J. Korean Math. Soc. **25**(1988), No. 2, pp. 199-206

ON THE NUMBER OF PRIMES BETWEEN TWO FUNCTION-VALUES $f(n)$ AND $f(n+1)$

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1. Introduction

Let $\pi(x)$ be the number of primes p not exceeding x, then the well-known Bertrand's postulate can be written as

$$\pi(2n)-\pi(n)\geq 1 \qquad \text{for all } n\in N\equiv\{1,2,3,\cdots\}.$$

This is first proved by Chebychev [1]. Actually, it can be shown that for any fixed real number $r\in(7/12, 1]$

$$(1) \qquad \pi(n+n^r)-\pi(n)\sim\frac{n^r}{\ln n} \quad \text{as } n\to\infty.$$

The purpose of this paper is to investigate the number of primes between two function-values $f(n)$ and $f(n+1)$, or more formally, to investigate

$$\Delta_f(n)\equiv\pi(f(n+1))-\pi(f(n)), \qquad n\in N, n\geq a,$$

where f is a positive function defined on an interval $[a,\infty)$. Recently, we obtain Theorem A below via the following heuristic result:

$$(2) \qquad p_{n+1}-p_n=O((\ln p_n)^\beta) \qquad (\text{as } n\to oo) \text{ for some } \beta\geq 2,$$

in which p_n denotes the nth prime. (Too [8, Theorem 2].)

THEOREM A. *Let $r>1$ and the function $f_r(x)=x^r, x\geq 1$. Then, under the hypothesis* (2), *there exists an $n_0(r)$ such that*

$$(3) \qquad \Delta_{f_r}(n)\geq 1 \qquad \textit{for all } n\geq n_0(r).$$

For polynomial functions

$$(4) \qquad f_k(x)=x^k, \quad x\geq 1, \quad \text{where } k=2,3,\cdots,$$

Hu and Lin [3] obtained the following asymptotic behavior of Δ_{f_k} by an elementary proof.

THEOREM B. *Let f_k be a function defined in* (4). *Then, if the ratio*

$$\Delta_{f_k}(n)/(n^{k-1}/\ln n)$$

Received September 22, 1987.

tends to a limit as $n\to\infty$, the limit must be 1.

In what follows, we are concerned with the larger class of functions (in comparison to (4))

(5) $\qquad f_{\alpha,\beta}(x)=x^{\alpha}\ln^{\beta}x, \qquad x\geq 1,$

where the exponents α and β satisfy either (i) $\alpha=1$, $\beta\geq 1$ or (ii) $\alpha>1$, $\beta\in(-\infty,\infty)$. For the case $\alpha>\frac{12}{5}$, we find the rate of convergence of $\Delta_{f_{\alpha,\beta}}$ (Theorem 1), a more precise estimate than (3); for the rest (in fact, for both cases (i) and (ii)), we obtain a result similar to Theorem B by an elementary proof (Theorem 2).

THEOREM 1. *Let* $\alpha>\frac{12}{5}$ *and* $\beta\in(-\infty,\infty)$, *then*

(6) $\qquad \Delta_{f_{\alpha,\beta}}(n)\sim n^{\alpha-1}\ln^{\beta-1}n$ *as* $n\to\infty$.

THEOREM 2. *Assume* (i) $\alpha=1$, $\beta\geq 1$ *or* (ii)′ $1<\alpha\leq\frac{12}{5}$, $\beta\in(-\infty,\infty)$. *Then, for* $f=f_{\alpha,\beta}$, *if the ratio*

(7) $\qquad \Delta_f(n)/(f'(n)/\ln f(n))$

tends to a limit as $n\to\infty$, *the limit must be* 1.

Finally, for general functions f, we have two interesting results concerning Δ_f (Theorems 3 and 4). The identities (9) and (11) hold for any function $f_{\alpha,\beta}$ defined in (5) since it satisfies all the conditions of Theorems 3 and 4 with $r\geq 1$ and a sufficiently large (see the proof of Theorem 2). Hence Theorem 4 is an extension of Lemma 1 of Hu and Lin [3] who considered the special case $r=1$ and the polynomial functions f_k defined in (4).

THEOREM 3. *Let f be a positive function defined on an interval* $[a,\infty)$ *such that f' is positive and nondecreasing. Assume, in addition, that*

(8) $$\sum_{i=[a]}^{n}\frac{(f'(i+1))^2}{f(i)f(i+1)}=o\,(\ln\ln f(n+1))\ \text{ as } n\to\infty.$$

Then

(9) $$\lim_{n\to\infty}\frac{1}{\ln\ln f(n+1)}\sum_{i=[a]}^{n}\frac{\Delta_f(i)}{f(i)}=1.$$

THEOREM 4. *Let f be a posititve function defined on an interval* $[a,\infty)$ *such that f' is positive and nondecreasing. Assume further that for a real* *number* $r \geq 1$

$$(10) \qquad \sum_{i=[a]}^{n} \frac{(f'(i+1))^2 \ln^r f(i)}{f(i) f(i+1)} = o(\ln^r f(n+1)) \qquad \text{as } n \to \infty.$$

Then

$$(11) \qquad \lim_{n\to\infty} \frac{r}{\ln^r f(n+1)} \sum_{i=[a]}^{n} \frac{\Delta_f(i) \ln^r f(i)}{f(i)} = 1.$$

2. Lemmas

To prove the theorems above, we need the next two lemmas. Lemma 1 is due to Huxley [4]. Lemma 2 is interesting in its own right since it is an extension of the well-known Mertens' first theorem written as

$$(12) \qquad \sum_{p \leq x} \frac{\ln p}{p} = \ln x + O(1) \qquad \text{as } x \to \infty.$$

(see Mertens [6] or Hardy and Wright [2] or Yaglom and Yaglom [9, p. 40]).

Lemma 1. *Let* $\varepsilon > 0$, *then as* $x \to \infty$

$$(13) \qquad \pi(x) - \pi(x-y) \sim \frac{y}{\ln x} \qquad \text{if } y \in [x^{\frac{7}{12}+\varepsilon}, \frac{x}{2}].$$

Lemma 2. *Let* $r \geq 1$ *be a real number. Then*

$$(14) \qquad \sum_{p \leq x} \frac{\ln^r p}{p} = \frac{1}{r} \ln^r x + O(\ln^{r-1} x) \qquad \text{as } x \to \infty.$$

Proof. It suffices to prove that (14) holds for $r > 1$ since (14) is exactly (12) when $r = 1$. Define

$$M_r(x) = \sum_{p \leq x} \frac{\ln^r p}{p}, \quad x > 1,$$

and set

$$(15) \qquad M_1(x) = \ln x + R(x),$$

so that, by (12),

$$(16) \qquad R(x) = O(1) \qquad \text{as } x \to \infty.$$

Then

$$(17) \qquad M_r(x) = \sum_{2 \leq n \leq x} (\ln^{r-1} n)(M_1(n) - M_1(n-1)).$$

Inserting (15) into (17) yields

$$(18) \qquad M_r(x) = \sum_{2 \leq n \leq x} (\ln^{r-1} n) \ln \frac{n}{n-1}$$

$$+\sum_{2\leq n\leq x}(\ln^{r-1}n)(R(n)-R(n-1))$$
$$\equiv S_1(x)+S_2(x).$$

We first estimate $S_2(x)$. Summing by parts,

$$S_2(x)=(\ln^{r-1}[x])R([x])$$
$$+\sum_{2\leq n\leq x-1}R(n)\{\ln^{r-1}n-\ln^{r-1}(n+1)\}.$$

From (16) it follows that as $n\to\infty$

$$R(n)\{\ln^{r-1}n-\ln^{r-1}(n+1)\}=O(\ln^{r-1}n-\ln^{r-1}(n+1)),$$

so that as $x\to\infty$

(19) $$S_2(x)=(\ln^{r-1}[x])R([x])+O(\ln^{r-1}x)$$
$$=O(\ln^{r-1}x).$$

As for $S_1(x)$ in (18), setting $\ln\dfrac{n}{n-1}=\dfrac{1}{n}+E(n)$, we have $E(n)=O(n^{-2})$ as $n\to\infty$, hence

(20) $$S_1(x)=\sum_{2\leq n\leq x}\frac{\ln^{r-1}n}{n}+\sum_{2\leq n\leq x}E(n)\ln^{r-1}n$$
$$=\frac{1}{r}\ln^r x+O(\ln^{r-1}x)\qquad\text{as } x\to\infty.$$

Combining (19) and (20), we have proved that

$$M_r(x)=S_1(x)+S_2(x)$$
$$=\frac{1}{r}\ln^r x+O(\ln^{r-1}x)\qquad\text{as } x\to\infty,$$

which is the desired result.

3. Proofs of theorems

Proof of Theorem 1. For any fixed real numbers $\alpha>\frac{12}{5}$ and β, let us define

$$x_n=f_{\alpha,\beta}(n),\quad y_n=x_{n+1}-x_n,\qquad n\in N.$$

Then, by the mean-value theorem, there exists $\theta_n\in(0,1)$ such that for sufficiently large n

$$y_n=f'_{\alpha,\beta}(n+\theta_n)$$
$$=\alpha(n+\theta_n)^{\alpha-1}\ln^\beta(n+\theta_n)+\beta(n+\theta_n)^{\alpha-1}\ln^{\beta-1}(n+\theta_n)$$
$$=(\alpha\ln(n+\theta_n)+\beta)(n+\theta_n)^{\alpha-1}\ln^{\beta-1}(n+\theta_n)$$
$$\geq\alpha n^{\alpha-1}\ln^{\beta-1}(n+\theta_n)\equiv y_n^*.$$

Choosing $0<\varepsilon<\frac{5}{12}-\frac{1}{\alpha}$, we obtain that for all sufficiently large n

$$x_{n+1}^{\frac{7}{12}+\varepsilon} \leq y_n^* \leq y_n \leq \frac{1}{2}x_{n+1},$$

and hence, by Lemma 1,

$$\Delta_{f_{\alpha,\beta}}(n) \sim \frac{y_n}{\ln x_{n+1}} \qquad \text{as } n\to\infty,$$

or equivalently

$$\Delta_{f_{\alpha,\beta}}(n) \sim \frac{f'_{\alpha,\beta}(n)}{\ln f_{\alpha,\beta}(n)} \sim n^{\alpha-1}\ln^{\beta-1}n \qquad \text{as } n\to\infty.$$

Proof of Theorem 2. To prove this theorem, we may apply either Theorem 3 or Theorem 4. It is seen that $f>0$, $f'>0$ and $f''>0$ on $[a,\infty)$ for some sufficiently large integer a. Also, for this f the LHS of (8) is

$$O\left(\sum_{i=a}^{n} i^{-2}\right)=O(1) \qquad \text{as } n\to\infty.$$

Therefore, the function f satisfies all the conditions of Theorem 3 and hence the identity (9). Namely,

$$(21) \qquad 1=\lim_{n\to\infty}\frac{1}{\ln\ln f(n+1)}\sum_{i=a}^{n}\frac{1}{i\ln i}\cdot\frac{\Delta_f(i)}{i^{\alpha-1}\ln^{\beta-1}i}.$$

Now, suppose that the ratio (7) tends to a limit c, say, as $n\to\infty$, namely,

$$\lim_{n\to\infty}\frac{\Delta_f(n)\ln f(n)}{f'(n)}=c,$$

or equivalently

$$(22) \qquad \lim_{n\to\infty}\frac{\Delta_f(n)}{n^{\alpha-1}\ln^{\beta-1}}=c.$$

Then we want to prove $c=1$. Note that

$$(23) \qquad \sum_{i=a}^{n}\frac{1}{i\ln i} \sim \ln\ln n \sim \ln\ln f(n+1) \qquad \text{as } n\to\infty.$$

Combining (22) and (23) yields

$$(24) \qquad \lim_{n\to\infty}\frac{1}{\ln\ln f(n+1)}\sum_{i=a}^{n}\frac{1}{i\ln i}\frac{\Delta_f(i)}{i^{\alpha-1}\ln^{\beta-1}i}=c.$$

Therefore, $c=1$ by (21) and (24). The proof is complete.

Proof of Theorem 3. For convenience, denote $I_f(i)=(f(i),f(i+1)]$ and assume without loss of generality that a is a positive integer. At first, the monotonicity of f implies that for all $n\geq a$

$$\sum_{i=a}^{n}\sum_{p\in I_f(i)}\frac{1}{f(i+1)}\leq\sum_{i=a}^{n}\sum_{p\in I_f(i)}\frac{1}{p}\leq\sum_{i=a}^{n}\sum_{p\in I_f(i)}\frac{1}{f(i)}$$

and hence

$$\sum_{i=a}^{n}\frac{\Delta_f(i)}{f(i+1)}\leq\sum_{p\in(f(a),f(n+1)]}\frac{1}{p}\leq\sum_{i=a}^{n}\frac{\Delta_f(i)}{f(i)}. \tag{25}$$

Secondly, pay attention to the second summation in (25). From Mertens' second theorem (see, e.g., Yaglom and Yaglom [9, p. 41]) written as

$$\sum_{p\leq x}\frac{1}{p}=\ln\ln x+O(1)\qquad\text{as } x\to\infty$$

it follows that

$$\lim_{n\to\infty}\frac{1}{\ln\ln f(n+1)}\sum_{p\in(f(a),f(n+1)]}\frac{1}{p}=1. \tag{26}$$

Finally, in view of (9), (25) and (26) it remains to prove that the difference between two sides of (25) is $o(\ln\ln f(n+1))$ as $n\to\infty$. As expected, this difference is

$$\begin{aligned}D&=\sum_{i=a}^{n}\frac{\Delta_f(i)}{f(i)f(i+1)}(f(i+1)-f(i))\\&\leq\sum_{i=a}^{n}\frac{(f(i+1)-f(i))^2}{f(i)f(i+1)}\\&\leq\sum_{i=a}^{n}\frac{(f'(i+1))^2}{f(i)f(i+1)}\\&=o(\ln\ln f(n+1))\qquad\text{as } n\to\infty,\end{aligned}$$

in which the last equality follows from the assumption (8). The proof is complete.

Proof of Theorem 4. By the conditions on f, $\lim_{n\to\infty}f(n)=\infty$, so that $f(x)>e^r$ for all $x\geq b$, where b is some sufficiently large integer greater than a. The rest of the proof is similar to that of Theorem 3. From the monotonicity of the function $(\ln^r x)/x$ on (e^r,∞) it follows that for all $n\geq b$

$$\begin{aligned}\sum_{i=b}^{n}\sum_{p\in I_f(i)}\frac{\ln^r f(i+1)}{f(i+1)}&\leq\sum_{i=b}^{n}\sum_{p\in I_f(i)}\frac{\ln^r p}{p}\\&\leq\sum_{i=b}^{n}\sum_{p\in I_f(i)}\frac{\ln^r f(i)}{f(i)},\end{aligned}$$

and hence

$$\sum_{i=b}^{n}\frac{\Delta_f(i)\ln^r f(i+1)}{f(i+1)} \tag{27}$$

$$\leq \sum_{p\in(f(b),f(n+1)]} \frac{\ln^r p}{p} \leq \sum_{i=b}^{n} \frac{\Delta_f(i)\ln^r f(i)}{f(i)}.$$

Now, applying (14) to the second summation in (27), we obtain that

$$\lim_{n\to\infty} \frac{r}{\ln^r f(n+1)} \sum_{p\in(f(b),f(n+1)]} \frac{\ln^r p}{p} = 1. \tag{28}$$

Finally, in view of (11), (27) and (28) it remains to prove that the difference between two sides of (27) is $o(\ln^r f(n+1))$ as $n\to\infty$. As expected, this difference is

$$\begin{aligned} D^* &= \sum_{i=b}^{n} \frac{\Delta_f(i)}{f(i)f(i+1)} \{f(i+1)\ln^r f(i) - f(i)\ln^r f(i+1)\} \\ &\leq \sum_{i=b}^{n} \frac{\Delta_f(i)}{f(i)f(i+1)} (f(i+1) - f(i))\ln^r f(i) \\ &\leq \sum_{i=b}^{n} \frac{(f(i+1)-f(i))^2 \ln^r f(i)}{f(i)f(i+1)} \\ &\leq \sum_{i=b}^{n} \frac{(f'(i+1))^2 \ln^r f(i)}{f(i)f(i+1)} \\ &= o(\ln^r f(n+1)) \qquad \text{as } n\to\infty, \end{aligned}$$

in which the last equality follows from the assumption (10). The proof is complete.

4. Remarks

Applying Lemma 1 directly we can prove that (1) holds for $r\in(\frac{7}{12}, 1]$; but the question, whether (1) holds for $r\in(0, \frac{7}{12}]$, is still open. It will be worth while to mention that if the Riemann hypothesis is true, then (1) holds for $r\in(\frac{1}{2}, \frac{7}{12}]$ and (6) holds for $\alpha>2$ and $\beta\in(-\infty, \infty)$ (see Ingham [5] and Titchmarsh [7, p. 77]).

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