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DERIVATIONS AND HOMOMORPHISMS ON BANACH ALGEBRAS (I)

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1. Introduction

A derivation on a Banach algebra A is a linear mapping D of A into itself such that $Dab=a(Db)+(Da)b$ $(a,b\in A)$. If S is a linear operator from a Banach space X into a Banach space Y, then the separating space $\mathfrak{S}(S)$ of S is defined by

$$\mathfrak{S}(S)=\{y\in Y\,|\,\text{there are } x_n\to 0 \text{ in } X \text{ with } Sx_n\to y \text{ in } Y\}$$

In Section 2, we show that if D is a derivation on a commutative Banach algebra A with the radical R, then for any positive integer m, $\mathfrak{S}(D^n|R)\subseteq R$ for all $n\leqslant m$ if and only if $\mathfrak{S}(D^n)\subseteq R$ for all $n\leqslant m$. From this result we know that if the restriction $D|R$ is continuous then the range of a derivation D on A is contained in R. This is a generalization of Singer and Wermer's Theorem [10]. And we also prove that every derivation on a commutative Banach algebra has a nilpotent separating space if and only if every derivation on a commutative semiprime Banach algebra is continuous.

In Section 3, we discuss the continuity of homomorphism mapping C^*-algebras into commutative Banach algebras. We prove that if A is a C^*-algebra and B is a commutative Banach algebra with radical R and if R is an integral domain such that $\bigcap_{n\geq 1} R^n=\{0\}$, then every homomorphism from A into B is continuous.

2. Derivations on Commutative Banach Algebras

In this section we suppose that A is a commutative Banach algebra with identity, R the radical of A, Φ_A the set of all multiplicative

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linear functionals of A, and D a derivation on A. In [6] Johnson proved the following theorem.

THEOREM. *If A has an identity e, then there exist orthogonal idempotent elements $e_0, e_1, \cdots e_m$ in A with sum e such that $D(e_0A)$ is contained in the radical of e_0A and such that each algebra $e_1A, \cdots e_mA$ has just one maximal ideal.*

LEMMA 2.1. *Suppose that A has a unique maximal ideal which is its radical R. Then for any positive integer m, $\mathfrak{S}(D^n|R) \subseteq R$ for all $n \leqslant m$ if and only if $\mathfrak{S}(D^n) \subseteq R$ for all $n \leqslant m$.*

Proof. One half of the proof is obvious. For the other half we use the induction. If $y \in \mathfrak{S}(D)$, then there exists a sequence $\{x_n\}$ in A with $x_n \to 0$ such that $Dx_n \to y$. Suppose that $y \notin R$. Since R is a unique maximal ideal, let $\Phi_A = \{\phi\}$. Then $\phi(y) \neq 0$. For each x in A, we have $\phi(xy - \phi(x)y) = 0$. Thus for each n, there exists r_n in R such that $r_n = x_ny - \phi(x_n)y$ and $r_n \to 0$. Since $x_ny = \phi(x_n)y + r_n$,

$$(Dx_n)y = \phi(x_n)Dy + Dr_n - x_nDy$$

and so there exists the limit of the sequence $\{Dr_n\}$. Thus

$$y^2 = \lim_{n\to\infty} Dr_n \in \mathfrak{S}(D|R) \subseteq R,$$

and $(\phi(y))^2 = \phi(y^2) = 0$. Hence $\phi(y) = 0$, and so $y \in R$. This is a contradiction, and so $\mathfrak{S}(D) \subseteq R$. Suppose that $\mathfrak{S}(D^l|R) \subseteq R$ for all $l \leqslant m$ and $\mathfrak{S}(D^l) \subseteq R$ for all $l \leqslant m-1$. If $y \in \mathfrak{S}(D^m)$, there exists a sequence $\{x_n\}$ in A with $x_n \to 0$ such that $D^mx_n \to y$. Suppose that $y \notin R$. Then $\phi(y) \neq 0$. For each x in A, we have $\phi(xy - \phi(x)y) = 0$. Thus there exists a sequence $\{r_n\}$ in R such that $r_n \to 0$ and $r_n = x_ny - \phi(x_n)y$. Since $x_ny = \phi(x_n)y + r_n$, we have

$$\begin{aligned} D^m(x_ny) &= \sum_{i=0}^{m} \binom{m}{i} D^{m-i}x_nD^iy \\ &= (D^mx_n)y + \sum_{i=1}^{m} \binom{m}{i} D^{m-i}x_nD^iy \\ &= \phi(x_n)D^my + D^mr_n. \end{aligned}$$

Since $\mathfrak{S}(D^l|R) \subseteq R$ for all $l \leqslant m$ and $\mathfrak{S}(D^l) \subseteq R$ for all $l \leqslant m-1$, $\mathfrak{S}(\phi \circ (D^l|R)) = [\phi\mathfrak{S}(D^l|R)]^- = \{0\}$ for all $l \leqslant m$ and $\mathfrak{S}(\phi \circ D^l) = [\phi\mathfrak{S}(D^l)]^- = \{0\}$ for all $l \leqslant m-1$. Then $\phi \circ (D^l|R)$ is continuous for all $l \leqslant m$, and $\phi \circ D^l$ is continuous for all $l \leqslant m-1$. Thus we have

$$\begin{aligned}\phi(y^2) &= \phi(\lim_{n\to\infty}(D^m x_n)y) \\ &= \lim_{n\to\infty} \phi\ [\phi(x_n)D^m y + D^m r_n - \sum_{i=1}^{m}\binom{m}{i} D^{m-i}x_n D^i y] \\ &= \lim_{n\to\infty} [\phi(x_n)\phi\circ D^m y + \phi\circ D^m r_n - \sum_{i=1}^{m}\binom{m}{i}(\phi\circ D^{m-i}x_n)(\phi\circ D^i y)] \\ &= 0.\end{aligned}$$

Therefore $y^2 \in R$, and so $y \in R$. This is a contradiction. Thus $\mathfrak{S}(D^m) \subseteq R$, which completes the proof.

Now we extend the above result to the general case.

THEOREM 2.2. *Let A be a commutative Banach algebra. For any positive integer m, $\mathfrak{S}(D^n|R) \subseteq R$ for all $n \leqq m$ if and only if $\mathfrak{S}(D^n) \subseteq R$ for all $n \leqq m$. In particular $\mathfrak{S}(D^n|R) \subseteq R$ for all n if and only if $\mathfrak{S}(D^n) \subseteq R$ for all n.*

Proof. We may assume that A has an identity e. By Johnson's theorem, there exist orthogonal idempotents $e_0, e_1, \cdots, e_s$ with sum e such that $D(e_0A)$ is contained in the radical of e_0A and each algebra $e_1A, \cdots, e_sA$ has a unique maximal ideal. Since $e_iA \cap R = e_iR$, e_iR is the radical of e_iA for each i. Then for each n, $D^n(e_0A) \subseteq D^{n-1}(e_0R) \subseteq e_0R$ and $\mathfrak{S}(D^n|e_0A) \subseteq \overline{D^n(e_0A)} \subseteq e_0R$. By the orthogonality of the idempotents $e_0, e_1 \cdots, e_s$ and the linearity of D, we have

$$\mathfrak{S}(D^n|R) = \mathfrak{S}(D^n|e_0R) + \cdots + \mathfrak{S}(D^n|e_sR).$$

Since $D^n e_i x = e_i D^n x$ for each $n \geqq 1$, we get the derivation $D^n : e_iA \to e_iA$ for each $i = 0, 1, \cdots, s$ and $n \geqq 1$, and $e_i\mathfrak{S}(D^n) \subseteq \mathfrak{S}(D^n|e_iA)$.

Now if $\mathfrak{S}(D|R) \subseteq R$, then $\mathfrak{S}(D|e_iR) \subseteq e_iR$ for each i. Since e_iR is the unique maximal ideal in e_iA, by Lemma 2.1 $\mathfrak{S}(D|e_iA) \subseteq e_iR$ for $i = 1, 2, \cdots, s$. Hence

$$\begin{aligned}\mathfrak{S}(D) &= e_0\mathfrak{S}(D) + \cdots + e_s\mathfrak{S}(D) \\ &\subseteq \mathfrak{S}(D|e_0A) + \cdots + \mathfrak{S}(D|e_sA) \\ &\subseteq e_0R + \cdots + e_sR = R.\end{aligned}$$

If $\mathfrak{S}(D^l|R) \subseteq R$ for each $l \leqq m$ and $\mathfrak{S}(D^l) \subseteq R$ for each $l \leqq m-1$, then for each $l \leqq m$

$$\begin{aligned}\mathfrak{S}(D^l|R) &= \mathfrak{S}(D^l|e_0R) + \cdots + \mathfrak{S}(D^l|e_sR) \\ &\subseteq R\end{aligned}$$

Hence $\mathfrak{S}(D^l|e_iR) \subseteq e_iR$ for each i. By Lemma 2.1, $\mathfrak{S}(D^l|e_iA) \subseteq e_iR$ for each i and $l \leqq m$. Hence

$$\begin{aligned}\mathfrak{S}(D^m) &= e_0\mathfrak{S}(D^m)+\cdots+e_s\mathfrak{S}(D^m)\\ &\subseteq \mathfrak{S}(D^m\,|\,e_0A)+\cdots+\mathfrak{S}(D^m\,|\,e_sA)\\ &\subseteq e_0R+\cdots+e_sR=R.\end{aligned}$$

Thus we get the result.

Corollary 2.3. *If* $\mathfrak{S}(D|R)=\{0\}$, *then* $DA\subseteq R$. *In particular if* R *is finite dimensional, then* $DA\subseteq R$.

Proof. Since $\mathfrak{S}(D|R)=\{0\}\subseteq R$, $\mathfrak{S}(D)\subseteq R$ by Theorem 2.2. Then $\phi\circ D$ is continuous for all $\phi\in\Phi_A$. Thus

$$[\phi\mathfrak{S}(D^2|R)]^- = \mathfrak{S}(\phi\circ D\circ D|R) = [\phi\circ D\mathfrak{S}(D|R)]^- = \{0\}$$

for all $\phi\in\Phi_A$, and so $\mathfrak{S}(D^2|R)\subseteq R$. By the induction, $\mathfrak{S}(D^n|R)\subseteq R$ for all $n\geqq 1$. Thus $\mathfrak{S}(D^n)\subseteq R$ for all $n\geqq 1$. Then $\phi\circ D^n$ is continuous for all $n\geqq 1$. By Theorem 1′ in [7], we have $DA\subseteq R$.

Note that if $K_D(I)=\{x\in I : D^n x\in I \text{ for all } n\geqq 1\}$ where I is an ideal of A, then $K_D(I)$ is an ideal, and if I is a prime ideal then $K_D(I)$ is a prime ideal [5].

Theorem 2.4. *The following conditions are equivalent.* (1) *Every derivation on a commutative Banach algebra has a nilpotent separating space.*

(2) *Every derivation on a commutative semiprime Banach algebra is continuous.*

Proof. Obviously (1) implies (2). Assume (2) and suppose that (1) is false. Then there exists a derivation D on a commutative Banach algebra A such that $\mathfrak{S}(D)$ is non-nilpotent. Thus there exists a minimal prime ideal P such that $\mathfrak{S}(D)\subsetneq P$ and P is closed, by Theorem 2.5 in [2]. Then $K_D(P)$ is a prime ideal and $K_D(P)\subseteq P$. By the minimality of P, $K_D(P)=P$, and so $D(P)\subseteq P$. Then we can define a derivation $\bar{D}$ on a commutative semiprime Banach algebra A/P by $\bar{D}(x+P)=Dx+P$. By the assumption, $\bar{D}$ is continuous. Then by Lemma 1.4 in [8] $\mathfrak{S}(D)\subseteq P$. This is a contradiction. Therefore (2) implies (1).

3. Homomorphism from C^*-Algebras.

Remark. By "Prime Ideal Theorem" in [1], we know that if θ is a discontinuous homomorphism from a Banach algebra A onto a dense subalgebra of a commutative Banach algebra B, then there exists a discontinuous linear operator $\theta_0=\theta(a_0)\theta$ for some $a_0\in A$ such that

(1) For each $a\in A$, either $\overline{\theta(a)\mathfrak{S}(\theta_0)}=\mathfrak{S}(\theta_0)$ or $\theta(a)\mathfrak{S}(\theta_0)=\{0\}$

* (2) A/I_0 is an integral domain where

$$I_0=\{a\in A\,|\,\theta(a)\mathfrak{S}(\theta_0)=\{0\}\}$$

LEMMA 3.1. *If θ is a homomorphism from a C^*-algebra A onto a dense subalgebra of a commutative Banach algebra B with radical R, then* $\mathfrak{S}(\theta)=R$.

Proof. By Theorem 4.1 in [9], $R\subseteq\mathfrak{S}(\theta)$ and there exists a homomorphism ν from $\bar{\tau}$ to $\mathfrak{S}(\theta)$ such that $\mathfrak{S}(\theta)=\nu(\bar{\tau})^-$ where $\tau=\{a\in A:\theta(a)\mathfrak{S}(\theta)=\{0\}\}$, and $\nu(a)$ is quasi-nilpotent for each a in $\bar{\tau}$. Since B is commutative. $R=\{b\in B: r(b)=0\}$ and $r(\nu(a))=0$ for each a in $\bar{\tau}$ where r is the spectral radius. Thus we have $\mathfrak{S}(\theta)=\nu(\bar{\tau})^-\subseteq R$.

REMARK. In [4], Esterle showed that if A is a commutative radical Banach algebra and also an integral domain then $x\in\overline{xaA}$ for some nonzero elements $x, a\in A$ if and only if $\bigcap_{n\geq 1} a^nA\neq\{0\}$. From this result we obtain the following lemma.

LEMMA 3.2. *Let A be a C^*-algebra, B a commutative Banach algebra with the radical R, and θ a homomorphism from A onto a dense subalgebra B. If there exists an element $a_1\in A$ such that $0\neq\theta(a_1)\in R$ and R is an integral domain, then we have $\bigcap_{n\geq 1}\theta(a_1)^nR\neq\{0\}$.*

Proof. Since $R\neq\{0\}$, θ is discontinuous by Lemma 3.1. By the Remark preceding Lemma 3.1, there exists a discontinuous linear operator $\theta_0=\theta(a_0)\theta$ for some a_0 in A such that for each a in A, either $\overline{\theta(a)\mathfrak{S}(\theta_0)}=\mathfrak{S}(\theta_0)$ or $\theta(a)\mathfrak{S}(\theta_0)=\{0\}$, and A/I_0 is an integral domain. Since $\mathfrak{S}(\theta)=R$ and $\mathfrak{S}(\theta_0)=\overline{\theta(a_0)\mathfrak{S}(\theta)}$, we have $\mathfrak{S}(\theta_0)\subseteq R$. Since R is an integral domain, $\theta(a_1)\mathfrak{S}(\theta_0)\neq\{0\}$. Note that $\theta(a_0)\mathfrak{S}(\theta_0)\neq\{0\}$ from $\theta(a_0)\mathfrak{S}(\theta)\neq\{0\}$. Then $a_1, a_0\notin I_0$ and so $a_0a_1\notin I_0$ because A/I_0 is an integral domain. Thus $\overline{\theta(a_0a_1)\mathfrak{S}(\theta_0)}=\mathfrak{S}(\theta_0)$, and so

$$\begin{aligned}\theta(a_0a_1)\in\mathfrak{S}(\theta_0)&=\overline{\theta(a_0a_1)\overline{\theta(a_1)\mathfrak{S}(\theta_0)}}\\&=\overline{\theta(a_0a_1)\theta(a_1)\mathfrak{S}(\theta_0)}\\&\subseteq\overline{\theta(a_0a_1)\theta(a_1)R}\end{aligned}$$

By the above Remark, we have $\bigcap_{n\geq 1}\theta(a_1)^nR\neq\{0\}$.

THEOREM 3. 3. *Let A be a C^*-algebra and B a commutative Banach algebra with the radical R. If $\bar{R}$ is an integral domain such that* $\bigcap_{n\geq 1} R^n = \{0\}$, *then every homomorphism θ from A into B is continuous.*

Proof. If not, then $\theta : A \to \overline{\theta(A)}$ is a discontinuous homomorphism. By Theorem 4. 1 in [9], there exists a discontinuous homomorphism ν from $\bar{\tau}$ into $\mathfrak{S}(\theta) = \nu(\bar{\tau})^-$. Lemma 3. 1 implies that $\mathfrak{S}(\theta) = R \cap \overline{\theta(A)}$. Take $\nu(a) \in \mathfrak{S}(\theta)$ with $\nu(a) \neq 0$. By Lemma 3. 2,

$$\{0\} \neq \bigcap_{n\geq 1} \nu(a)^n (R \cap \overline{\theta(A)}) \subseteq \bigcap_{n\geq 1} R^n$$

This is a contradiction. Thus θ is continuous.

EXAMPLE. It is well-known that for a Banach algebra of formal power series the radical R is an integral domain and satisfies $\bigcap_{n\geq 1} R^n = \{0\}$. So every homomorphism from a C^*-algebra into a Banach algebra of formal power series is continuous.

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