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REDUCIBILITY OF A COMPLEX VECTOR FIELD TO A MIZOHATA TYPE OPERATOR

JONGSIK KIM AND YONG JING SUK

§1. Introduction

In this article, we shall show that a complex vector field

$$L=\frac{\partial}{\partial t}+ib(x,t)\frac{\partial}{\partial x}$$

satisfying the property M_k (for definition cf. next page) is reducible to a Mizohata type operator

$$L=\frac{\partial}{\partial t}+it^k\frac{\partial}{\partial x}$$

where k is an odd positive integer by a local change of coordinate in a neighborhood of the origin if and only if the partial differential equation

$$Lu=0$$

has a C^k solution u with derivative du, not vanishing in a neighborhood of the origin (cf. Theorem 2).

Whether such an equivalence holds or not, was raised and positively answered by Treves in [5] for the case $k=1$.

The generalization, however, to arbitrary odd positive integers has turned out to require a deep analysis in solutions of $Lu=0$ (cf. Theorem 1).

Let Ω be an open neighborhood of the origin in R^2. We denote a point in R^2 by (x,t).

Let L be a C^∞ complex vector field given by

$$L=\frac{\partial}{\partial t}+ib(x,t)\frac{\partial}{\partial x} \tag{1}$$

where $b(x,t)$ is a real valued C^∞ function in Ω.

For an integer $j\geqq 0$, we define Σ_j, a subset of Ω, such as

$$\Sigma_j=\{(x,t)\in\Omega\,|\,\frac{\partial^j b}{\partial t^j}(x,t)=0\}.$$

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For each integer $k \geqq 1$, we introduce the condition M_k on $b(x, t)$ such that

(M_k) (i) Σ_j $(j=0, 1, \ldots, k-1)$ are all equal and coincide with one dimensional C^∞ manifold Σ of Ω, and
(ii) $\frac{\partial^k b}{\partial t^k}(x, t) \neq 0$ for all $(x, t) \in \Sigma$.

Also we introduce the condition $(\tilde{M}_k)$ such as

$(\tilde{M}_k)$ (i) Σ_j $(j=0, 1, \ldots, k-1)$ are all equal and coincide with $\Sigma = \{(x, t) \in \Omega \mid t=0\}$, and
(ii) $\frac{\partial^k b}{\partial t^k}(x, 0) \neq 0$ for all $(x, 0) \in \Sigma$.

In this paper we prove that the linear partial differential operator defined by (1) is locally

$$\frac{\partial}{\partial t} + it^k g(x, t) \frac{\partial}{\partial x},$$

where $g(x, t)$ is a nowhere vanishing real valued C^∞ function, if L satisfies the condition $\tilde{M}_k$ for some nonnegative integer k.

We also prove that when L satisfies the condition M_k for $k=2n+1$, $(n=0, 1, 2, \ldots)$, in order that $Lu=0$ has a C^k solution u such that $du \neq 0$ in an open neighborhood V of $w_0 \in \Sigma$, it is necessary and sufficient that there be a local chart (U, x, t) in V centered at w_0 in which

$$L = g(x, t)\left(\frac{\partial}{\partial t} + it^k \frac{\partial}{\partial x}\right),$$
$$\Sigma \cap U = \{(x, t) \in U \mid t=0\}$$

with $g \in C^\infty(U)$ nowhere zero.

§ 2. Main theorems

LEMMA 1. *Let L be a C^∞ complex vector field defined by* (1). *We assume that $\tilde{M}_k$ (k: nonnegative integer) holds for $b(x, t)$. If $w_0 \in \Sigma$, then there exists an open neighborhood V of w_0 such that*

$$b(x, t) = t^k g(x, t),$$

where $g(x, t)$ is a nowhere vanishing real valued C^∞ function in V.

Proof. By the translation of coordinates if necessary, we may assume that w_0 is the origin in R^2.

Since $\frac{\partial^j b}{\partial t^j}(x, 0) = 0$ for $j=0, 1, \ldots, k-1$ and $\frac{\partial^k b}{\partial t^k}(x, 0) \neq 0$, by the

Malgrange preparation theorem, there exists V, an open neighborhood of the origin, such that in V,

$$b(x,t)=g(x,t)\,(t^k+\alpha_1(x)t^{k-1}+\cdots+\alpha_k(x)),$$

where $\alpha_i(x)$ $(i=1,2,\ldots,k)$ is a real valued C^∞ function in V and $g(x,t)$ is a nowhere vanishing real valued C^∞ function in V.

By the condition $\tilde{M}_k$, for each fixed x,

$$t^k+\alpha_1(x)t^{k-1}+\cdots+\alpha_k(x)$$

has a k-multiple root $t=\phi(x)$ for some function ϕ.

Therefore,

$$t^k+\alpha_1(x)t^{k-1}+\cdots+\alpha_k(x)=(t-\phi(x))^k.$$

Since $-k\phi(x)=\alpha_1(x)$ and $\alpha_1(x)$ is a real valued C^∞ function in V, $\phi(x)$ is a real valued C^∞ function in V.

Thus

$$b(x,t)=g(x,t)\,(t-\phi(x))^k.$$

From the fact that $b(x,0)=0$ for all $(x,0)\in V$, we have

$$\phi(x)\equiv 0.$$

REMARK 1. The class of operator satisfying the condition $\tilde{M}_k$ with odd integer $k\geqq 1$ includes the generalized Mizohata operator

$$\frac{\partial}{\partial t}+it^k\frac{\partial}{\partial x}\quad (k\geqq 1,\ \text{odd integer}),$$

where Σ is the x-axis.

LEMMA 2. *Let L be a C^∞ complex vector field in Ω defined by* (1). *We assume that M_k (k: odd nonnegative integer) holds for $b(x,t)$.*

Let $u=A(x,t)+iB(x,t)$ with A, B, real valued, be the C^k solution of

$$Lu=0$$

such that $A_x\neq 0$ at every point in an open neighborhood V of $w_0\in\Sigma$. Then

$$H_{A^j}B=0\ \text{for}\ 1\leqq j\leqq k$$

$$H_{A^{k+1}}B\neq 0$$

in $\Sigma\cap V$. Here H_A denotes the Hamiltonian vector field defined by A.

Proof. Note first that $Lu=0$ implies $A_t=bB_x$, $B_t=-bA_x$.

$$H_AB=-\{A,B\}=A_tB_x-A_xB_t$$
$$=bB_x^2+bA_x^2=b(A_x^2+B_x^2).$$

Now,

$$H_{A^2}B=\{A,\{A,B\}\}$$
$$=(-1)b_tA_x(A_x^2+B_x^2)+b\alpha_2(x,t)$$

for suitable $\alpha_2(x, t)$. By induction,

$$H_{A^i}B = (-1)^{i-1}\frac{\partial^{i-1}b}{\partial t^{i-1}}A_x^{i-1}(A_x^2+B_x^2) + \frac{\partial^{i-2}b}{\partial t^{i-2}}\alpha_{i,1}(x,t) + \cdots + b\alpha_{i,i-1}(x,t).$$

$$H_{A^{k+1}}B = (-1)^k\frac{\partial^k b}{\partial t^k}A_x^k(A_x^2+B_x^2) + \cdots + b\alpha_{k+1,k}(x,t).$$

Thus on $\Sigma \cap V$

$$H_{A^j}B=0 \text{ for } j \leq k,$$
$$H_{A^{k+1}}B \neq 0.$$

REMARK 2. We note that $\Sigma \cap V = \{(x,t) \in V \mid H_A B(x,t)=0\}$.

Also we note that if $du \neq 0$ at every point in V, then there exists a solution $u=A+iB$ such that $A_x \neq 0$ at every point in V. In sequel, u is such a solution.

THEOREM 1. *Let L be a C^∞ complex vector field in Ω defined by* (1). *We assume that M_k(k : odd nonnegative integer) holds for $b(x,t)$.*

Let $u=A(x,t)+iB(x,t)$ be the C^k solution of $Lu=0$ such that $du \neq 0$ in an open neighborhood V of $w_0 \in \Sigma$. Then one can choose local coordinates (x, y) in an open neighborhood $U(\subset V)$ of w_0, vanishing at w_0, such that

$$u(x,y)-u(0,0) = \{1+ir(x, y^{k+1})\}(x+i\varepsilon_0 y^{k+1}). \tag{2}$$

Proof. For the sake of simplicity we assume that w_0 is the origin of R^2.

Let $H_{A^k}B=\tau$.

As $\{A, H_{A^k}B\} \neq 0$ at w_0, we may write (cf. (6))

$$H_{A^{k-1}}B = Af_{k-1}(A) + \tau g_{k-1}(A,\tau).$$

Considering the case $\tau=0$, we see that $Af_{k-1}(A)=0$.

Let us apply H_A on both sides to get

$$\tau = H_{A^k}B = (H_A\tau)g_{k-1}(A,\tau) + \tau(H_A g_{k-1}(A,\tau)).$$

As $H_A\tau \neq 0$, we get

$$g_{k-1}(A,\tau) = (H_A\tau)^{-1}\tau[1-H_A g_{k-1}(A,\tau)].$$

Therefore

$$H_{A^{k-1}}B = \tau^2[1-H_A g_{k-1}(A,\tau)](H_A\tau)^{-1} = \tau^2 h_{k-1}(A,\tau).$$

Now we can write $H_{A^{k-2}}B$ as

$$H_{A^{k-2}}B = Af_{k-2}(A) + \tau g_{k-2}(A, \tau)$$
$$= \tau g_{k-2}(A, \tau).$$

Then

$$H_{A^{k-1}}B = (H_A\tau)g_{k-2}(A, \tau) + \tau(H_A g_{k-2}(A, \tau)).$$

That is,

$$\tau^2 h_{k-1}(A, \tau) = (H_A\tau)g_{k-2}(A, \tau) + \tau(H_A g_{k-2}(\mathrm{A}, \tau)).$$

Hence

$$g_{k-2}(A, \tau) = (H_A\tau)^{-1}\tau[\tau h_{k-1}(A, \tau) - H_A g_{k-2}(A, \tau)].$$

Therefore

$$H_{A^{k-2}}B = \tau^2\tilde{h}_{k-2}(A, \tau).$$

Let us apply H_A again to both sides.

$$H_{A^{k-1}}B = 2\tau(H_A\tau)\tilde{h}_{k-2}(A, \tau) + \tau^2(H_A\tilde{h}_{k-2}(A, \tau)).$$

That is,

$$\tau^2 h_{k-1}(A, \tau) = 2\tau(H_A\tau)\tilde{h}_{k-2}(A, \tau) + \tau^2(H_A\tilde{h}_{k-2}(A, \tau)).$$

Hence

$$\tilde{h}_{k-2}(A, \tau) = \{2(H_A\tau)\}^{-1}\tau[h_{k-1}(A, \tau) - H_A\tilde{h}_{k-2}(A, \tau)].$$

Summing up,

$$H_{A^{k-2}}B = \tau^3 h_{k-2}(A, \tau).$$

Now let

$$H_{A^{k-3}}B = \tau g_{k-3}(A, \tau).$$

Then

$$H_{A^{k-2}}B = (H_A\tau)g_{k-3}(A, \tau) + \tau(H_A g_{k-3}(A, \tau))$$

or

$$\tau^3 h_{k-2} = (H_A\tau)g_{k-3}(A, \tau) + \tau(H_A g_{k-3}(A, \tau)).$$

Therefore

$$g_{k-3}(A, \tau) = \tau q_{k-3}(A, t)$$

and hence

$$H_{A^{k-3}}B = \tau^2 q_{k-3}(A, \tau).$$

Apply H_A to both sides. Then

$$\tau^3 h_{k-2} = H_{A^{k-2}}B = 2\tau(H_A\tau)q_{k-3} + \tau^2(H_A q_{k-3}).$$

Therefore

$$q_{k-3} = \{2(H_A\tau)\}^{-1}\tau[\tau h_{k-2} - H_A q_{k-3}].$$

Hence

$$H_{A^{k-3}}B = \tau^3\tilde{h}_{k-3}(A, \tau).$$

Applying H_A again we see that

$$H_{A^{k-3}}B = \tau^4 h_{k-3}(A, \tau).$$

Repeating the same process, we can get

(3) $$H_A B = H_{A^{k-(k-1)}} B = \tau^k h_1(A, \tau).$$

Now let us write B as

$$B = Af(A) + \tau g_0(A, \tau)$$
$$= \tau g_0(A, \tau).$$

Then

$$H_A B = (H_A \tau) g_0(A, \tau) + \tau (H_A g_0(A, \tau)).$$

Repeating the same argument, we get

$$B = Af(A) + \tau^{k+1} h(A, \tau).$$

Now let us apply H_A both sides. Then

$$\tau^k h_1(A, \tau) = H_A B = (k+1) \tau^k (H_A \tau) h(A, \tau) + \tau^{k+1} (H_A h(A, \tau)).$$

Hence

$$h(A, \tau) = \frac{1}{(k+1)(H_A \tau)} [h_1(A, \tau) - \tau H_A h(A, \tau)].$$

We notice that $h_1(A, \tau) \neq 0$ in V from (3).

Therefore

$$h(A, \tau) = [(k+1)(H_A \tau)]^{-1} h_1(A, \tau) \{1 - \tau \frac{H_A h(A, \tau)}{h_1(A, \tau)}\}$$
$$= [(k+1)(H_A \tau)]^{-1} h_1(A, \tau) \{1 + \phi(A, \tau)\}.$$

Let us set

$$K = \frac{1}{(k+1)} \left| \frac{h_1(A, \tau)}{H_A \tau} \right|$$

and

$$\varepsilon_0 = \frac{h_1(A, \tau)}{H_A \tau} \left| \frac{H_A \tau}{h_1(A, \tau)} \right|.$$

Then

$$B = Af(A) + \varepsilon_0 K \tau^{k+1} (1 + \phi(A, \tau)),$$

where $\phi(0, 0) = 0$.

Also

$$u = A + iB$$
$$= A + iAf(A) + i\varepsilon_0 K \tau^{k+1} (1 + \phi(A, \tau))$$
$$= \{1 + if(A)\} [A + i\varepsilon_0 K \tau^{k+1} \frac{1 + \phi(A, \tau)}{1 + if(A)}]$$
$$= \{1 + if(A)\} [A + i\varepsilon_0 K \tau^{k+1} \frac{1 + \phi(A, \tau) - if(A)(1 + \phi(A, \tau))}{1 + [f(A)]^2}]$$
$$= \{1 + if(A)\} [A + \frac{(1 + \phi(A, \tau))}{1 + [f(A)]^2} f(A) \varepsilon_0 K \tau^{k+1} + i\varepsilon_0 K \tau^{k+1} \frac{(1 + \phi(A, \tau))}{1 + [f(A)]^2}].$$

We can choose our new coordinates x, y such that

$$x=A+\varepsilon_0 K\tau^{k+1}\frac{(1+\phi(A,\tau))}{1+[f(A)]^2}f(A),$$

$$y=\{\frac{K(1+\phi(A,\tau))}{1+[f(A)]^2}\}^{1/k+1}\tau.$$

We note that

(4) $$x=A+\varepsilon_0 y^{k+1}f(A).$$

We can solve the equation (4) with respect to A to get

$$A=A(x, y^{k+1}).$$

Then we set

$$r(x, y^{k+1})=f(A(x, y^{k+1}))$$

to get

(5) $$u=\{1+ir(x, y^{k+1})\}[x+i\varepsilon_0 y^{k+1}].$$

Let the local chart (U, x, y) be as in the theorem. Then two points (x, y) and (x', y') in U satisfy

$$u(x, y)=u(x', y')$$

if and only if

$$x=x' \text{ and } y=\pm y'.$$

We also note that $\Sigma\cap U=\{(x, y)\in U\mid y=0\}$.

Definition 1. Let D be a subset of R^2 and C be the field of complex numbers. Let u be a mapping from D into C. The *fiber* of u at $z\in u(D)$ is the set of points (x, y) in D such that $u(x, y)=z$.

Definition 2. Let D and u be as in the definition 1.

We say that D is *u-symmetric* if $(x, y)\in D$ if and only if $(x', y')\in D$ for any (x', y') satisfying $u(x', y')=u(x, y)$.

In the theorem 1 we can choose U to be u-symmetric and small enough so that the following holds.

(*) The fibers of u in U consist of either of a single point, when they are contained in $U\cap\Sigma$, or else of a pair of points, (x, y) and $(x, -y)$, when they do not interset Σ.

Hence forth we assume that U has the property (*) and also

(**) $u(U^+)$ is connected and simply connected,

where $$U^+=\{(x, y)\in U\mid y>0\}.$$

Now let ϕ be a holomorphism of the upper half disk

$$D^+=\{z\in C\mid |z|<1,\ \mathrm{Im}\ z>0\}$$

onto $u(U^+)$, which extends as a C^∞ map up to the piece of boundary

$$I=\{(x,0)\in C\mid |x|<1\}$$

and maps diffeomorphically I onto $u(\Sigma\cap U)$ with $\phi(0)=u(w_0)$. Such a holomorphism exists by the Riemann mapping theorem.

Set then

$$u_0=\phi^{-1}\circ u.$$

Then u_0 is a C^∞ map of U into C and

$$u_0(U)=D^+\ \cup\ I.$$

We note from the theorem that for $j=1, 2, \ldots, k$

$$\frac{\partial^j u_0}{\partial y^j}(x,0)=0.$$

Thus we derive that

$$\begin{aligned} u_0(x,y)&=u_0(x,0)+i\varepsilon_0 y^{k+1}u_1(x,y)\\ &=i\varepsilon_0 y^{k+1}u_1(x,y). \end{aligned}$$

But we have

$$\begin{aligned} \frac{1}{(k+1)y^k}\frac{\partial u}{\partial y}&=[1+ir(0,0)]i\varepsilon_0\\ &=i\varepsilon_0 u_1(0,0)\frac{\partial\phi}{\partial z}(0). \end{aligned}$$

In fact, from (5) we get

$$\begin{aligned} \frac{\partial u}{\partial y}=\frac{\partial\phi}{\partial z}\frac{\partial u_0}{\partial y}&=ir_y(x,y^{k+1})(k+1)y^k(x+i\varepsilon_0 y^{k+1})\\ &\quad+(1+ir(x,y^{k+1}))(i\varepsilon_0(k+1)y^k). \end{aligned}$$

Hence

$$\frac{1}{(k+1)y^k}\frac{\partial u}{\partial y}\bigg|_{(0,0)}=[1+ir(0,0)](i\varepsilon_0)$$

and, as $u=\phi\circ u_0$, we get from (6)

$$\frac{\partial u}{\partial y}=\frac{\partial\phi}{\partial z}\frac{\partial u_0}{\partial y}=\left\{i\varepsilon_0(k+1)y^k u_1(x,y)+i\varepsilon_0 y^{k+1}\frac{\partial u_1}{\partial y}\right\}\frac{\partial\phi}{\partial z}.$$

Hence

$$\frac{1}{(k+1)y^k}\frac{\partial u}{\partial y}\bigg|_{(0,0)}=i\varepsilon_0 u_1(0,0)\frac{\partial\phi}{\partial z}(0).$$

From the above, we have

$$1+r(0,0)=u_1(0,0)\frac{\partial\phi}{\partial z}(0).$$

Now ϕ is a diffeomorphism of the interval I onto the arc of curve

$u(\Sigma\cap U)$ which means that

$$\phi(x)=u(t(x),0)$$

with $x\to t(x)$ a diffeomorphism of I onto an open interval containing $t=0$.

Thus

$$\begin{aligned}\phi(x)&=u(t(x),0)\\&=\{1+ir(t(x),0)\}t(x).\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial\phi}{\partial z}(0)&=\frac{\partial\phi}{\partial x}(0)\\&=\{1+ir(0,0)\}t'(0)+ir_x(t(0),0)t(0)\\&=\{1+ir(0,0)\}t'(0)\end{aligned}$$

as $\qquad \phi(0)=t(0)=0=u(w_0).$

We note that $t'(0)>0$ as ϕ preserves orientation.

We reach the conclusion that

$$\begin{aligned}u_1(0,0)&=\{1+ir(0,0)\}\frac{1}{[1+ir(0,0)]}[t'(0)]^{-1}\\&=[t'(0)]^{-1}\end{aligned}$$

and therefore, that

$$u_0(x,y)=u_0(x,0)+i\frac{1}{t'(0)}y^{k+1}[1+\psi(x,y)]$$

with $\psi(0,0)=0$.

Let $\qquad \psi(x,y)=\alpha(x,y)+i\beta(x,y).$

Then

$$u_0(x,y)=u_0(x,0)-\frac{\beta(x,y)y^{k+1}}{t'(0)}+i\frac{1}{t'(0)}y^{k+1}[1+\alpha(x,y)]$$

with $\alpha(0,0)=0$, $\beta(0,0)=0$.

Therefore

$$\begin{aligned}X&=u_0(x,0)-\frac{\beta(x,y)y^{k+1}}{t'(0)},\\Y&=\sqrt[k+1]{\frac{1+\alpha(x,y)}{t'(0)}}\cdot|y|\end{aligned}$$

is a *bona fide* change of coordinates in a neighborhood of w_0.

In this new coordinates we have

$$\begin{aligned}u_0&=X+iY^{k+1},\\u&=\phi(X+iY^{k+1}).\end{aligned}$$

THEOREM 2. *Let L be a C^∞ complex vector field in Ω defined by* (1) *and let $b(x, t)$ satisfy the condition M_k for $k=2n+1$,* $(n=0, 1, 2, \ldots)$. *In order that*

$$Lu=0$$

has a C^k solution u such that $du \neq 0$ at every point in an open neighborhood V of w_0 in Σ it is necessary and sufficient that there be a local chart (U, x, t) in V centered at w_0 in which

$$L=g\left(\frac{\partial}{\partial t}+it^k\frac{\partial}{\partial x}\right)$$

$$\Sigma \cap U=\{(x, t) \in U \mid t=0\}$$

with $g \in C^\infty(U)$ nowhere zero.

Proof. The sufficiency is self-evident. To prove the necessity, we represent the solution u as

$$u=\phi(X+iY^{k+1}).$$

We see, by taking U small enough, that we must have

$$\phi'(X+iY^{k+1})L(X+iY^{k+1})=0,$$

and since $\phi'(0) \neq 0$,

$$LX+(k+1)iY^kLY=0.$$

Hence
$$L=(LX)\frac{\partial}{\partial X}+(LY)\frac{\partial}{\partial Y}$$
$$=(LY)\left(\frac{\partial}{\partial Y}-(k+1)iY^k\frac{\partial}{\partial X}\right).$$

It suffices to take $x=-\sqrt[k]{k+1}\ X$ and $t=\sqrt[k]{k+1}\ Y$. In fact,

$$\frac{\partial}{\partial X}=\frac{\partial}{\partial x}\frac{\partial x}{\partial X}=-\frac{\partial}{\partial x}\sqrt[k]{k+1},$$
$$\frac{\partial}{\partial Y}=\frac{\partial}{\partial t}\frac{\partial t}{\partial Y}=\frac{\partial}{\partial t}\sqrt[k]{k+1}.$$

Therefore

$$L=(LY)\sqrt[k]{k+1}\left(\frac{\partial}{\partial t}+it^k\frac{\partial}{\partial x}\right).$$

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Seoul National University
Seoul 151-742, Korea