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A DUALITY THEOREM FOR FUZZY LIMIT ALGEBRAS

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1. Introduction

The wellknown Gelfand-Naimark Theorem has been extended in many directions. Among them a spectral duality (a generalized Gelfand-Naimark duality) has been developed by many researchers [1, 2, 3, 7, 9, 14]. In limit spaces Binz [1] obtained a spectral duality: the category of c-embedded limit spaces (containing all completely regular Hausdorff spaces) is dual to the category of all algebras of continuous real-valued functions carrying the canonical function space structure of continuous convergence. He first recognized the utility of cartesian closedness of a category for the topological aspects of the duality theory.

The purpose of this paper is to obtain a spectral duality in "fuzzy topology". For this purpose, we introduce a notion of fuzzy limitierung [13] which is defined in terms of prefilters as an appropriate "fuzzy version" of limitierung [2, 4]. In fuzzy limit spaces there exists a natural function space structure, while it is not the case in fuzzy topological spaces. In fact, the category **FLim** of fuzzy limit spaces and fuzzy continuous maps is a cartesian closed topological category. Utilizing the cartesian closedness of **FLim** we obtain a spectral duality in fuzzy limit spaces: The category of embeddable fuzzy limit spaces is dually equivalent to the category of all fuzzy limit algebras of fuzzy continuous real-valued functions carrying the natural function space structure. For this we will use categorical abstract results. An essential part in the development of our duality is ontoness of the counit of the basic adjunction. We note that the notion of fuzzy limitierung is a good generalization of those of both fuzzy topolgy and limitierung. As a matter of fact, the category **FLim** contains the category **FTop** of fuzzy topological

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spaces and fuzzy continuous maps as a bireflective subcategory and the category **Lim** of limit spaces and continuous maps as a bicoreflective subcategory.

Throughout this paper, we use Lowen's definition for a fuzzy topological space [10]. For general categorical background we refer to Herrlich and Strecker [6] and for cartesian closed topological categories to Herrlich [5].

2. Preliminaries

We recall some basic definitions from [12, 15, 16, 17].

Let X be a set. A *fuzzy set* A in X is characterized by a memership function μ_A from X into $I=[0,1]$. We say that B *includes* A and write $A\subseteq B$ if $\mu_A(x)\leq\mu_B(x)$ for all $x\in X$. A *fuzzy point* p in X is a fuzzy set in X given by $\mu_p(x)=\lambda$ for $x=x_0$ $(0<\lambda<1)$ and $\mu_p(x)=0$ for $x\neq x_0$. We call x_0 the *support* of p and λ the *value* of p. We denote a fuzzy point p in X by (x_0,λ) if it is necessary to indicate the support and the value. We say that $p=(x,\lambda)$ *belongs to* A, denoted by $p\in A$, if $\lambda<\mu_A(x)$.

Let X be a set. A nonempty subset $\mathcal{F}$ of I^X is called a *prefilter* on X if it satisfies the following:

(F1) if $A\in\mathcal{F}$ and $A\subseteq B$, then $B\in\mathcal{F}$,

(F2) for all $A, B\in\mathcal{F}$, we have $A\cap B\in\mathcal{F}$,

(F3) $\underline{0}\notin\mathcal{F}$, where $\underline{0}$ is the constant map with value 0.

Let X be a fuzzy topological space (fts for short). A fuzzy set N in X is called a *neighborhood* of a fuzzy point p in X if there exists an open fuzzy set U such that $p\in U\subseteq N$. For each fuzzy point p in X, the *neighborhood system* $\mathcal{U}(p)$ (the collection of all neighborhoods of p) of p is a prefilter on X. A subset $\mathcal{B}$ of a prefilter $\mathcal{F}$ on a set X is called a *base* for $\mathcal{F}$ if for all $A\in\mathcal{F}$ there is $B\in\mathcal{B}$ such that $B\subseteq A$. For a set X a nonempty subset $\mathcal{B}$ of I^X is a base for a prefilter if and only if (B1) for all $A, B\in\mathcal{B}$, there exists $C\in\mathcal{B}$ such that $C\subseteq A\cap B$, (B2) $0\notin\mathcal{B}$. In this case, we say that the prefilter $\langle\mathcal{B}\rangle=\{A\in I^X : B\subseteq A \text{ for some } B\in\mathcal{B}\}$ is *generated by* $\mathcal{B}$.

We now introduce a notion of fuzzy limitierung from [13].

Let X be a set, $\mathcal{F}(X)=$the collection of all prefilters on X and $\mathfrak{X}=$

the set of all fuzzy points in X. A *fuzzy limitierung* Δ is a map from X into $\mathcal{P}(\mathcal{F}(X))$, the power set of $\mathcal{F}(X)$, subject to the following axioms: for each $p=(x,\lambda)$,

(L0) $\mathcal{F}\in\Delta(p)\Rightarrow\underline{\alpha}\in\mathcal{F}$ for all $\alpha>\lambda$

(L1) $\langle p\rangle=\{A\in I^X : p\in A\}\in\Delta(p)$,

(L2) $\mathcal{F}\in\Delta(p)$ and $\mathcal{F}\subseteq\mathcal{G}\Rightarrow\mathcal{G}\in\Delta(p)$,

(L3) $\mathcal{F},\mathcal{G}\in\Delta(p)\Rightarrow\mathcal{F}\cap\mathcal{G}\in\Delta(p)$.

The pair (X,Δ) is called a *fuzzy limit space* (fls for short). If $\mathcal{F}\in\Delta(p)$, we say that $\mathcal{F}$ *converges* to p and p is a *limit* of $\mathcal{F}$. We sometimes write $\mathcal{F}\rightarrow p$ instead of $\mathcal{F}\in\Delta(p)$.

Let (X,δ) be a fts. For each $p\in X$, let $\Delta_\delta(p)=\{\mathcal{F}\in\mathcal{F}(X) : \mathcal{U}(p)\subseteq\mathcal{F}\}$. Then Δ_δ is a fuzzy limitierung on X. Hence any fuzzy topology can be interpreted as a fuzzy limitierung. On the other hand, every fuzzy limitierung induces a fuzzy topology. A fuzzy set U in a fls (X,Δ) is said to be *open* if, whenever $\mathcal{F}\in\Delta(p)$ and $p\in U$, then $U\in\mathcal{F}$. The collection δ_Δ of all open sets in a fls X forms a fuzzy topology on X. Since a fuzzy set U in a fts X is open iff it is a neighborhood of p for each $p\in U$, it is easy to see that $\delta=\delta_{\Delta_\delta}$.

A map $f:(X,\Delta)\rightarrow(Y,\Delta')$ is said to be *fuzzy continuous* at a fuzzy point p in X if, whenever $\mathcal{F}\in\Delta(p)$, then $f(\mathcal{F})\in\Delta'(f(p))$, where $f(\mathcal{F})=<\{f(A)\in I^Y : A\in\mathcal{F}\}>$. A map $f:X\rightarrow Y$ is said to be *fuzzy continuous* if it is fuzzy continuous at every fuzzy point p in X. Clearly, identity map and composition of two fuzzy continuous maps are fuzzy continuous.

Let **FLim** (**FTop**) denote the category of fuzzy limit (topological) spaces and fuzzy continuous maps, respectively. Define a functor $\hat{L}$: **FTop**→**FLim** by $\hat{L}(X,\delta)=(X,\Delta_\delta)$ and $\hat{L}(f)=f$ and a functor $\hat{R}$: **FLim** →**FTop** by $\hat{R}(X,\Delta)=(X,\delta_\Delta)$ and $\hat{R}(f)=f$. Then $\hat{R}$ is a left adjoint to $\hat{L}$. We note that $\hat{L}$ preserves products. In fact, $\hat{L}$ is a full embedding functor. Hence

THEOREM 1. ***FTop*** *is a bireflective subcategory of* ***FLim***.

Let X be a set, $\{(Y_i,\Delta_i)\}_I$ a family of fls's and for each $i\in I$ let $f_i : X\rightarrow Y_i$ a map. Define a map $\Delta : X\rightarrow\mathcal{P}\mathcal{F}(X))$ as follows: for each $p=(x,\lambda)$, $\mathcal{F}\in\Delta(p)$ if $\underline{\alpha}\in\mathcal{F}$ for all $\alpha>\lambda$ and $f_i(\mathcal{F})\in\Delta_i(f_i(p))$ for each $i\in I$. Then Δ is the initial fuzzy limitierung on X w.r.t. the family $\{f_i\}_I$. Hence (1) the category **FLim** admits initial fuzzy limitierung. Moreover, by definition of fuzzy limitierung, (2) for any set X the class of all fuzzy limitierung on X is a set and (3) on any singleton there exists precisely one fuzzy limitierung. Hence

THEOREM 2. ***FLim** is a topological category.*

Let X and Y be fls's, and let $C(X, Y)$ be the set of all fuzzy continuous maps from X into Y. Let $\mathscr{H}$ be a prefilter on $C(X, Y)$ and $\mathscr{A}$ a prefilter on X. For $H\in\mathscr{H}$ and $A\in\mathscr{A}$, define a fuzzy set $H(A) : Y\to I$ by $\mu_{H(A)}(y)=\sup_{(x,g)\in ev^{-1}(y)}\mu_H(g)\wedge\mu_A(x)$ if $ev^{-1}(y)\neq\phi$ and 0, otherwise, where $ev : X\times C(X, Y)\to Y$ is the evaluation map. Let $\mathscr{H}(\mathscr{A})$ be the prefilter on Y generated by $\{H(A)\in I^Y : H\in\mathscr{H},\ A\in\mathscr{A}\}$. Let $\mathscr{C}(X, Y)$ be the set of all fuzzy points in $C(X, Y)$. Define a map $\Delta : \mathscr{C}(X, Y)\to\mathscr{P}(\mathscr{F}(C(X, Y)))$ as follows: for each fuzzy point (f, λ) in $C(X, Y)$, $\mathscr{H}\in\Delta((f, \lambda))$ if (1) $\underline{\alpha}\in\mathscr{H}$ for all $\alpha>\lambda$ and (2) for any fuzzy point p in X with value λ, $\mathscr{A}\to p$ in $X\Rightarrow\mathscr{H}(\mathscr{A})\to f(p)$ in Y. Then Δ is a fuzzy limitierung on $C(X, Y)$, which is called the *continuous fuzzy limitierung* on $C(X, Y)$. Moreover, the evaluation map $ev : X\times C(X, Y)\to Y$ is fuzzy continuous. Let X, Y and Z be fls's and $f : X\times Z\to Y$ be fuzzy continuous. Then there exists a unique fuzzy continuous map $f^* : Z\to C(X, Y)$ suchthat $ev\circ(id\times f^*)=f$. Hence

THEOREM 3. *The topological category **FLim** is cartesian closed, i.e., for each $X\in$ **FLim**, the endo functor $X\times\_$ has a right adjoint $C(X, \_)$.*

3. Duality theorem

we first recall an intimate relationship between fuzzy topological spaces and topological spaces from [10, 11]. Let $(X, \mathscr{T})$ be a topological space. Then we obtain a natural fuzzy topology $\delta_{\mathscr{T}}=C(X, I_r)$ (=the set of all continuous maps of X into I_r) on X, where I_r is the unit interval with the topology $\{(\alpha, 1] : \alpha\in 1\}\cup\{I\}$. Define a functor F : **Top**→ **FTop** by $F(X, \mathscr{T})=(X, \delta_{\mathscr{T}})$ and $F(f)=f$. Then F is full embedding and has a right adjoint T. ($T(X, \delta)=(X, \mathscr{T}_\delta)$, $\mathscr{T}_\delta$ is the initial topology w.r.t. the family $\{\mu_U : X\to I_r : U\in\delta\}$.) Moreover, F preserves initial sources, in particular, products.

Assume that the field R of reals equipped with a fuzzy limitierung induced by the fuzzy topology $\delta_{\mathscr{U}}$, where $\mathscr{U}$ is the usual topology on

R, i. e., $R=\hat{L}\circ F(R)$.

DEFINITION 1. A fuzzy limit space A is called a *fuzzy algebra* (fla for short) over R if it is an algebra over R whose operations are fuzzy continuous.

DEFINITION 2. Let A and B be fla's. A map $h: A\rightarrow B$ is called a *homomorphism* if it is a fuzzy continuous map preserving all operations. (Note that $h(0_A)=0_B$, $h(1_A)=1_B$.)

Let **FLA** denote the category of fuzzy limit algebras over R and homomorphisms. Since $(R, \mathcal{U})$ is a topological algebra over R, F and $\hat{L}$ preserves products, it is easy to see that the fuzzy limit space R is a fuzzy limit algebra over R. Since **FLim** is a cartesian closed topological categroy, we obtain a *function algebra functor* C : $\mathbf{FLim}^{op}\rightarrow\mathbf{FLA}$, $C(X)=C(X, R)$ with the usual operations and continuous fuzzy limitierung and $C(f)=C(f, R)$, where $C(f, R)(\_)=(\_)\circ f$. For a fla A, let $S(A:R)$ be the subspace of $C(A, R)$ consisting of all homomorphisms. Then we obtain a *spectral space functors* S : $\mathbf{FLA}\rightarrow\mathbf{FLim}^{op}$ by $S(A)=S(A:R)$ and $S(h)=(h:R)$ where $S(h:R)(\_)=(\_)\circ h$. (Cf. [7], [14])

Now, by applying the results in [7] and [14], we obtain the following theorems.

THEOREM 1. *The function algebra functor C is right adjoint to the spectral space functor S, the unit and counit of the adjunction being respectively*

$$\eta_A: A\rightarrow C\circ S(A) \quad (\eta_A(a)(h)=h(a))$$
$$\varepsilon_X: X\rightarrow S\circ C(X) \quad (\varepsilon_X(x)(f)=f(x)).$$

Let E be the class of all epimorphisms, i. e. onto fuzzy continuous maps and M the class of all embeddings. (embedding=1−1, initial map). Then since **FLim** is a topological category, it is an (E, M)-category,

DEFINITION 3. A fls X is called *embeddable* if the counit $\varepsilon_X: X\rightarrow S\circ C(X)$ is an embedding.

THEOREM 2. *The full subcategory* ***Emb*** *of* ***FLim*** *formed by all embeddable spaces is an epi-reflective subcategory of* ***FLim***.

THEOREM 3. *Spaces of the form $S(A)$, $C(X, R)$, $C(C(X, R), C(Y, R))$*

$S(C(X,R):C(Y,R))$ are embeddable.

THEOREM 4. *The following statements are equivalent:*

(1) *X is embeddable,*

(2) *$C(W,X)$ is embeddable for all W,*

(3) *the canonical map of $C(W,X)$ into $S(C(W,R):C(X,R))$ is an embedding for all W,*

(4) *X is a subspace of $C(C(X,R),R)$.*

Let **Fix** ε (resp. **Fix** η) denote the full subcategory of fls's X (resp. fla's A) for which ε_X (resp. η_A) is an isomorphism.

LEMMA. *For every fls X, the counit $\varepsilon_X : X \to S\circ C(X)$ is onto.*

Proof. We first assume that X is a fuzzy topological space. Take any homomorphism $h: C(X)\to R$. We will show that there exists $p\in X$ such that $h(f)=0$, i.e., $f\in\ker(h)$ implies $f(p)=0$.

Suppose for every $x\in X$ there exists $u\in\ker(h)$ such that $u(x)\neq 0$. Take any λ with $0<\lambda<1$. Then for every $x\in X$ there exists $f_x\in\ker(h)$ such that $\underline{0}\leq f_x\leq \underline{1}$ and the characteristic function $1_{f_x^{-1}(1)}$ is a neighborhood of (x,λ) : Since $u(x)\neq 0$ and $u\in\ker(h)$, $u^2(x)\neq 0$, $u^2\geq 0$ and $u^2\in\ker(h)$. Let $\alpha=\frac{1}{2}u^2(x)$. Define a map $w: X\to R$ by $w(t)=\frac{u^2(t)}{\alpha}$ and let $v=(w\wedge\underline{1})\wedge(w\vee\underline{1})^{-1}$. Then w and v are fuzzy continuous. Define a map $f_x=v\cdot w$. Then $f_x\in\ker(h)$ and $\underline{0}\leq f_x\leq\underline{1}$. We note that $f_x(t)=1 \Leftrightarrow u^2(t)\geq\alpha$ and hence $f_x(x)=1$. Moreover, $1_{f_x^{-1}(1)}$ is a neighborhood of (x,λ): Indeed, $(x,\lambda)\in 1_{(\alpha,\infty)}\circ u^2=1_{\{t\in X\,:\,u^2(t)>\alpha\}}\subseteq 1_{f_x^{-1}(1)}$ and $1_{(\alpha,\infty)}\circ u^2$ is open in X, since u^2 is fuzzy continuous and $1_{(\alpha,^{-1}\infty)}$ is open in $R(=FR)$

For $f\in\ker(h)$, let $W_f=\{g\in\ker(h) : g|_{f^{-1}(1)}=1\}$. Then $W_f\neq\phi$. Let $\mathscr{B}=\{\lambda_{w_f} : f\in\ker(h),\ \underline{0}\leq f\leq\underline{1},\ 1_{f^{-1}(1)}$ is a neighborhood of some fuzzy point in $X\}$, where $\lambda_{w_f}=\lambda\cdot 1_{w_f}$. Then $\mathscr{B}$ is a base for a prefilter on $C(X)$, since $W_f\cap W_g=W_{f\vee g}$ and hence $\lambda_{w_f}\cap\lambda_{w_g}=\lambda_{w_{f\vee g}}$. (Note that $f\vee g=\frac{1}{2}(f+g+|f-g|)$ and $|f-g|=\sqrt{(f-g)^2}$. Hence $f\vee g\in\ker(h)$, $\underline{0}\leq f\vee g\leq\underline{1}$, and $1_{(f\vee g)^{-1}(1)}$ is a neighborhood of some fuzzy point in X, since $1_{f^{-1}(1)}\subseteq 1_{(f\vee g)^{-1}(1)}$.)

Let $\mathscr{F}$ be the prefilter on $C(X)$ generated by $\mathscr{B}$, Then $\mathscr{F}\to(\underline{1},\lambda)$ in $C(X,R)$: For each $z\in X$, $\mathscr{F}(\mathscr{U}((z,\lambda)))\to(1,\lambda)$ in R, since for each $U\in\mathscr{U}((1,\lambda))$ in R, $\lambda_{w_{f_z}}(1_{f_z^{-1}(1)})=\lambda_{\{1\}}\subseteq U$. Now, since h is fuzzy continuous, $h(\mathscr{F})\to(h(\underline{1}),\lambda)=(1,\lambda)$ in R. This is a contradiction since

$h(\lambda_{w_f})=\lambda_{h(w_f)}=\lambda_{\{0\}}$ and hence $h(\mathcal{F})\supseteq\mathcal{U}((1,\lambda))$.

Therefore there exists $p\in X$ such that $h(f)=0$ implies $f(p)=0$. In fact, $\varepsilon_X(p)=h$: $h(f)=\alpha\Rightarrow h(f-\alpha\cdot\underline{1})=0\Rightarrow(f-\alpha\cdot\underline{1})(p)=0\Rightarrow f(p)=\alpha\Rightarrow\varepsilon_X(p)(f)=\alpha$. Therefore ε_X is onto.

Now, since **FTop** is a bireflective subcategory of **FLim** and $R\in$**FTop** the counit ε_X is onto for all $X\in$**FLim**, by Lemma 1 in [8].

Hence by the results in [7] and [14] we obtain

THEOREM 5. (1) *The category **Fix** ε coincides with the category S(**FLA**) of spectral spaces and also with the category of embeddable spaces. In particular, S(**FLA**) inherits the nice properties of embeddable spaces mentioned above.*

(2) *The category **Fix** η coincides with the category C(**FLim**^{op}) of function algebras.*

(3) *S(**FLA**) is dual to C(**FLim**^{op}), i.e., S(**FLA**)$^{op}\cong C$(**FLim**^{op}). (This equivalence is called a spectral duality). In particular, embeddable spaces X and Y are homeomorphic in **FLim** iff $C(X)$ and $C(Y)$ are isomorphic in **FLA***

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