|^2 + \frac{2t}{\lambda}\sum_{k=1}^{n} e^{-2k\omega t}(y_t - x, J(y_t - S(kt)z)). \end{aligned} \tag{3.3}$$

We also note that for $z \in C$,

$$\begin{aligned} &(y_t - x, J(y_t - z)) - (y_t - x, J(y_t - S(kt)z)) \\ &= \|y_t - z\|^2 + (z - x, J(y_t - z)) - \|y_t - S(kt)z\|^2 \end{aligned}$$

$$
\begin{aligned}
&-(S(kt)z-x, J(y_t-S(kt)z)) \\
&\leq \|S(kt)z-z\|\{\|y_t-z\|+\|y_t-S(kt)z\|\} \\
&\quad +(z-x, J(y_t-z)-J(y_t-S(kt)z)) \\
&\quad -(S(kt)z-z, J(y_t-S(kt)z)).
\end{aligned}
$$

Thus given $\varepsilon>0$,

$$
\begin{aligned}
&|(y_t-x, J(y_t-z))-(y_t-x, J(y_t-S(kt)z))| \\
&\leq \alpha(T,z)M_1+|(z-x, J(y_t-z)-J(y_t-S(kt)z))| \\
&\quad +\alpha(T,z)M_2<\varepsilon \quad \text{if } T \text{ is small enough.} \qquad (3.4)
\end{aligned}
$$

Let $\{y_{t_m}\}$ be a subsequence of $\{y_t\}$ with $t_m\downarrow 0$. Then, since $\{y_{t_m}\}$ is bounded, for a Banach generalized limit LIM through $N=\{1, 2, \cdots\}$; i. e., a Banach limit, we can define a real valued function ϕ on C by $\phi(z)=\mathrm{LIM}\|y_{t_m}-z\|^2$ for each $z\in C$.

We will now show that if $t_m\downarrow 0$, then there is a subsequence of $\{y_{t_m}\}$ that is strongly convergent.

To this end, denote y_{t_m} by y_m. Let L denote $\inf\{\phi(z) : z\in C\}$ and consider a sequence $\{u_j\}\subset C$ such that $\lim_{j\to\infty}\phi(u_j)=L$ and $\phi(u_j)\leq\phi(tu_j+(1-t)x)$ for all $0\leq t\leq 1$ and all j. Then, by Lemma 2. 1, we have

$$
\mathrm{LIM}(x-u_j, J(y_m-u_j))\leq 0 \qquad (3.5)
$$

for all j. Since $\phi(z)\to\infty$ as $\|z\|\to\infty$, the sequence $\{u_j\}$ is bounded. Thus, using (3. 3) and (3. 4) with $z=u_j$ for all j, for given $\varepsilon>0$, we can find a $T>0$ such that for $\omega\geq 0$,

$$
\begin{aligned}
&\frac{2(T-\alpha_m)}{\lambda}(y_m-x, J(y_m-u_j)) \\
&=\frac{2t_m n_m}{\lambda}\ \frac{1}{n_m}\sum_{k=1}^{n_m}(y_m-x, J(y_m-u_j)) \\
&\leq e^{2\omega T}\frac{2t_m n_m}{\lambda}\ \frac{1}{n_m}\sum_{k=1}^{n_m}e^{-2k\omega t}t_m(y_m-S(kt_m)u_j))+\frac{2t_m n_m}{\lambda}\varepsilon \\
&\leq e^{2\omega T}\|y_m-u_j\|^2-\|y_m-S(t_m n_m)u_j\|^2+\frac{2T}{\lambda}\varepsilon
\end{aligned}
$$

where n_m is an integer such that $T=n_m t_m+\alpha_m$ and $0\leq\alpha_m<t_m$. Hence we have

$$
\begin{aligned}
&\frac{2T}{\lambda}\mathrm{LIM}(y_m-x, J(y_m-u_j)) \\
&\leq e^{2\omega T}n\phi)_j)-\phi(S(T)u_j)+\frac{2T}{\lambda}\varepsilon
\end{aligned}
$$

for all j. Since $\phi(u_j) \leq L + \frac{2T}{\lambda}\varepsilon$ for all $j \geq n_0(\varepsilon)$, we obtain

$$\begin{aligned} &\text{LIM}\ (y_m - x, J(y_m - u_j)) \\ &\leq \frac{(e^{2\omega T}-1)\lambda}{2T} L + (e^{2\omega T}+1)\varepsilon \end{aligned} \tag{3.6}$$

for all $j \geq n_0(\varepsilon)$. Combining (3.5) and (3.6) and letting $T \downarrow 0$, we have $L \leq \phi(u_j) \leq \lambda\omega L$ for all $j \geq n_0(\varepsilon)$ and hence $L=0$. Thus $\{u_j\}$ is a Cauchy sequence and hence $\phi(u)=0$ for some $u \in C$. For $\omega<0$, by setting $\omega=-\omega'$ with $\omega'>0$, we have the same result. Thus there is a subsequence $\{y_{m_k}\}$ of $\{y_m\}$ such that $m_k \downarrow 0$ and $\{y_{m_k}\}$ is strongly convergent to u.

Now suppose that strong $\lim_{t_m \downarrow 0} y_{t_m} = u$ and strong $\lim_{t_{m'} \downarrow 0} y_{t_{m'}} = v$. We complete the proof of the theorem by showing that $u=v$.

To this end, let $\varepsilon>0$ be given and define integer $\{n_m\}$ by $0<T= n_m t_m + \alpha_m$, where $0 \leq \alpha_m < t_m$. For $1 \leq k \leq n_m$, we have

$$\begin{aligned} &(y_{t_m} - x, J(y_{t_m} - S(kt_m)\, y_{t_{m'}})) - (u-x, J(u-v)) \\ &= (y_{t_m} - u, J(y_{t_m} - S(kt_m)\, y_{t_{m'}})) + (u-x, J(y_{t_m} - S(kt_m)\, y_{t_{m'}})) \\ &\quad - (u-x, J(u-v)), \end{aligned}$$

and

$$\begin{aligned} \|S(kt_m)\, y_{t_{m'}} - v\| &\leq \|S(kt_m)\, y_{t_{m'}} - S(kt_m)\, v\| + \|S(kt_m)\, v - v\| \\ &\leq e^{k\omega t_m}\|y_{t_{m'}} - v\| + \alpha(T, v). \end{aligned}$$

Consequently,

$$|(y_{t_m} - x, J(y_{t_m} - S(kt_m)\, y_{t_{m'}})) - (u-x, J(u-v))| < \varepsilon$$

if $t_m, t_{m'}$, and T are small enough. It follows that for $\omega \geq 0$,

$$\begin{aligned} &\frac{2(T-\alpha_m)}{\lambda}(u-x, J(u-v)) \\ &= \frac{2n_m t_m}{\lambda} \frac{1}{n_m} \sum_{k=1}^{n_m} (u-x, J(u-v)) \\ &\leq e^{2\omega T} \frac{2n_m t_m}{\lambda} \frac{1}{n_m} \sum_{k=1}^{n_m} e^{-2\omega t_m} (y_{t_m} - x, J(y_{t_m} - S(kt_m)\, y_{t_{m'}})) + \frac{2n_m t_m}{\lambda}\varepsilon \\ &\leq e^{2\omega T}\|y_{t_m} - y_{t_{m'}}\|^2 - \|y_{t_m} - S(n_m t_m)\, y_{t_{m'}}\|^2 + \frac{2T}{\lambda}\varepsilon, \end{aligned}$$

and

$$\frac{2T}{\lambda}(u-x, J(u-v))$$

$$\leq e^{2\omega T}\|u-y_{t_{m'}}\|^2+\|u-S(T)y_{t_{m'}}\|^2+\frac{2T}{\lambda}\varepsilon.$$

This last inequality holds all small enough T and $t_{m'}$. Choosing $T=t_{m'}$, we obtain

$$\frac{2T}{\lambda}(u-x, J(u-v))$$
$$\leq e^{2\omega T}\|u-y_T\|^2-\|u-y_T-\frac{T}{\lambda}(y_T-x)\|^2+\frac{2T}{\lambda}\varepsilon$$
$$\leq e^{2\omega T}\|u-y_T\|^2-\|u-y_T\|^2+\frac{2T}{\lambda}(y_T-x, J(u-y_T))+\frac{2T}{\lambda}\varepsilon$$

so that

$$\{(u-x, J(u-v))-(y_T-x, J(u-y_T))\}$$
$$\leq\frac{(e^{2\omega T}-1)\lambda}{2T}\|u-y_T\|^2+\varepsilon.$$

Letting $T\downarrow 0$, we obtain $\|u-v\|^2\ (1-\lambda\omega)\leq 0$, and hence $\|u-v\|^2\leq 0$. For $\omega<0$, we have same result. Thus $u=v$ and the strong $\lim_{t\downarrow 0} y_t$ exists. This completes the proof.

Notation. We write $J_\lambda x=\lim_{t\downarrow 0}(I+\frac{\lambda}{t}(I-S(t)))^{-1}x=\lim_{t\downarrow 0} J_{\lambda,t}x$ for $\lambda>0$ such that $\lambda\omega<1$.

4. Generators of Semigroups

Recall that a Banach space E possesses the Radon-Nikodym property if and only if every function $g:[0,1]\to E$ of bounded variation is differentiable almost everywhere (cf. [8]).

The following theorems are partially extensions of [13, Theorem 3.1] and [14, Theorem 3]. We include proofs for completeness.

THEOREM 4.1. *Let E be a Banach space with a uniformly Gâteaux differentiable norm. Let C be a closed convex subset of E and $S\in Q_\omega(C)$. If $S(t)x$ is uniformly continuous on bounded (t, x) sets and E has the Radon-Nikodym property, then the strong generators A_0 of $S(t)$ has a dense domain in C.*

Proof. For positive T, let $T=n_t t+\alpha_t$ where n_t is an integer and

$0\leq\alpha_t<t$. For $1\leq k\leq n_t$, we have

$$\begin{aligned}&\|S(kt)J_{\lambda,t}x-J_{\lambda,t}x\|\\ &\leq e^{(k-1)\omega t}\|S(t)J_{\lambda,t}x-J_{\lambda,t}x\|+\|S((k-1)t)J_{\lambda,t}x-J_{\lambda,t}x\|\\ &=\frac{te^{(k-1)\omega t}}{\lambda}\|J_{\lambda,t}x-x\|+\|S((k-1)t)J_{\lambda,t}x-J_{\lambda,t}x\|.\end{aligned}$$

Summing, we have

$$\|S(T-\alpha_t)J_{\lambda,t}x-J_{\lambda,t}x\|\leq\frac{tn_t}{\lambda}\ \frac{1}{n_t}\sum_{k=0}^{n_t}e^{k\omega t}\|J_{\lambda,t}x-x\|,$$

so that

$$\|S(T)J_\lambda x-J_\lambda x\|\leq\frac{T}{\lambda}\left(\frac{1}{T}\int_0^T e^{\sigma\omega}d\sigma\right)\|J_\lambda x-x\|=\frac{e^{\omega T}-1}{\lambda\omega}\|J_\lambda x-x\|.$$

This means that $S(t)J_\lambda x$ is Lipschitzian, and hence differentiable almost everywhere. From inequality (3.1), (3.2), $\lim_{\lambda\downarrow 0}J_\lambda x=x$. Thus given $x\in C$ and $\varepsilon>0$, we first find λ such that $0<\lambda$ and $\lambda\omega<1$ with $\|J_\lambda x-x\|<\frac{\varepsilon}{2}$, and then a small enough t_0 such that $S(t)J_\lambda x$ is differentiable at t_0 and $\|S(t_0)J_\lambda x-J_\lambda x\|<\frac{\varepsilon}{2}$. We obtain $\|S(t_0)J_\lambda x-x\|<\varepsilon$ and $S(t_0)J_\lambda x\in D(A_0)$. Since x is arbitrary, $\mathrm{cl}(D(A_0))=C$.

THEOREM 4.2. *Let E, C, S and A_0 be as in Theorem 4.1. Let*

$$\tilde{A}=\bigcup_{\substack{\lambda>0\\ \lambda\omega>1}}\left\{\left[J_\lambda x,\ \frac{x-J_\lambda x}{\lambda}\right]:x\in C\right\}\cup A_0.$$

If A is $cl(\tilde{A})$ in $E\times E$, then A has following properties:

(i) *$A_0\subset A$, $cl(D(A))=C$ and A is closed*

(ii) *$A+\omega I$ is accretive*

(iii) *$C\subset R(I+\lambda A)$ for $\lambda>0$ such that $\lambda\omega<1$*

(iv) *$S(t)x=\lim_{n\to\infty}\left(I+\frac{t}{n}A\right)^{-n}x$ for all $x\in C$.*

Proof. Clearly $A_0\subset A$, $D(A)\subset C$, $C\subset R(I+\lambda A)$ for all $\lambda>0$, and A is closed. For $x\in C$, (3.1), (3.2) imply that $\lim_{\lambda\downarrow 0}J_\lambda x=x$, and hence $\mathrm{cl}(D(A))=C$.

Our next step is to prove that $A+\omega I$ is accretive. Let $\varepsilon>0$ be given and define integers $\{n_t\}$ by $0<T=n_tt+\alpha_t$, $0\leqq\alpha_t<t$. Then from (3.3) and (3.4), we have, for all $x,z\in C$ and $\omega\geqq 0$ (if $\omega<0$, put $\omega=-\omega$ with $\omega'>0$),

$$\frac{2(T-\alpha_t)}{\lambda}(J_{\lambda,t}x-x,\ J(J_{\lambda,t}x-z))$$

$$=\frac{2tn_t}{\lambda}\frac{1}{n_t}\sum_{k=1}^{n_t}(J_{\lambda,t}x-x, J(J_{\lambda,t}x-z))$$
$$\leq e^{2\omega T}\frac{2tn_t}{\lambda}\frac{1}{n_t}\sum_{k=1}^{n_t}e^{-2\omega t}(J_{\lambda,t}x-x, J(J_{\lambda,t}x-S(kt)z))+\frac{2tn_t}{\lambda}\varepsilon$$
$$\leq e^{2\omega T}\|J_{\lambda,t}x-z\|^2-\|J_{\lambda,t}x-S(tn_t)z\|^2+\frac{2T}{\lambda}\varepsilon$$

and

$$\frac{2T}{\lambda}(J_\lambda x-x, J(J_\lambda x-z))$$
$$\leq e^{2\omega T}\|J_\lambda x-z\|^2-\|J_\lambda x-S(T)z)\|^2+\frac{2T}{\lambda}\varepsilon.$$

Consequently, it follows that

$$\frac{2}{\lambda}(J_\lambda x-x, J(J_\lambda x-z))+2\left(\frac{z-S(T)z}{T}, J(J_\lambda x-z)\right)$$
$$\leq\left(\frac{e^{2\omega T}-1}{T}\right))\|J_\lambda x-z\|^2+\frac{2\varepsilon}{\lambda}. \tag{4.1}$$

Thus, if we take $z=J_{\mu,T}x'$ in (4.1), then, by letting $T\downarrow 0$, we have

$$\frac{2}{\lambda}(J_\lambda x-x, J(J_\lambda x-J_\mu x'))-\frac{2}{\mu}(J_\mu x'-x', J(J_\lambda x-J_\mu x'))$$
$$\leq 2\omega\|J_\lambda x-J_\mu x'\|^2,$$

and hence

$$\left(\frac{1}{\lambda}(x-J_\lambda x)+\omega J_\lambda x-\frac{1}{\mu}(x'-J_\mu x')-\omega J_\mu x', J(J_\lambda x-J_\mu x')\right)\geq 0. \tag{4.2}$$

On the one hand, if we take $z=x'\in D(A_0)$ in (4.1), then, by letting $T\downarrow 0$, we have

$$\frac{2}{\lambda}(J_\lambda x-x,\ J(J_\lambda x-x'))+2(A_0x',\ J(J_\lambda x-x))$$
$$\leq 2\omega\|J_\lambda x-x'\|^2$$

and hence

$$\left(\frac{1}{\lambda}(x-J_\lambda x)+\omega J_\lambda x-A_0x'-\omega x',\ J(J_\lambda x-x')\right)\geq 0. \tag{4.3}$$

On the other hand, if $x, x'\in D(A_0)$, then we have

$$(S(t)x-S(t)x', J(x-x'))\leq e^{\omega t}\|x-x'\|^2$$

and hence

$$\left(\frac{S(t)x-x}{t}-\frac{S(t)x'-x'}{t},\ J(x-x')\right)\leq\frac{e^{\omega t}-1}{t}\|x-x'\|^2.$$

Letting $t\downarrow 0$, we have

$$(A_0x+\omega x-A_0x'-\omega x', J(x-x'))\geq 0. \qquad (4.4)$$

Therefore we conclude from (4.2), (4.3) and (4.4) that $A+\omega I$ is accretive.

Finally we complete the proof of the theorem by showing (iv).

To this end, let $x\in D(A_0)$ and consider $u(t)=S(t)x$. Then the function $t\to u(t)$ is locally Lipschitzian on $[0, \infty)$, and hence differentiable a.e. on $[0, \infty)$. Thus we have $du/dt+A_0u=0$ a.e. on $[0, \infty)$. From this fact,

$$\frac{du(t)}{dt}+Au(t)\ni 0 \quad \text{a.e. on } [0, \infty).$$

On the other hand, by Theorem 2.1, A generates a continuous semigroup $\tilde{S}$ of type ω via the exponential formula (2.3). On the other hand, by Theorem 2.2, we have for all $x\in D(A_0)$, $u(t)=\tilde{S}(t)x$ because $x\in D(A)$ and $u(t)$ is a strong solution of $du(t)/dt+Au(t)\ni 0$. Therefore for all $x\in D(A_0)$, we have $S(t)x=\tilde{S}(t)x$. Thus, by continuity, $S=\tilde{S}$. This completes the proof.

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