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SEMI-INVARIANT SUBMANIFOLDS WITH PARALLEL RICCI TENSOR OF A COMPLEX HYPERBOLIC SPACE

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0. Introduction

A Kaehlerian manifold of constant holomorphic sectional curvature c is called a complex space form. The complete and simply connected complex space form is a complex projective space, a complex Euclidean space or a complex hyperbolic space according as $c>0$, $c=0$ or $c<0$ respectively.

Using the properties of the almost contact metric structure Takagi ([16]) has completely classified homogeneous real hypersurfaces of a complex projective space, which satisfy the condition that the number of distinct constant principal curvatures does not exceed 4. In the light of Ricci tensor, Kimura ([11]) has proved that there are no real hypersurfaces of a complex projective space with parallel Ricci tensor on which the structure vector P is principal.

Real hypersurfaces of a complex hyperbolic space have also been investigated from different points of view ([3], [10], [14], [15] etc).

In particular, it has been proved in [14] that there are no Einstein real hypersurfaces in a complex hyperbolic space.

Recently, Ki ([8]) has proved that there are no real hypersurfaces with parallel Ricci tensor of a complex space form, which extends the Kimura's theorem ([11]) and the Montiel's theorem ([14]).

On the other hand, semi-invariant submanifolds of a Kaehlerian manifold have been studied by Blair and Ludden [1], Ki [5], [8], [19], Tashiro [17], Yano [1], [19] and others. It is well known that the almost contact metric structure is induced on these submanifolds ([19]).

In the present paper, utilizing the properties of this induced structure, we study a semi-invariant submanifold of codimension 3 in a complex

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space form which satisfy the condition that the distinguished normal is parallel in the normal bundle. The submanifolds are investigated on each case of harmonic curvature, parallel Ricci tensor and parallel normal curvature. The main purpose of the present paper is to extend the Montiel's theorem ([14]) and the Ki's theorem ([8]) to the semi-invariant submanifolds of codimension 3. We prove that there are no semi-invariant submanifolds of codimension 3 in a complex hyperbolic space with parallel Ricci tensor and the parallel distinguished normal in the normal bundle.

1. Preliminaries

Let $\overline{M}$ be a real $(2n+2)$-dimensional Kaehlerian manifold covered by a system of coordinate neighborhoods $\{U\,;\,y^A\}$ with almost complex structure J and Riemannian metric G, where J is parallel and G is J-Hermitian. The components $J_B{}^A$ and G_{BA} of J and G satisfy $J_A{}^C J_C{}^B = -\delta_A{}^B$ and $J_B{}^D J_A{}^C G_{DC} = G_{BA}$ respectively, $\delta_A{}^B$ being the Kronecker delta.

Let M be a real $(2n-1)$-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V\,;\,x^h\}$ and immersed isometrically in $\overline{M}$ by the immersion $\phi : M \rightarrow \overline{M}$. When the argument is local, M does not need to be distinguished from $\phi(M)$ and the immersion is represented locally by $y^A = y^A(x^h)$. We now put $B_i{}^A = \partial_i y^A$, $\partial_i = \partial/\partial x^i$, then $B_i = (B_i{}^A)$ are $(2n-1)$-linearly independent vectors of $\overline{M}$ tangent to M. Since the immersion is isometric, the components g_{ji} of the metric tensor g of M induced from that of $\overline{M}$ are given by $g_{ji} = G_{BA} B_j{}^B B_i{}^A$. Three mutually orthogonal unit normals to M will be denoted by C^A, D^A and E^A.

Denoting by ∇_j the operator of van der Waerden–Bortolotti covariant differentiation with respect to the induced Riemannian metric, the equation of Gauss for M of $\overline{M}$ is obtained:

$$(1.1) \qquad \nabla_j B_i{}^A = h_{ji} C^A + k_{ji} D^A + l_{ji} E^A,$$

where h_{ji}, k_{ji} and l_{ji} are the components of the second fundamental forms in the direction of normals C^A, D^A and E^A respectively.

The equations of Weingarten are also given by

$$(1.2)\qquad \begin{cases} \nabla_j C^A = -h_j{}^h B_h{}^A + l_j D^A + m_j E^A, \\ \nabla_j D^A = -k_j{}^h B_h{}^A - l_j C^A + n_j E^A, \\ \nabla_j E^A = -l_j{}^h B_h{}^A - m_j C^A - n_j D^A, \end{cases}$$

where $h_j{}^h = h_{jt}g^{th}$, $k_j{}^h = k_{jt}g^{th}$, $l_j{}^h = l_{jt}g^{th}$, l_j, m_j and n_j being the components of the third fundamental tensors and $(g^{ji}) = (g_{ji})^{-1}$.

The vector field C is said to be *parallel* in the normal bundle if $\nabla_j{}^{\perp}C = 0$, that is, l_j and m_j vanish identically.

On the other hand, a submanifold M is called a *CR-submanifold* [19] of a Kaehlerian manifold $\bar{M}$ if there exists a differentiable distribution $T: x \to T_x \subset M_x$ on M satisfying the following conditions, where M_x denotes the tangent space to M at each point x in M:

(1) T is invariant, that is, $JT_x = T_x$ for each point x in M, (2) the complementary orthogonal distribution $T^{\perp}: x \to T_x{}^{\perp} \subset M_x$ is totally real, that is, $JT_x{}^{\perp} \subset M_x{}^{\perp}$ for each x in M, where $M_x{}^{\perp}$ denotes the normal space to M at $x \in M$. In particular, M is said to be a *semi-invariant* submanifold provided that $\dim T^{\perp} = 1$. Then the unit normal vector field in $JT^{\perp}$ is called the *distinguished normal* to the semi-invariant submanifold and denoted by N^A ([17] [19]).

Let M be a semi-invariant submanifold of codimension 3 in a $(2n+2)$-dimensional Kaehlerian manifold $\bar{M}$. We take the distinguished normal N^A as C^A. Then we have ([19])

$$(1.3)\qquad J_B{}^A B_i{}^B = f_i{}^h B_h{}^A + p_i C^A, \quad J_B{}^A C^B = -p^h B_h{}^A,$$

$$(1.4)\qquad J_B{}^A D^B = -E^A, \quad J_B{}^A E^B = D^A,$$

where we have put $f_{ji} = G(JB_j, B_i)$, $p_i = G(JB_i, C)$ in M, p^h being associated components of p_h. By the properties of the almost Hermitian structure (J, G), it follows from (1.3) and (1.4) that (f, g, P) defines an almost contact metric structure. Since J is parallel tensor, by differentiating (1.3) covariantly and taking account of (1.1) and (1.2), we find

$$(1.5)\qquad \nabla_j f_i{}^h = -h_{ji}p^h + h_j{}^h p_i,$$

$$(1.6)\qquad \nabla_j p_i = -h_{jr} f_i{}^r.$$

From now on we suppose that the distinguished normal C is parallel in the normal bundle. Then we can verify from (1.2) and (1.4) that ([9])

$$(1.7)\qquad k_{jr} f_i{}^r = l_{ji}, \quad l_{jr} f_i{}^r = -k_{ji}.$$

Thus, it follows that

(1.8) $k_{jr}p^r=0,\ l_{jr}p^r=0,\ k=l=0,$

where $k=k_{ji}g^{ji}$ and $l=l_{ji}g^{ji}$.

To write our formulas in a convention form, the components T_{ji}^m of a tensor field T^m and a function T_m on M for any integer $m(\geq 2)$ are introduced as follows:

$$T_{ji}^m=T_{ji_1}T_{i_2}{}^{i_1}\cdots T_i{}^{i_{m-1}},\ T_m=\sum_i T_{ii}{}^m.$$

In our notation, it is easily seen from (1.7) that

(1.9) $k_{jr}l_i{}^r+k_{ir}l_j{}^r=0,$

(1.10) $k_{ji}{}^2=l_{ji}{}^2.$

In the sequel, the ambient Kaehlerian manifold $\overline{M}$ is assumed to be of constant holomorphic sectional curvature c, which is called a *complex space form* and denoted by $M_{n+1}(c)$. Then the curvature tensor of $M_{n+1}(c)$ takes the following form:

$$R_{DCB}{}^A=\frac{c}{4}(\delta_D{}^AG_{CB}-\delta_C{}^AG_{DB}+J_D{}^AJ_{CB}-J_C{}^AJ_{DB}-2J_{DC}J_B{}^A).$$

Thus, the Gauss equation of M is obtained:

(1.11) $$R_{kji}{}^h=\frac{c}{4}(\delta_k{}^hg_{ji}-\delta_j{}^hg_{ki}+f_k{}^hf_{ji}-f_j{}^hf_{ki}-2f_{kj}f_i{}^h)$$
$$+h_k{}^hh_{ji}-h_j{}^hh_{ki}+k_k{}^hk_{ji}-k_j{}^hk_{ki}+l_k{}^hl_{ji}-l_j{}^hl_{ki}.$$

And the equations of Codazzi for M are given by

(1.12) $\nabla_kh_{ji}-\nabla_jh_{ki}=\frac{c}{4}(p_kf_{ji}-p_jf_{ki}-2p_if_{kj}),$

(1.13) $D_kk_{ji}=D_jk_{ki},\ D_kl_{ji}=D_jl_{ki},$

where we have denoted

(1.14) $D_kk_{ji}=\nabla_kk_{ji}-n_kl_{ji},\ D_kl_{ji}=\nabla_kl_{ji}+n_kk_{ji}.$

Those of Ricci are given by

(1.15) $h_{jr}k_i{}^r=h_{ir}k_j{}^r,\ h_{jr}l_i{}^r=h_{ir}l_j{}^r,$

(1.16) $\nabla_jn_i-\nabla_in_j+2k_{jr}l_i{}^r=\frac{c}{2}f_{kj}.$

Let S_{ji} be the components of the Ricci tensor S of M, then the Gauss equation implies

(1.17) $S_{ji}=\frac{c}{4}\{(2n+1)g_{ji}-3p_jp_i\}+hh_{ji}-h_{ji}{}^2-2k_{ji}{}^2,$

where we have used (1.10) and the fact that $\nabla_j{}^\perp C=0$.

2. Semi-invariant submanifolds with harmonic curvature

Let M be a $(2n-1)$-dimensional semi-invariant submanifold with $\nabla_j{}^\perp C=0$ and harmonic curvature in a complex space form $M_{n+1}(c)$,

$c\neq 0$, that is, the Ricci tensor S of M satisfies $\nabla_k S_{ji}=\nabla_j S_{ki}$. Then, we easily, using the second Bianchi identity, see that the scalar curvature r of M is constant everywhere. Moreover the Ricci formula for S_{ji} gives rise to

$$\nabla_m\nabla_k S_{ji}=\nabla_j\nabla_i S_{mk}-R_{mjkr}S_i{}^r-R_{mjir}S_k{}^r,$$

which together with the first Bianchi identity and the Ricci formula implies that

$$(2.1)\qquad R_{mkir}S_j{}^r+R_{kjir}S_m{}^r+R_{jmir}S_k{}^r=0,$$

where $S_j{}^h=S_{ji}g^{ih}$. Therefore it follows that

$$f^{kj}R_{kjih}S_m{}^h+2f^{rk}R_{kmih}S_r{}^h=0$$

and hence, in consequence of (1.11)

$$\left(-n+\frac{3}{2}\right)cS_{jr}f_i{}^r+\frac{c}{2}\{S_{ir}f_j{}^r-(r-A_1)f_{ji}-p_i(S_{rt}p^r)f_j{}^t-2p_j(S_{tr}p^r)f_i{}^t\}$$
$$+2(h_{tr}h_{is}f^{rs}S_j{}^t-h_{jt}h_{ir}f^{sr}S_s{}^t+2k_{tr}l_i{}^rS_j{}^t-k_{jt}l_i{}^sS_s{}^t+l_{jt}k_i{}^sS_s{}^t)=0,$$

where we have put $A_1=S_{ji}p^jp^i$. By the way, the last term of this is reduced to $-\frac{3}{2}cp_j(h_{rt}h_{is}p^t)f^{rs}$ by virtue of (1.17). Accordingly we have

$$S_{ir}f_j{}^r-(2n-3)S_{jr}f_i{}^r-(r-A_1)f_{ji}$$
$$-S_{tr}p^r(p_if_j{}^t+2p_jf_i{}^t)-3h_{rt}p^th_{is}f^{rs}p_j=0$$

because of the fact $c\neq 0$ is assumed, which implies

$$3h_{rt}p^th_{is}f^{rs}+(2n-1)S_{rt}p^tf_i{}^r=0.$$

Thus the last equation can be written as

$$(2.2)\qquad (2n-3)\{S_{jr}f_i{}^r-(S_{tr}p^r)f_i{}^tp_j\}-S_{ir}f_j{}^r$$
$$+(S_{rt}p^t)p_if_j{}^r+(r-A_1)f_{ji}=0,$$

from which, taking the symmetric part,

$$(2.3)\qquad S_{jr}f_i{}^r+S_{ir}f_j{}^r=S_{tr}p^r(p_jf_i{}^t+p_if_j{}^t).$$

Hence, the relationship (2.2) turns out to be

$$2(n-1)\{S_{jr}f_i{}^r-(S_{tr}p^r)p_jf_i{}^t\}+(r-A_1)f_{ji}=0.$$

Transforming this by $f_k{}^i$ and using the properties of the almost contact metric structure (f, g, P), it is reduced to

$$(2.4)\qquad 2(n-1)\{S_{ji}-p_iS_{jr}p^r-p_jS_{ir}p^r\}-(r-A_1)g_{ji}$$
$$+\{r+(2n-3)A_1\}p_jp_i=0,$$

which implies immediately that

$$(2.5)\qquad 2(n-1)(S_2-2A_2-A_1{}^2)=(r-A_1)^2$$

where $A_2=S_{ji}^2p^jp^i$.

PROPOSITION 2.1. *Let M be a $(2n-1)$-dimensional semi-invariant sub-*

manifold with harmonic curvature and $\nabla_j{}^{\perp}C=0$ in a complex space form $M_{n+1}(c)$, $c\neq 0$. If the structure vector P is an eigenvector of the Ricci tensor, namely, if

$$(2.6)\qquad S_{jr}p^r=A_1p_j,$$

then M is of Ricci-parallel.

Proof. By means of (2.6), the relationship (2.3) is reduced to

$$(2.7)\qquad 2(n-1)S_{ji}-(r-A_1)g_{ji}+\{r-(2n-1)A_1\}p_jp_i=0,$$

which implies

$$(2.8)\qquad 2(n-1)S_{ji}^2-\{r+(2n-3)A_1\}S_{ji}+A_1(r-A_1)g_{ji}=0.$$

Differentiating (2.7) covariantly, we find

$$(2.9)\qquad 2(n-1)\nabla_kS_{ji}+(\nabla_kA_1)g_{ji}-(2n-1)(\nabla_kA_1)p_jp_i + \{r-(2n-1)A_1\}\{(\nabla_kp_j)p_i+(\nabla_kp_i)p_j\}=0$$

because the scalar curvature r is constant. Since the Ricci tensor is of Codazzi type, it is seen that

$$(2.10)\qquad (\nabla_kA_1)g_{ji}-(\nabla_jA_1)g_{ki}-(2n-1)\{(\nabla_kA_1)p_j-(\nabla_jA_1)p_k\}p_i + \{r-(2n-1)A_1\}\{(\nabla_kp_j-\nabla_jp_k)p_i+(\nabla_kp_i)p_j-(\nabla_jp_i)p_k\}=0.$$

If we contract this to j and i, then we obtain

$$\nabla_kA_1-(2n-1)(p^r\nabla_rA_1)p_k+\{r-(2n-1)A_1\}p^r\nabla_rp_k=0$$

and hence $p^r\nabla_rA_1=0$. Thus, it follows that $\nabla_kA_1+\{r-(2n-1)A_1\}p^r\nabla_rp_k=0$. Transvecting (2.10) with p^jp^i and taking account of the last equation, we can verify that A_1 is constant everywhere. Therefore, by differentiating (2.8) covariantly, we find

$$2(n-1)\nabla_kS_{ji}^2-\{r+(2n-3)A_1\}\nabla_kS_{ji}=0,$$

which shows that S_{ji}^2 is of Codazzi type. Thus the Ricci tensor S is parallel because the scalar curvature of M is constant (See Umehara Theorem 1.3 of [18]). This completes the proof of Proposition 2.1.

PROPOSITION 2.2. *Let M be a $(2n-1)$-dimensional semi-invariant submanifold with harmonic curvature and $\nabla_j{}^{\perp}C=0$ of a complex space form $M_{n+1}(c)$, $c\neq 0$. Then the structure vector P is an eigenvector of the Ricci tensor, namely $S_{jr}p^r=A_1p_j$, if and only if the structure vector P is principal with respect to the distinguished normal C, that is, $h_{jr}p^r=\alpha p_j$, where $\alpha=h_{rs}p^rp^s$.*

Proof. We now prove "only if" part. Because of (2.6), the equation (2.3) is reduced to

$$S_{jr}f_i^r+S_{ir}f_j^r=0.$$

Differentiating this covariantly along M and using (1.5) and (2.6), we find

$$(2.11)\qquad (\nabla_k S_{jr})f_i^r+(\nabla_k S_i^r)f_{jr}-A_1h_{ki}p_j + (S_{jr}h_k^r)p_i-A_1h_{kj}p_i+(S_{ir}h_k^r)p_j=0.$$

Since S is of Codazzi type, by taking the skew-symmetric part with respect to indices k and j, we have

$$(\nabla_k S_{ir})f_j^r-(\nabla_j S_{ir})f_k^r-A_1(h_{ki}p_j-h_{ji}p_k) + (S_{jr}h_k^r-S_{kr}h_j^r)p_i+(S_{ir}h_k^r)p_j-(S_{ir}h_j^r)p_k=0$$

for all indices k, j and i and hence

$$(\nabla_i S_{kr})f_j^r-(\nabla_j S_{kr})f_i^r-A_1(h_{ik}p_j-h_{jk}p_i) + (S_{jr}h_i^r-S_{ir}h_j^r)p_k+(S_{kr}h_i^r)p_j-(S_{kr}h_j^r)p_i=0.$$

Adding this equation to (2.11), we get

$$(2.12)\qquad 2(\nabla_k S_{ir})f_j^r-2A_1h_{ki}p_j+(S_{jr}h_k^r-S_{kr}h_j^r)p_i + (S_{ir}h_k^r+S_{kr}h_i^r)p_j+(S_{jr}h_i^r-S_{ir}h_j^r)p_k=0.$$

Contracting (2.12) to k and i and making use of (2.6), we obtain

$$S_{jr}h_s^r p^s=A_1h_{jr}p^r.$$

Thus it is not hard, using (2.12), to see that

$$(2.13)\qquad S_{ir}h_j^r+S_{jr}h_i^r=2A_1h_{ji},$$

which together with (1.17) yields

$$\frac{c}{2}(2n+1)h_{ji}-\frac{3}{4}c\{(h_{jr}p^r)p_i+(h_{ir}p^r)p_j\} +2h_{jr}(hh_i^r-h_i^{r2})-4h_{jr}{}^2k_i^{r2}=2A_1h_{ji}.$$

Transvecting p^i and making use of (2.6), we can find $h_{jr}p^r=\alpha p_j$ because $c\neq 0$ is assumed.

The converse assertion is clear from the definition of A_1 and (1.17). This completes the proof.

As a quite similar method as that used in [8], we can see from (2.4) that

$$(2.14)\qquad 4(n-1)^2S_{ji}^3-4(n-1)\{r+(n-2)A_1\}S_{ji}^2 +\{(r-A_1)(r+(4n-5)A_1)-4(n-1)^2(A_2-A_1^2)\}S_{ji} -\beta(r-A_1)g_{ji}=0,$$

or, equivalently

$$\left(S_j^r-\frac{r-A_1}{2(n-1)}\delta_j^r\right)\{2(n-1)S_{ir}^2-\gamma S_{ir}+\beta g_{ir}\}=0,$$

where we have put $\beta=A_1(r-A_1)-2(n-1)(A_2-A_1^2)$ and

$\gamma = r + (2n-3)A_1$. Thus the minimal polynomial for S tells us that there exist at most three Ricci curvatures of M : $(r-A_1)/2(n-1)$, $(\gamma \pm \sqrt{D})/4(n-1)$,
where

(2.15) $\quad D = \{r-(2n-1)A_1\}^2 + 16(n-1)^2(A_2 - A_1^2)$.

And their multiplicities are respectively denoted by $(2n-1-l_1-l_2)$, l_1 and l_2. Therefore the scalar curvature r of M satisfies

(2.16) $\quad (l_1+l_2-2)\{r-(2n-1)A_1\} = \sqrt{D}(l_1-l_2)$.

We also have

$$4(n-1)^2 S_2 = \frac{1}{4}(\gamma^2 + D)(l_1+l_2) + \frac{1}{2}\gamma\sqrt{D}(l_1-l_2) + (r-A_1)^2(2n-1-l_1-l_2),$$

which together with (2.5), (2.15) and (2.16) implies that

(2.17) $\quad (A_2 - A_1^2)(l_1+l_2-2) = 0$.

Now suppose that the number of distinct Ricci curvatures does not exceed 2. Then we can easily see that $A_2 = A_1^2$ because of (2.15) and (2.16). Thus, it follows that $S_{jr}p^r = A_1 p_j$. According to Proposition 2.1, we have

PROPOSITION 2.3. *Let M be a $(2n-1)$-dimensional semi-invariant submanifold with harmonic curvature and $\nabla_j^{\perp} C = 0$ of a complex space form $M_{n+1}(c)$, $c \neq 0$. Then the number of the distinct Ricci curvature is at most 3. In particular, if it does not exceed 2, then M is of Ricci-parallel.*

3. Semi-invariant submanifolds with parallel Ricci tensor

In this section we consider the semi-invariant submanifold with parallel Ricci tensor and $\nabla_j^{\perp} C = 0$ of a complex space form $M_{n+1}(c)$, $c \neq 0$. Since the Ricci tensor S is assumed to be parallel, we have (2.14) and hence

$$\begin{aligned} &4(n-1)^2 S_3 - 4(n-1) r S_2 - 4(n-1)(n-2) S_2 A_1 + r(r-A_1)^2 \\ &+4(n-1) r A_1 (r-A_1) + 2(n-1) r (A_2 - A_1^2) \\ &-2(n-1)(2n-1) A_1 (A_2 - A_1^2) - (2n-1) A_1 (r-A_1)^2 = 0, \end{aligned}$$

which together with (2.5) yields

$$\frac{1}{2(n-1)}(r-A_1)^3 + 2(n-1)A_1^3 + 3rA_1(r-A_1) - 3(2n-3)S_2A_1$$

$$-3rS_2+4(n-1)S_3=0.$$

Thus, A_1 is a root of the cubic equation with constant coefficients because S_i is constant for each number i. Accordingly A_1 is constant. By the definition of A_1, it is not hard to see that

(3.1) $$S_{ir}p^i\nabla_k p^r=0$$

because the Ricci tensor is parallel. By differentiating (2.4) covariantly, we find

(3.2) $$2(n-1)\{(\nabla_k p_i)S_{jr}p^r+(\nabla_k p_j)S_{ir}p^r+p_iS_{jr}\nabla_k p^r+p_jS_{ir}\nabla_k p^r\} = \{r+(2n-3)A_1\}\{(\nabla_k p_j)p_i+(\nabla_k p_i)p_j\}.$$

If we apply p^j to this and sum for j, and make use of (3.1), we obtain $2(n-1)S_{ir}\nabla_k p^r=(r-A_1)\nabla_k p_i$. Thus, (3.2) turns out to be

$$(\nabla_k p_i)S_{jr}p^r+(\nabla_k p_j)S_{ir}p^r=A_1(p_i\nabla_k p_j+p_j\nabla_k p_i).$$

Transvecting the last equation with $S_t{}^j p^t$ and using (3.1), we get

(3.3) $$(A_2-A_1{}^2)\nabla_k p_i=0.$$

If we suppose that P is parallel along M, then the equation (1.6) becomes $h_{jr}f_i{}^r=0$. Thus it is not hard to see that $h_{ji}=hp_jp_i$ because of the properties of the almost contact metric structure. Hence it follows that $\nabla_k h_{ji}=(\nabla_k h)p_jp_i$, which together with (1.12) gives

$$\frac{c}{4}(p_kf_{ji}-p_jf_{ki}-2p_if_{kj})=\{(\nabla_k h)p_j-(\nabla_j h)p_k\}p_i.$$

By transvecting p^if^{kj}, we have $c(n-1)=0$. Thus the assumption $c\neq0$ will produce a contradiction. Thus it follows that $A_2=A_1{}^2$ and hence $S_{jr}p^r=A_1p_j$. Therefore the relationship (2.4) is reduced to

$$2(n-1)S_{ji}=(r-A_1)g_{ji}-\{r-(2n-1)A_1\}p_jp_i.$$

Since the Ricci tensor of M is parallel, it is seen that

$$\{r-(2n-1)A_1\}(p_i\nabla_k p_j+p_j\nabla_k p_i)=0$$

and hence $r-(2n-1)A_1=0$. Thus, M is Einstein, which means that its Ricci tensor is a scalar multiple of the identity at each point. Therefore we can state the following fact:

PROPOSITION 3.1. *Let M be a $(2n-1)$-dimensional semi-invariant submanifold with parallel Ricci tensor of a complex space form $M_{n+1}(c)$, $c\neq0$. If the distinguished normal is parallel in the normal bundle, then M is Einstein.*

4. Semi-invariant submanifolds of a complex hyperbolic space

In this section we devote to investigate a $(2n-1)$-dimensional semi-invariant Einstein submanifold M in a complex hyperbolic space $H_{n+1}(-4)$ such that the distinguished normal is parallel in the normal bundle. Then the components S_{ji} of the Ricci tensor S of the submanifold are obtained as follows; $S_{ji}=A_1 g_{ji}$, where A_1 is constant. Thus we can get from (1.17) that

$$A_1=-2(n-1)+h\alpha-\alpha^2 \tag{4.1}$$

and the scalar curvature r is given by $r=(2n-1)A_1$. Therefore (1.17) turns out to be

$$2k_{ji}^2=(\alpha^2-3-\alpha h)g_{ji}+3p_jp_i+hh_{ji}-h_{ji}^2. \tag{4.2}$$

If we transform this by $f_i{}^k$ and make use of (1.7), we can get

$$2k_{jr}l_i{}^r=(3+\alpha h-\alpha^2)f_{ji}+hh_{jr}f_i{}^r-h_{jr}{}^2f_i{}^r, \tag{4.3}$$

which implies

$$h_{jr}{}^2f_i{}^r+h_{ir}{}^2f_j{}^r=h(h_{jr}f_i{}^r+h_{ir}f_j{}^r) \tag{4.4}$$

because of (1.9).

By means of Proposition 2.2, it is seen that

$$h_{jr}p^r=\alpha p_j. \tag{4.5}$$

If we differentiate this covariantly along M and take account of (1.6), then we obtain

$$(\nabla_k h_{jr})p^r-h_{jr}h_{ks}f^{rs}=\alpha_k p_j-\alpha h_{kr}f_j{}^r,$$

where $\alpha_k=\nabla_k\alpha$, which together with (1.12) implies that

$$2f_{kj}-2h_{jr}h_{ks}f^{rs}=\alpha_k p_j-\alpha_j p_k-\alpha(h_{kr}f_j{}^r-h_{jr}f_k{}^r). \tag{4.6}$$

LEMMA 4.1. *Let M be a semi-invariant Einstein submanifold of codimension 3 with $\nabla_j^{\perp}C=0$ in $H_{n+1}(-4)$. Then α and h are constant everywhere.*

Proof. By the properties of the almost contact metric structure, (4.6) implies $\alpha_i=\varepsilon p_i$ for some function ε on M. Differentiating this covariantly, we find $\nabla_j\alpha_i=(\nabla_j\varepsilon)p_i-\varepsilon h_{jr}f_i{}^r$ and hence $\varepsilon(h_{jr}f_i{}^r-h_{ir}f_j{}^r)=0$. Let M_0 be a set of consisting of points the function ε does not vanish and suppose that M_0 is not empty. Then M_0 is an open submanifold of M and we have

$$h_{jr}f_i{}^r-h_{ir}f_j{}^r=0$$

and hence $h=\alpha$ on M_0. Furthermore, (4.6) implies

$$f_{kj}-h_{jr}{}^2f_k{}^r=0$$

on this set. Hence, by the properties of the almost contact metric structure it follows that $h_2=2(n-1)+\alpha^2$ on M_0.

On the other hand, from (4. 2) we have

$$2k_2=-6(n-1)+\alpha^2-h_2$$

because of the fact that $h=\alpha$. Combining the last two equations it follows that $k_2+4(n-1)=0$, which will produce a contradiction. By the definition of ε, we see that α is constant everywhere. Therefore, Lemma 4. 1 is proved.

According to Lemma 4. 1, the relationship (4. 6) is reduced to

$$h_{jr}h_{ks}f^{rs}=\frac{1}{2}\alpha(h_{kr}f_j{}^r-h_{jr}f_k{}^r)+f_{kj}. \tag{4.7}$$

If we apply $h_k{}^i$ to (4. 3) and utilize (4. 7), then we obtain

$$2k_{jr}l_s{}^r h_k{}^s=(3+h\alpha-\alpha^2)h_{kr}f_j{}^r+h_{jr}f_k{}^r$$
$$+\left(\frac{1}{2}\alpha-h\right)h_{jr}h_{ks}f^{rs}-\frac{1}{2}\alpha h_{jr}{}^2 f_k{}^r,$$

which together with (1. 15) implies

$$(4+h\alpha-\alpha^2)(h_{kr}f_j{}^r+h_{jr}f_k{}^r)=\frac{1}{2}\alpha(h_{jr}{}^2f_k{}^r+h_{kr}{}^2f_j{}^r).$$

Combining this and (4. 4), it is clear that

$$(h\alpha-2\alpha^2+8)(h_{jr}f_i{}^r+h_{ir}f_j{}^r)=0, \tag{4.8}$$

which together with (4. 7) gives

$$(h\alpha-2\alpha^2+8)(h_2-\alpha h+2(n-1))=0. \tag{4.9}$$

We notice here that the shape operator A with respect to the distinguished normal and f commute each other if and only if $h_2-\alpha h+2(n-1)=0$.

Let X be an eigenvector of A orthogonal to P with the eigenvalue λ, namely

$$AX=\lambda X,\quad X\neq 0.$$

Then we get from (4. 7)

$$(2\lambda-\alpha)AfX=(\lambda\alpha-2)fX. \tag{4.10}$$

In the sequel we denote

$$\lambda_1=\ldots=\lambda_s=\frac{\alpha}{2},\ \lambda_{s+1}\neq\frac{\alpha}{2},\cdots,\ \lambda_{2n-2}\neq\frac{\alpha}{2},\ 0\leq s\leq 2(n-1).$$

We are going to prove that $Af=fA$. If $s=2(n-1)$, then the property $Af=fA$ holds trivially.

Now, suppose that $0<s<2(n-1)$. Then there exists at least one eigenvector of A associated with the eigenvalue $\frac{\alpha}{2}$. Thus, equation (4. 10) tells us that $\alpha^2=4$. Hence, the relationship (4. 9) is reduced to

(4.11) $$h\{h_2-\alpha h+2(n-1)\}=0.$$

But h cannot be zero. In fact, if $h=0$, then it is not hard to, taking account of (4.4), see that $h_2=2(n+1)$. From this and (4.7) it follows that

$$||Af-fA||^2+8=0,$$

which is contradictory. Thus relationship (4.11) means $Af=fA$. Developed above, we may only consider the case where $s=0$. The eigenvalue corresponding to fX will be denoted by μ. Since we have $s=0$, the equation (4.10) implies $\mu=(\alpha\lambda-2)/(2\lambda-\alpha)$. We also have from (4.4) $(\lambda+\mu-h)(\lambda-\mu)=0$. Now, suppose that $\lambda\neq\mu$. Then we have

(4.12) $$\lambda^2-h\lambda+\alpha^2-5=0,\quad h\alpha=2(\alpha^2-4)$$

because of (4.9). Thus the shape operator A in the direction of the distinguished normal C has three principal curvatures

$$\alpha,\ \frac{h+\sqrt{D}}{2},\ \frac{h-\sqrt{D}}{2},\quad D=h^2-4\alpha^2+20.$$

And their multiplicities are respectively denoted by $2n-1-l_1-l_2, l_1$ and l_2. Thus, the trace of the shape operator A is given by

$$h=\alpha(2n-1-l_1-l_2)+\frac{h}{2}(l_1+l_2)+\frac{\sqrt{D}}{2}(l_1-l_2).$$

Using this fact, h_2 satisfies the following:

$$h_2=h^2-3(l_1+l_2)-(2n-1)(\alpha^2-8).$$

On the other hand, we have from (4.2)

$$2k_2=-6(n-1)-(\alpha^2-8)(2n-1)+h^2-h_2$$

with the aid of the second equation of (4.12). Combining the last two equations, it follows that

$$2k_2+3\{2(n-1)-l_1-l_2\}=0,$$

which is a contradiction. Thus we have $h\alpha-2\alpha^2+8\neq0$ on M. Accordingly (4.8) means that $Af=fA$.

Summing up we have

LEMMA 4.2. *On a semi-invariant Einstein submanifold of codimension 3 with $\nabla_j{}^{\perp}C=0$ in $H_{n+1}(-4)$, we have*

(4.13) $$Af=fA.$$

From (4.7) and this, it is seen that

$$h_{ji}{}^2=\alpha h_{ji}-(g_{ji}-p_jp_i).$$

Thus, the shape operator A in the direction of the distinguished normal has three constant principal curvatures α, $\frac{\alpha+\sqrt{D_1}}{2}$ and $\frac{\alpha-\sqrt{D_1}}{2}$ with multiplicities 1, $n-1$ and $n-1$ respectively, where $D_1=\alpha^2-4$. Consequently we have $h=n\alpha$. Thus (4.2) turns out to be

$$(4.14)\qquad 2k_{ji}{}^2=-\{(n-1)\alpha^2+2\}g_{ji}+2p_jp_i+(n-1)\alpha h_{ji}.$$

It is proved in [7] that

$$(4.15)\qquad \nabla_k h_{ji}+p_j f_{ik}+p_i f_{jk}=0$$

provided that $Af=fA$.

If we differentiate (4.14) covariantly and take account of (4.15), we get

$$(4.16)\qquad 2(k_{jr}\nabla_k k_i{}^r+k_{ir}\nabla_k k_j{}^r)$$
$$=2(p_i\nabla_k p_j+p_j\nabla_k p_i)+(n-1)\alpha(p_jf_{ki}+p_if_{kj}),$$

from which, taking the skew symmetric part with respect to indices k and j and making use of (1.13) and (1.14),

$$2(k_{jr}\nabla_i k_k{}^r-k_{kr}\nabla_i k_j{}^r-2k_{jr}l_k{}^r n_i)$$
$$=2(2p_i\nabla_k p_j+p_j\nabla_k p_i-p_k\nabla_j p_i)+(n-1)\alpha(2p_if_{kj}+p_jf_{ki}-p_kf_{ji})$$

for all indices k, j and i. Thus, it follows that

$$2(k_{jr}\nabla_k k_i{}^r-k_{ir}\nabla_k k_j{}^r-2n_kk_{jr}l_i{}^r)$$
$$=2(2p_k\nabla_i p_j+p_j\nabla_i p_k-p_i\nabla_j p_k)+(n-1)\alpha(2p_kf_{ij}+p_jf_{ik}-p_if_{jk}).$$

Adding this to (4.15), we obtain

$$2k_{jr}D_kk_i{}^r=-2(p_kh_{ir}f_j{}^r+p_ih_{kr}f_j{}^r)+(n-1)\alpha(p_kf_{ij}+p_if_{kj})$$

because of (1.14), which together with (4.14) implies that

$$(4.17)\qquad -\{2+(n-1)\alpha^2\}D_kk_{ji}+2p_j(p^rD_kk_{ir})+(n-1)\alpha h_{jr}D_kk_i{}^r$$
$$=2(p_kh_{ir}l_j{}^r+D_ih_{kr}l_j{}^r)+(n-1)\alpha(p_kl_{ji}+p_il_{jk}).$$

On the other hand, by differentiating the first equation of (1.8) covariantly, we find

$$(4.18)\qquad (D_kk_{jr})p^r=-h_{kr}l_j{}^r.$$

Thus (4.17) turns out to be

$$-\{2+(n-1)\alpha^2\}D_kk_{ji}+(n-1)\alpha h_{jr}D_kk_i{}^r$$
$$=2(p_kh_{ir}l_j{}^r+p_ih_{kr}l_j{}^r+p_jh_{kr}l_j{}^r)+(n-1)\alpha(p_kl_{ji}+p_il_{jk}).$$

Transvecting this with $h_m{}^j$ and using (4.11), we find

$$D_kk_{ji}+p_jh_{kr}l_i{}^r+p_kh_{jr}l_i{}^r+p_ih_{jr}l_k{}^r=0$$

because of the fact that $(n^2-1)\alpha^2+4\neq 0$.

From this and (4.18), it is clear that

$$||D_k k_{ji}||^2 - 3l_{ji}^2 h^{ji2} = 0$$

and hence $||D_k k_{ji}||^2 + \frac{3}{2}\{\alpha^2(n-1)^2 + 2||h_{ji} - \alpha p_j p_i||^2\} = 0$ by virtue of (4.11) and (4.14). It is contradictory. Thus according to Proposition 3.1 we can conclude

THEOREM 4.3. *There are no semi-invariant submanifolds of codimension 3 with parallel Ricci tensor of a complex hyperbolic space such that the distinguished normal is parallel in the normal bundle.*

Because of Proposition 2.1, Proposition 2.2 and Proposition 2.3, we also have

COROLLARY 4.4. *There are no semi-invariant submanifolds of codimension 3 with harmonic curvature of a complex hyperbolic space satisfying one of the following conditions provided that the distinguished normal is parallel in the normal bundle:*

(1) *P is an eigenvector corresponding to the Ricci tensor,*
(2) *P is principal with respect to the distinguished normal C,*
(3) *the number of distinct Ricci curvatures does not exceed 2.*

5. Semi-invariant submanifolds with parallel normal curvature

Let M be a semi-invariant submanifold with $\nabla_j^{\perp} C = 0$ of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$.

In the sequel, suppose that the normal curvature tensor $R^{\perp}$ of M in the normal bundle is parallel, namely $\nabla_j R^{\perp} = 0$. If we put $A_{ji} = \nabla_j n_i - \nabla_i n_j$, then we see that $\nabla_k A_{ji} = 0$ because of the distinguished normal assumed to be parallel in the normal bundle. The Ricci equation (1.16) gives rise to

$$A_{ji} = 2(k_{ir} l_j^r + \frac{c}{4} f_{ji}) \tag{5.1}$$

and consequently

$$A_{jr} p^r = 0. \tag{5.2}$$

Differentiating (5.2) covariantly along M and taking account of (1.6), we find $A_{jr} h_{ks} f^{rs} = 0$ because the normal curvature is parallel. Since we easily see from (5.1) that $A_{jr} f_i^r = A_{ir} f_j^r$, it follows that $A_{jr} h_i^r = 0$. Therefore, the equation (5.1) implies

$$k_{ir}l_s{}^r h_j{}^s + \frac{c}{4} f_{ri} h_j{}^r = 0,$$

which together with (1.9) yields

$$h_{jr} f_i{}^r + h_{ir} f_j{}^r = 0.$$

As it is already shown in section 4, we get

(5.3) $$h_{ji}^2 = \alpha h_{ji} + \frac{c}{4}(g_{ji} - p_j p_i),$$

(5.4) $$\nabla_k h_{ji} = \frac{c}{4}(p_j f_{ik} + p_i f_{jk}).$$

From (5.2), (5.3) and the fact that $A_{jr} h_i{}^r = 0$, it is seen that $A_{ji} = 0$. Hence the equation (5.1) leads to $k_{ir} l_j{}^r + \frac{c}{4} f_{ji} = 0$ which together with (1.7) gives

(5.5) $$k_{ji}^2 + \frac{c}{4}(g_{ji} - p_j p_i) = 0.$$

Thus, it follows that $k_2 + \frac{c}{2}(n-1) = 0$ and consequently c is negative. That is, the ambient space is a complex hyperbolic space.

Substituting (5.3) and (5.5) into (1.17), the Ricci tensor is given by

$$S_{ji} = \frac{c}{2}(n+1) g_{ji} - c p_j p_i + (n-1)\alpha h_{ji}$$

because of the fact that $h = n\alpha$. Differentiating this covariantly and making use of (1.6) and (5.4), we find

$$\nabla_k S_{ji} = c\{p_j h_{kr} f_i{}^r + p_i h_{kr} f_j{}^r\} + \frac{c}{4}(n-1)\alpha(p_j f_{ik} + p_i f_{jk})$$

and consequently the Ricci tensor is of cyclic parallel. Therefore, the semi-invariant submanifold M is of Ricci-parallel provided that M is of harmonic curvature.

According to Proposition 3.1 and Theorem 4.3 we have

THEOREM 5.1. *There are no semi-invariant submanifolds of codimension 3 with harmonic curvature of a complex space form $M_{n+1}(c)$, $c \neq 0$ on which the normal curvature and the distinguished normal are parallel in the normal bundle.*

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