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GENERAL MULTIPLICITY AS A MAPPING

CHAN-BONG PARK AND MI-KYUNG KIM

I. Introduction

A first treatment of multiplicities was given by Chevalley ([3]), who studied geometrically them on local rings. The theory of multiplicities in general Noetherian rings was initiated by Samuel ([6]), who based his theory on a generalization of the Hilbert function. M. Auslander and D. A. Bushsbaum([1]) have taken up and extended an approach first suggested by J. P. Serre. They employed the methods of homology. K. Blackburn studied multiplicity with the other method not Hilbert function or homology methods ([2]), and D. J. Wright who got suggestions in Auslander and Bushsbaum's thesis([7]) investigated K. Blackburn's results in the general viewpoint.

Throughout this note, R is a commutative Noetherian ring with unity, and all modules are finitely generated over R, unitary and E', E, E'', ect. will always denote R-modules. $I_{1,2,\cdots S}$ and $I'_{1,2,\cdots S}$ of R will denote ideals generated by $r_1, r_2, \cdots, r_S$ and $r_1', r_2', \cdots, r_S'$ respectively, and $I_{\hat{i},\hat{j}}$ the ideal generated by the set which r_i, r_j are deleted from the set $\{r_1, r_2, \cdots, r_i, r_j, \cdots, r_S\}$ and $I_{\sigma(1),\cdots,\sigma(S)}$ the ideal generated by $r_{\sigma(1)}, \cdots, r_{\sigma(S)}$ for some permutation σ in symetric group on S letters. Most of others will be taken from Northcott's book([5]).

The purpose of this note is to investigate whether the multiplicity can be defined or not as a mapping from $\mathcal{C}_R$ to $\boldsymbol{Z}_0$, and to study its properties, where $\mathcal{C}_R$ is a Category of all finitely generated R-modules and $\boldsymbol{Z}_0$ is nonnegative integers.

Accordingly, it suggests that the authors' result can be refered to the multiplicity as the mapping.

II. Preliminaries

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DEFINITION 1. $\gamma_1, \gamma_2, \cdots, \gamma_S$ $(S \geq 0)$ of R will be said to form a multipicity system on E if $L_R(E/I_{1,2,\cdots,S}E) < \infty$. When $S=0$ this condition is to be understood as meaning that $L_R(E) < \infty$.

DEFINITION 2. Let $\gamma_1, \gamma_2, \cdots, \gamma_S$ be a multiplicity system on E. The multiplicity of $\gamma_1, \gamma_2, \cdots, \gamma_S$ on E denoted by $e_R^{I_{1,2,\cdots,S}}(E)$ is defined as the followings; when $S=0$, we put $e_R^0(E) = L_R(E)$. Assume that $S \geq 1$ and the multiplicity symbol has been defined for E and multiplicity system with only $S-1$ elements. We define $e_R^{I_{1,2,\cdots,S}}(E)$, to be $e_R^{I_{\hat{1}}}(E/\gamma_1 E) - e_R^{I_{\hat{1}}}(0 :_E \gamma_1)$.

We first list well known properties without proof.

LEMMA 1. *Let* $0 \longrightarrow E_S \longrightarrow E_{S-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow 0$ *be an exact sequence. Then*

$$\sum_{\mu=0}^{S} (-1)^\mu L_R(E_\mu) = 0$$ ([5], *Theorem* 20, *p.* 58).

LEMMA 2. *Let* $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ *be an exact sequence and let* $\gamma \in R$. *Then an exact sequence of the form*

$$0 \longrightarrow 0 :_{E'} \gamma \longrightarrow 0 :_E \gamma \longrightarrow 0 :_{E''} \gamma \longrightarrow E'/\gamma E' \longrightarrow E/\gamma E \longrightarrow E''/\gamma E'' \longrightarrow 0$$

can be constructed ([5], *Lemma* 3, *p.* 301).

LEMMA 3. *Let* $0 \longrightarrow E_p \longrightarrow \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow 0$ *be an exact sequence and suppose that* $\gamma_1, \cdots, \gamma_S$ *is a multiplicity system on each term. Then*

$$\sum_{i=0}^{P} (-1)^i e_R^{I_{1,2,\cdots,S}}(E_i) = 0$$ ([5], *Corollary* 1, *p.* 302).

LEMMA 4. *Let* $\gamma_1, \gamma_2, \cdots, \gamma_S (S \geq 2)$ *be a multiplicity system on E. If* σ *is a permutation of* $\{1, 2, \cdots, S\}$. *Then*

$$e_R^{I_{1,2,\cdots,S}}(E) = e_R^{I_{\sigma(1),\cdots,\sigma(S)}}(E)$$ ([7], *Theorem* 2, *p.* 274).

LEMMA 5. *Let* $\gamma_1, \gamma_2, \cdots, \gamma_S$ *and* $\gamma_1', \gamma_2', \cdots, \gamma_S'$ *be elements of R such that* $I_{1,2,\cdots,S}R = I'_{1,2,\cdots,S}R$. *Then*

$$e_R^{I_{1,2,\cdots,S}}(E) = e_R^{I'_{1,2,\cdots,S}}(E)$$ ([7], *Theorem* 4, *p.* 278)

III. The important properties of the multiplicity and the main theorem

PROPOSITION 1. *Let* $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ *be an exact sequence and suppose that* $\gamma_1, \gamma_2, \cdots, \gamma_S$ *is a multiplicity system on each term. Them*

$$e_R{}^{I_{1,2,\cdots,S}}(E) = e_R{}^{I_{1,2,\cdots,S}}(E') + e_R{}^{I_{1,2,\ldots,S}}(E'').$$

Proof. If $S=0$, then by Lemma 1,

$$L_R(E') - L_R(E) + L_R(E'') = 0$$
$$L_R(E) = L_R(E') + L_R(E'').$$

Since $e_R{}^0(E) = L_R(E)$,

$$e_R{}^0(E) = e_R{}^0(E') + e_R{}^0(E'').$$

Assume that $S \geq 1$ and this theorem is true for S-1.

By Lemma 2, we get an exact sequence

$$0 \longrightarrow 0 : {}_{E'}\gamma_1 \longrightarrow 0 : {}_E\gamma_1 \longrightarrow 0 : {}_{E''}\gamma_1 \longrightarrow E'/\gamma_1 E' \longrightarrow E/\gamma_1 E \longrightarrow E''/\gamma_1 E'' \longrightarrow 0.$$

By Lemma 3,

$$e_R{}^{I_{\hat{1}}}(E''/\gamma_1 E'') - e_R{}^{I_{\hat{1}}}(E/\gamma_1 E) + e_R{}^{I_{\hat{1}}}(E'/\gamma_1 E') - e_R{}^{I_{\hat{1}}}(0 : {}_{E''}\gamma_1) + e_R{}^{I_{\hat{1}}}(0 : {}_E\gamma_1) - e_R{}^{I_{\hat{1}}}(0 : {}_{E'}\gamma_1) = 0.$$

$$e_R{}^{I_{\hat{1}}}(E/\gamma_1 E) - e_R{}^{I_{\hat{1}}}(0 : {}_E\gamma_1) = e_R{}^{I_{\hat{1}}}(E'/\gamma_1 E') - e_R{}^{I_{\hat{1}}}(0 : {}_{E'}\gamma_1) + e_R{}^{I_{\hat{1}}}(E''/\gamma_1 E'') - e_R{}^{I_{\hat{1}}}(0 : {}_{E''}\gamma_1)$$

$$e_R{}^{I_{1,2,\cdots,S}}(E) = e_R{}^{I_{1,2,\cdots,S}}(E') + e_R{}^{I_{1,2,\cdots,S}}(E''),$$

which completes the proof.

PROPOSITION 2. *Let* $\gamma_1, \gamma_2, \cdots, \gamma_S$ *be a multiplicity system on E. Assume that for some particular value of i, we have* $\gamma_i{}^m E = 0$, *where* $m \in \boldsymbol{Z}_+$. *Then*

$$e_R{}^{I_{1,2,\cdots,S}}(E) = 0.$$

Proof. The exchange property of the multiplicity symbol shows that we may assume that $i=1$. Having made this assumption we proceed to prove the proposition by induction on m.

Suppose that $m=1$.

The $\gamma_1 E = 0$ and therefore $E/\gamma_1 E = E$ and $0 : {}_E\gamma_1 = E$.

$$\begin{aligned} e_R{}^{I_{1,2,\cdots,S}}(E) &= e_R{}^{I_{\hat{1}}}(E/\gamma_1 E) - e_R{}^{I_{\hat{1}}}(0 : {}_E\gamma_1) \\ &= e_R{}^{I_{\hat{1}}}(E) - e_R{}^{I_{\hat{1}}}(E) \\ &= 0. \end{aligned}$$

Now assume that $m>1$ and that the desired result has already been established for smaller values of the inductive variable. By proposition 1 and the exact sequence

$$0 \longrightarrow \gamma_1 E \longrightarrow E \longrightarrow E/\gamma_1 E \longrightarrow 0,$$

we see that

$$e_R^{I_{1,2,\cdots,s}}(E)=e_R^{I_{1,2,\cdots,s}}(\gamma_1 E)+e_R^{I_{1,2,\cdots,s}}(E/\gamma_1 E).$$

But $\gamma_1^{m-1}(\gamma_1 E)=0$, $\gamma_1(E/\gamma_1 E)=0$.
Thus $e_R^{I_{1,2,\cdots,s}}(\gamma_1 E)=0$ and $e_R^{I_{1,2,\cdots,s}}(0:{}_E\gamma_1)=0$.
Accordingly, $e_R^{I_{1,2,\cdots,s}}(E)=0$.

LEMMA 6. *Let $K\subseteq L$ be submodules of E and let $\gamma\in R$. Then*

$$E/(K+\gamma E)\cong(E/K)/\gamma(E/K).$$

Proof. Consider the natural mapping $\phi: E\to E/K$. Since $K+\gamma E$ is mapped onto $\gamma(E/K)$, then $E/(K+\gamma E)\cong(E/K)/\gamma(E/K)$.

LEMMA 7. $(L:{}_E\gamma)/K=(L/K):{}_{E/K}\gamma$.

Proof. Consider the natural mapping $\phi: E\longrightarrow E/K$. If $e\in E$ then $\phi(e)\in(L/K):{}_{E/K}\gamma$ if and only if $e\in L:{}_E\gamma$. It follows that

$$(L:{}_E\gamma)/K=(L/K):{}_{E/K}\gamma.$$

PROPOSITION 3. *A useful special case of Lemma 7 is obtained by putting $L=K$. This yields*

$$(K:{}_E\gamma)/K=0:{}_{E/K}\gamma.$$

PROPOSITION 4. *Let $\gamma_1, \gamma_2, \cdots, \gamma_S$ of R be a multiplicity system on E. If γ_{i+1} is not zero divisor on $E/I_{\widehat{i+1},\widehat{i+2},\cdots,\hat{S}}E$, for $0\leq i\leq S-1$, then*

$$e_R^{I_{1,2,\ldots,s}}(E)=L_R(E/I_{1,2,\cdots,S}E).$$

Proof. Put $E_i=E/I_{\widehat{i+1},\widehat{i+2},\cdots,\hat{S}}E$. Then

$$\begin{aligned}0:{}_{E_i}\gamma_{i+1}&=0:{}_{E/I_{\widehat{i+1},\widehat{i+2},\cdots,\hat{S}}E}\gamma_{i+1}\\&=I_{\widehat{i+1},\widehat{i+2},\cdots,\hat{S}}E:{}_E\gamma_{i+1}/I_{\widehat{i+1},\cdots\hat{S}}E \quad \text{(by proposition 3)}\\&=0.\end{aligned}$$

$$\begin{aligned}E_i/\gamma_{i+1}E_i&=(E/I_{\widehat{i+2},\cdots\hat{S}}E)/\gamma_{i+1}(E/I_{\widehat{i+1},\cdots,\hat{S}}E)\\&\cong E/I_{\widehat{i+1},\cdots,\hat{S}}E+\gamma_{i+1}E \quad \text{(by Lemma 6)}\\&=E_{i+1}.\end{aligned}$$

$$\begin{aligned}e_R^{I_{1,2,\ldots,s}}(E)&=e_R^{I_{\hat{1}}}(E/\gamma_1 E)-e_R^{I_{\hat{1}}}(0:{}_E\gamma_1)\\&=e_R^{I_{\hat{1}}}(E_1) \quad (\because E/\gamma_1 E=E_1)\end{aligned}$$

$$
\begin{aligned}
&= e_R^{I_{\hat{1},\hat{2}}}(E_1/\gamma_2 E_1) - e_R^{I_{\hat{1},\hat{2}}}(0 : {}_{E_1}\gamma_2) \\
&= e_R^{I_{\hat{1},\hat{2}}}(E_2) \\
&\ \ \vdots \\
&= e_R^{0}(E_S) \\
&= L_R(E_S) \\
&= L_R(E/I_{1,2,\cdots,S}E).
\end{aligned}
$$

Accordingly, $e_R^{I_{1,2,\cdots,S}}(E) = L_R(E/I_{1,2,\cdots,S}E)$.

LEAMM 8. *$e_R^{I_{1,2,\cdots,S}}(E)$ is a non-negative integer.*

Proof. Using proposition 2 and the exact sequence

$$0 \longrightarrow 0 : {}_E\gamma_1^n \longrightarrow E \longrightarrow E/(0 : {}_E\gamma_1^n) \longrightarrow 0,$$

for some positive integer n, we obtain

$$e_R^{I_{1,2,\cdots,S}}(E) = e_R^{I_{1,2,\cdots,S}}(E/(0 : {}_E\gamma_1^n)).$$

Since the chain

$$0 : {}_E\gamma_1 \subseteq 0 : {}_E\gamma_1^2 \subseteq 0 : {}_E\gamma_1^3 \subseteq \cdots\cdots$$

must terminate, it follows that if n is large enough then γ_1 is not a zero divisor in $E/(0 : {}_E\gamma_1^n) = E^*$ (say). Theretore

$$e_R^{I_{1,2,\cdots,S}}(E) = e_R^{I_{1,2,\cdots,S}}(E^*) = L_R(E^*/\gamma_1 E^*).$$

Hence $e_R^{I_{1,2,\cdots,S}}(E)$ is a non-negative integer.

THEOREM 1. *Let $\mathcal{C}_R$ be a Category of all finitely generated R-modules and Z_0 is non-negative integer. $I_{1,2,\cdots,S}$ is an ideal of R such that $L_R(E/I_{1,2,\cdots,S}E) < \infty$ for $E \in \mathcal{C}_R$.*
A mapping $\varphi : \mathcal{C}_R \rightarrow Z_0$ by $\varphi_R^{I_{1,2,\cdots,S}}(E) = e_R^{I_{1,2,\cdots,S}}(E)$, for $I_{1,2,\cdots,S}$ of $E \in \mathcal{C}_R$, has the following properties.

(1) *Let $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ be an exact sequence, where E', E, $E'' \in \mathcal{C}_R$, then*

$$\varphi_R^{I_{1,2,\cdots,S}}(E) = \varphi_R^{I_{1,2,\cdots,S}}(E') + \varphi_R^{I_{1,2,\cdots,S}}(E'').$$

(2) *If $\gamma_i^m E = 0$ for some $m \in Z_+$, where Z_+ are positive integers, $1 \leq i \leq S$ and $E \in \mathcal{C}_R$, then*

$$\varphi_R^{I_{1,2,\cdots,S}}(E) = 0.$$

(3) *Let γ_{i+1} be not zero divisor on $E/I_{\widehat{i+1},\widehat{i+2},\cdots,\hat{S}}E$, for $0 \leq i \leq S-1$, then*

$$\varphi_R^{I_{1,2,\cdots,S}}(E) = L_R(E/I_{1,2,\cdots,S}E).$$

Proof. The proof of (1) is already discussed by proposition 1, (2) follows by proposition 2 and (3) is clear by proposition 4.

References

1. M. Auslander and D. A. Buchsbaum, *Codimension and Multiplicity*, Ann. of Math. **68**(1958) 625-57.
2. K. Blackburn, *An alternative approach to multiplicity theory*, Proc. London Math. Soc. (3) **14**(1964) 115-36.
3. C. Chevalley, *On the theory of local rings*, Ann. of Math. **44**(1943) 690-708.
4. Nagata, *The theory of multiplicity in general local rings*, International symposium on algebraic number theory (Tokyo and Nikko, 1955), 191-226.
5. Northcott, *Lessons on Rings, Modules and Multiplicities*, Cambridge (1968).
6. Pierre Samuel, *La Notion de Multiplicité en Algébre et en géométrie Algébrique*, Gauthier-villars, Imprimeur-éditeur, (1951).
7. D. J. Wright, *General Multiplicity theory*, Proc. London Math. Soc. 3 (1965), 269-288.

Won-Kwang University
Iri 570-749, Korea