J. Korean Math. Soc. 25(1988), No. 1, pp. 149-160

# ON THE RELATIVE EVALUATION SUBGROUPS OF A $CW$-PAIR

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D. H. Gottlieb has defined and studied the subgroups $G_n(X)$ of the homotopy groups $\pi_n(X)$ of a topological space $X$. Recently, $G_n(X)$ is generalized by Woo and Kim [11] as following;

Let $(X, A, x_0)$ be a triplet. Consider the class of continuous maps $F: A\times S^n\to X$ such that $F(a, s_0)=a$, then the map $h: (S^n, s_0)\to(X, x_0)$ defined by $h(s)=F(x_0, s)$ represents an element $[h]\in\pi_n(X, x_0)$. The set of all elements $[h]\in\pi_n(X, x_0)$ obtained in the above manner from some $F$ was denoted by $G_n(X, A, x_0)$.

The exactness of the sequence of homotopy groups combined with relative homotopy groups for a pair $(X, A)$ plays an important role in algebraic topology.

$$\begin{array}{ccccccccc}
\cdots\to & \pi_n(A) & \to & \pi_n(X) & \to & \pi_n(X, A) & \to & \pi_{n-1}(A) & \to\cdots \\
 & \cup & & \cup & & \cup & & \cup & \\
\cdots\to & G_n(A) & \to & G_n(X, A) & \to & ? & \to & G_{n-1}(A) & \to\cdots
\end{array}$$

If we can construct certain subgroups of the relative homotopy groups and find the exactness of the sequence of evaluation subgroups $G_n(A)$, $G_n(X, A)$ combined with subgroups constructed by us, it will not be difficult to calculate evaluation subgroups and solve some problems. Here, we raise a question. Can we construct the subgroups as above? We give affirmative answers.

In this paper, we will construct subgroups $G_n^{Rel}(X, A)$ of the relative homotopy groups $\pi_n(X, A)$ and show the exactness of the sequence of those groups for some topological pair $(X, A)$ and examine the relationship between the evaluation map from the mapping space of $X$ to $X$ and $G_n^{Rel}(X, A)$. We also show that $G_n^{Rel}(X, A)$ is an invariant of homotopy type in the category of pairs homotopically equivalent to $CW$-pairs and study the

Received October 31, 1987.
This research was supported by the research grant of the Ministry of Education 1986-87.

algebraic relations between $G_n(A)$, $G_n(X, A)$ and $G_n^{Rel}(X, A)$ and study $G_n^{Rel}$ of pairs of product spaces and $H$-pairs.

Let $I^n$ be the $n$-cube. The initial $(n-1)$-face of $I^n$ defined by $t_n=0$ will be indentified with $I^{n-1}$. The union of all remaining $(n-1)$-faces of $I^n$ is denoted by $J^{n-1}$. Then we have $\partial I^n=I^{n-1}\cup J^{n-1}$.

Consider a map

$$H:(X\times I^n,\ A\times\partial I^n,\ x_0\times J^{n-1})\to(X, A, x_0)$$

such that $H(x,u)=x$, when $x\in X$ and $u\in J^{n-1}$. Then the map $f$: $(I^n, \partial I^n, J^{n-1})\to(X, A, x_0)$ defined by $f(u)=H(x_0, u)$, where $x_0$ is a base point of $X$, represents an element $\alpha=[f]\in\pi_n(X, A, x_0)$

Definition. The set of all elements $\alpha\in\pi_n(X, A, x_0)$ obtained in the above manner from some $H$ will be denoted by $G_n^{Rel}(X, A, x_0)$.

Thus for $\alpha\in G_n^{Rel}(X, A, x_0)$, there is at least one map $H:(X\times I^n, A\times\partial I^n, x_0\times J^{n-1})\to(X, A, x_0)$ which satisfies the above conditions such that $[H(x_0,\cdot)]=\alpha$. We say that $H$ is a *weakly cyclic homotopy* affiliated to $\alpha$ and $H(x_0,\cdot)$ is the trace of $H$.

Theorem 1. *$G_n^{Rel}(X, A, x_0)$ is a subgroup of $\pi_n(X, A, x_0)$ for $n>1$.*

*proof*. Let $\alpha, \beta\in G_n^{Rel}(X, A, x_0)$. Then there exist their representative maps

$$f, g:(I^n, \partial I^n, J^{n-1})\to(X, A, x_0)$$

which are traces of weakly cyclic homotopies $H$ and $G$ respectively. Define

$$F:(X\times I^n, A\times\partial I^n, x_0\times J^{n-1})\to(X, A, x_0)$$

by
$$F(x, t_1, \ldots, t_n)=\begin{cases}H(x, 2t_1, \ldots, t_n), & 0\leqq t_1\leqq 1/2\\ G(x, 2t_1-1, \ldots, t_n), & 1/2\leqslant t_1\leqslant 1.\end{cases}$$

Then $F$ is the weakly cyclic homotopy with trace $f+g$. So

$$[f+g]=[f]+[g]\in G_n^{Rel}(X, A, x_0)$$

Let $\alpha\in G_n^{Rel}(X, A, x_0)$ and $H$ is a weakly cyclic homotopy affiliated to $\alpha$ such that $H(x_0, u)=f(u)$ which represents $\alpha$.
Define $K:(X\times I^n,\ A\times\partial I^n,\ x_0\times J^{n-1})\to(X, A, x_0)$.
by
$$K(x, t_1, \ldots, t_n)=H(x, (1-t_1), \ldots, t_n)$$
Then $K$ is a weakly cyclic homotopy with trace $f^{-1}$, where $f^{-1}(t_1, \ldots, t_n)=f(1-t_1, \ldots, t_n)$.

Thus $[K(x_0,\cdot)]=[f]^{-1}=\alpha^{-1}\in G_n^{Rel}(X, A, x_0)$.

Let $X^X$ be the mapping space from $X$ to itself with compact open topology and $A(X^X)$ be the subspace of $X^X$ which consists of all maps $f \in X^X$ such that $f(A) \subset A$. Consider the map $w : X^X \to X$ such that $w(f) = f(x_0)$, then $w$ induces a homomorphism

$$w_* : \pi_n(X^X, A(X^X), 1_X) \to \pi_n(X, A, x_0).$$

THEOREM 2. *$w_*(\pi_n(X^X, A(X^X), 1_X) = G_n^{Rel}(X, A, x_0)$ if $(X, A)$ is a $CW$-pair.*

*proof*. Let $\alpha \in \pi_n(X^X, A(X^X), 1_X)$. Then a map $f : (I^n, \partial I^n, J^{n-1}) \to (X^X, A(X^X), 1_X)$ represents $\alpha$. Consider the adjoint $\phi(f) : X \times I^n \to X$ which is defined by $\phi(f)\,(x, u) = f(u)\,(x)$ where $x \in X$ and $u \in I^n$. Then

$$\phi(f)\,(a \times \partial I^n) = f(\partial I^n)\,(a) \in A,$$
$$\phi(f)\,(x, u) = f(u)\,(x) = x \text{ for every } u \in J^{n-1},$$
$$\phi(f)\,(x_0, v) = f(v)\,(x_0) = wf(v).$$

where $a \in A$, $x_0$ is the base point of $X$ and $v \in I^n$.

Thus $\phi(f)$ is a weakly cyclic homotopy with the trace $wf$. Therefore, $w_*[f] = [wf] \in G_n^{Rel}(X, A, x_0)$.

Conversely, let $[g] \in G_n^{Rel}(X, A, x_0)$. Then there is a weakly cyclic homotopy $H$ affiliated to $[g]$.

Consider the adjoint $\phi^{-1}(H) : I^n \to X^X$ which is defined by $\phi^{-1}(H)\,(u)\,(x) = H(x, u)$. Then

$$\phi^{-1}(H)\,(\partial I^n)\,(A) = H(A \times \partial I^n) \subset A,$$
$$\phi^{-1}(H)\,(J^{n-1})\,(x) = H(x \times J^{n-1}) = x$$

since $w\phi^{-1}(H)\,(v) = \phi^{-1}(H)\,(v)\,(x_0) = H(x_0, v) = g(v)$ for any $v \in I^n$, we have $w_*[\phi^{-1}(H)] = [w\phi^{-1}(H)] = [g]$.

Because of Theorem 2, the subgroup $G_n^{Rel}(X, A, x_0)$ is called the *relative evaluation subgroup of* a topological pair $(X, A)$

Let $\sigma$ be a path in $A \subset X$ such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$. Then $\sigma$ induces an isomorphism

$$\sigma_* : \pi_n(X, A, x_1) \simeq \pi_n(X, A, x_0) \text{ given by } \sigma_*[f] = [f'],$$

where $f'$ is homotopic to $f$ by a homotopy $F : I^n \times I \to X$ that sends $\partial I^n \times I$ to $A$ and satisfies $F(u, t) = \sigma(1-t)$ for all $u \in J^{n-1}$ [8].

THEOREM 3. $\sigma_* : G_n^{Rel}(X, A, x_1) \cong G_n^{Rel}(X, A, x_0)$

*proof*. Let $[f]\in G_n^{Rel}(X, A, x_1)$. Then there is a weakly cyclic homotopy

$$H: (X\times I^n,\ A\times\partial I^n,\ x_1\times J^{n-1})\rightarrow(X, A, x_1) \text{ such that}$$

$H(x, u)=x$ for every $u\in J^{n-1}$ and $H(x_1, v)=f(v)$ for every $v\in I^n$. We define $F: I^n\times I\rightarrow X$ by $F(u, t)=H(\sigma(1-t), u)$. Then $F(\partial I^n\times I)\subset A$ because $\sigma(t)\subset A$ for every $t\in I$. Since $F(u, t)=H(\sigma(1-t), u)=\sigma(1-t)$ for all $u\in J^{n-1}$. Thus $F(\cdot, 1): (I^n, \partial I^n, J^{n-1})\rightarrow(X, A, x_0)$. Since $H$ is an affiliated map of $[F(\cdot, 1)]$, $[F(\cdot, 1)]\in G_n^{Rel}(X, A, x_0)$. Thus $\sigma_*[f]=[F(\cdot, 1)]\in G_n^{Rel}(X, A, x_0)$

Similarly, $\sigma_*^{-1}(G_n^{Rel}(X, A, x_0))\subset G_n^{Rel}(X, A, x_1)$, where $\sigma^{-1}$ is the inverse path of $\sigma$.

If we define a subspace $\Omega(X, A, x_0)=\{\alpha\in X^I\,|\,\alpha(0)\in A, \alpha(1)=x_0\}$ of $X^I$ with compact open topology, there is a natural isomorphism

$\theta: \pi_n(X, A, x_0)\rightarrow\pi_{n-1}(\Omega(X, A, x_0), *)$ given by $\theta([g])=[\bar{g}]$ for $n>1$, where $*(t)=x_0$ and $\bar{g}: I^{n-1}\rightarrow X^I$ is determined by $g: (I^n, \partial I^n, J^{n-1})\rightarrow(X, A, x_0)$ such that $\bar{g}(u)(t)=g(u, t)$ [5].

THEOREM 4. *$\theta$ maps $G_n^{Rel}(X, A, x_0)$ into $G_{n-1}(\Omega(X, A, x_0), *)$.*

*proof*. Let $[f]\in G_n^{Rel}(X, A, x_0)$. Then there is a weakly cyclic homotopy

$$H: (X\times I^n,\ A\times\partial I^n,\ x_0\times J^{n-1})\rightarrow(X, A, x_0) \text{ with trace } f.$$

Define $G: X^I\times I^{n-1}\rightarrow X^I$ by $G(p, u)(t)=H(p(t), u, t)$, where $p\in X^I$, $u\in I^{n-1}$ and $t\in I$. Since the map $\bar{G}: X^I\times I^{n-1}\times I\rightarrow X$ defined by $\bar{G}(p, u, t)=G(p, u)(t)$ is continuous, $G$ is continuous.

Now $G(\Omega(X, A, x_0)\times I^{n-1})\subset\Omega(X, A, x_0)$. We also have $G(p, u)(0)=H(p(0), u, 0)\in A$ and $G(p, u)(1)=H(p(1), u, 1)=x_0$, where $p\in\Omega(X, A, x_0)$ and $u\in I^{n-1}$. Since

$$G(p\times\partial I^{n-1})(t)=H(p(t), \partial I^{n-1}\times t)=p(t) \text{ and}$$
$$G(*, u)(t)=H(*(t), u, t)=f(u, t)=\bar{f}(u)(t),$$

$G$ is an affiliated map to $[\bar{f}]$. Thus $\theta[f]=[\bar{f}]\in G_{n-1}(\Omega(X, A, x_0), *)$.

DEFINITION. A $CW$-pair $(X, A)$ is called an *$H$-pair* if $X$ and $A$ are $H$-spaces with multiplication $\mu_X$ and $\mu_A$ respectively and $\mu_X|_{A\times A}$ is homotopic (and therefore may be assume equal) to $\mu_A$ [10].

THEOREM 5. *If $(X, A)$ is an $H$-pair, then we have*

$$G_n^{Rel}(X, A, x_0) = \pi_n(X, A, x_0),$$

*where $x_0$ is the base point of $X$ for $H$-structure.*

*proof.* It is sufficient to show that $\pi_n(X, A, x_0) \subset G_n^{Rel}(X, A, x_0)$. Let $[f] \in \pi_n(X, A, x_0)$, $h : (I^n, I^{n-1}\times 0) \equiv (I^n, J^{n-1})$ be a homeomorphism [5, p81] and $\mu : X\times X \to X$ be the $H$-structure such that $\mu(A\times A) \subset A$. Define

$$H : X\times I^{n-1}\times 0 \cup A\times I^{n-1}\times I \to X$$

by
$$H(x, u, 0) = x \text{ for } (x, u, 0) \in X\times I^{n-1}\times 0,$$
$$H(a, u, t) = \mu(a, (fh)(u, t)) \text{ for } (a, u, t) \in A\times I^{n-1}\times I.$$

Then $H$ is well-defined and continuous. By the absolute homotopy extension property, there is a map $F : X\times I^n \to X$ which is an extension of $H$. Since

$$F(x\times I^{n-1}\times 0) = x, \quad F(A\times\partial I^n) \subset A \text{ and}$$
$$F(x_0, u) = \mu(1\times fh)(x_0, u) = fh(u),$$

$\bar{F} = F\circ(1\times h)$ is a weakly cyclic homotopy affiliated to $[f]$. Thus $[f] \in G_n^{Rel}(X, A, x_0)$.

Let $r : (X, A) \to (Y, B)$ be a continuous map. Then *the right homotopy inverse* of $r$ is a continuous map $j : (Y, B) \to (X, A)$ such that $rj \sim 1_{(Y,B)}$.

THEOREM 6. *Let $r : (X, A) \to (Y, B)$ be a map which has a right homotopy inverse $j$. If $(Y, B)$ is a $CW$-pair and $A$ is a pathwise connected, then $r_*((G_n^{Rel}(X, A, x_0)) \subset G_n^{Rel}(Y, B, r(x_0))$.*

*proof.* Without loss of generality we may assume that $jr(x_0) = x_0$[11]. Let $\alpha \in G_n^{Rel}(X, A, x_0)$. Then there is a weakly cyclic homotopy $F$ affiliated to $\alpha$. Define

$$F' : Y\times I^n \to Y$$

by
$$F'(y, u) = r(F(j(y), u)).$$

Then
$$F'(B\times\partial I^n) = r(F(j(B)\times\partial I^n)) \subset rF(A\times\partial I^n) \subset r(A) \subset B,$$
$$F'(r(x_0)\times J^{n-1}) = r(F(jr(x_0)\times J^{n-1})) = r(x_0),$$
$$F'(y\times J^{n-1}) = r(F(j(y)\times J^{n-1}) = rj(y).$$

Since $rj \sim 1_{(Y,B)}$, there is a homotopy $H : (Y\times I, B\times I) \to (Y, B)$ such that $H(y, 0) = rj(y)$ and $H(y, 1) = y$. Define

$$\phi : B\times\partial I^n\times 0\cup B\times J^{n-1}\times I\to B$$

by
$$\phi(b,u,0)=F'(b,u) \text{ for } (b,u,0)\in B\times\partial I^n\times 0$$
$$\phi(b,v,t)=H(b,t) \text{ for } (b,v,t)\in B\times J^{n-1}\times I$$

Then $\phi$ is well-defined and continuous. By the absolute homotopy extension property, there is an extension $\bar{\phi} : B\times\partial I^n\times I\to B$ of $\phi$. Define

$$\psi : Y\times I^n\times 0\cup Y\times J^{n-1}\times I\cup B\times\partial I^n\times I\to Y$$

by
$$\psi(y,u,0)=F'(y,u) \text{ if } (y,u,0)\in Y\times I^n\times 0,$$
$$\psi(y,v,t)=H(y,t) \text{ if } (y,v,t)\in Y\times J^{n-1}\times I,$$
$$\psi(b,u,t)=\bar{\phi}(b,u,t) \text{ if } (b,u,t)\in B\times\partial I^n\times I.$$

Then $\psi$ is well-defined and continuous. By the absolute homotopy extension property, there is an extension $\bar{\psi} : Y\times I^n\times I\to Y$ of $\psi$. Let $\bar{H} : Y\times I^n\to Y$ by $\bar{H}(y,u)=\bar{\psi}(y,u,1)$. Then

$$\bar{H}(B\times\partial I^n)=\bar{\psi}(B\times\partial I^n\times 1)=\psi(B\times\partial I^n\times 1)\subset B,$$
$$\bar{H}(r(x_0)\times J^{n-1})=r(x_0),$$
$$\bar{H}(y,u)=y, \text{ where } y\in Y \text{ and } u\in J^{n-1}.$$

Thus $\bar{H}$ is a weakly cyclic homotopy and $[\bar{H}(r(x_0),\cdot)]\in G_n^{Rel}(Y,B,r(x_0))$. Note that $F'(r(x_0),u)=rF(jr(x_0),u)=rg(u)$, where $g$ is the trace of $F$. Let $\sigma$ be the loop at $r(x_0)$ defined by $\sigma(t)=\bar{\psi}(r(x_0),u,1-t)$ where $u\in J^{n-1}$ and $h : I^n\times I\to Y$ be the map given by $h(u,t)=\bar{\psi}(r(x_0),u,1-t)$. Then

$$h(\partial I^n\times I)=\bar{\psi}(r(x_0)\times\partial I^n\times I)\subset B$$
$$h(J^{n-1}\times t)=\bar{\psi}(r(x_0)\times J^{n-1}\times(1-t))=\sigma(1-t).$$

Since $rg(u)=F'(r(x_0),u)=\bar{\psi}(r(x_0),u,0)=h(u,1), rg=h(\cdot,1)$.

Therefore $r_*(\alpha)=r_*[g]=[rg]=[h(\cdot,1)]=\sigma_*[h(\cdot,0)]$
$=\sigma_*[\bar{H}(r(x_0,\cdot)]\in G_n^{\mathrm{Rel}}(Y,B,r(x_0))$.

Let us consider two given homotopic maps $f,g : (X,A)\to(Y,B)$. Choose a point $x_0\in X$ and denote $y_0=f(x_0)$ and $y_1=g(x_0)$. Let $H : (X\times I, A\times I)\to(Y,B)$ be a homotopy between them and $\sigma : I\to B\subset Y$ be the path defined by $\sigma(t)=H(x_0,t)$. Then $\sigma_* f_*=g_*$ [6].

THEOREM 7. *Let* $(X,A)$ *and* $(Y,B)$ *be topological pairs. Let* $(X,A)$ *be a CW-pair and* $B$ *a path connected subspace. If* $j : (X,A)\to(Y,B)$ *has a left homotopy inverse* $r$ *such that* $jr(y_0)=y_0$, *then* $j_*(\alpha)\in G_n^{\mathrm{Rel}}(Y,B,y_0)$ *implies* $\alpha\in G_n^{\mathrm{Rel}}(X,A,r(y_0))$.

*proof*. Since $rj\sim 1_{(X,A)}$, there is a homotopy $H : (X\times I, A\times I)\to(X,A)$ from $rj$ to $1_{(X,A)}$. Let $\sigma : I\to X$ be the closed path given by

$\sigma(t)=H(r(y_0),t)$. Then

$\sigma_*r_*j_*=1 : \pi_n(X, A, r(y_0))\to\pi_n(X, A, r(y_0))$.

If $j_*(\alpha)\in G_n^{\mathrm{Rel}}(Y, B, y_0)$, then $r_*j_*(\alpha)\in G_n^{\mathrm{Rel}}(X, A, r(y_0))$ by Theorem 6. Thus $\alpha=\sigma_*r_*j_*(\alpha)\in G_n^{\mathrm{Rel}}(X, A, r(y_0))$.

If $f : (X, A)\to(Y, B)$ is a homotopy equivalence, then $f$ induces the natural isomorphism $f_* : \pi_n(X, A, x_0)\to\pi_n(Y, B, f(x_0))$ [6].

THEOREM 8. *Let $(X, A)$ and $(Y, B)$ be both the same homotopy type of path-connected $CW$-pair. If $f : (X, A)\to(Y, B)$ is a homotopy equivalence, then $f_*$ carries $G_n^{\mathrm{Rel}}(X, A, x_0)$ isomorphically onto $G_n^{\mathrm{Rel}}(Y, B, f(x_0))$.*

*proof*. Since $(X, A)$ is a $CW$-pair, we can fined a homotopy inverse $g$: $(Y, B)\to(X, A)$ such that $gf(x_0)=x_0$. Thus we have $f_*(G_n^{\mathrm{Rel}}(X, A, x_0))\subset G_n^{\mathrm{Rel}}(Y, B, f(x_0))$. Similarly $g_*(G_n^{\mathrm{Rel}}(Y, B, f(x_0)))\subset G_n^{\mathrm{Rel}}(X, A, x_0)$. Thus $G_n^{\mathrm{Rel}}(Y, B, f(x_0))=f_*g_*(G_n^{\mathrm{Rel}}(Y, B, f(x_0)))\subset f_*(G_n^{\mathrm{Rel}}(X, A, x_0))$. Therefore, $f_*(G_n^{\mathrm{Rel}}(X, A, x_0))=G_n^{\mathrm{Rel}}(Y, B, f(x_0))$.

Recall that boundary operator $\partial : \pi_n(X, A, x_0)\to\pi_{n-1}(A, x_0)$ is a map given by $\partial[f]=[f|_{I^{n-1}}]\in\pi_{n-1}(A, x_0)$. Let $(X, A, x_0)$ be any given triplet. The inclusion maps $i : (A, x_0)\to(X, x_0)$, $j : (X, x_0)\to(X, A, x_0)$ induce homomorphisms $i_*$ and $j_*$ for each $n>0$. Together with the boundary operator $\partial$, they form a long exact sequence;

$$\xrightarrow{j_*}\pi_{n+1}(X, A, x_0)\xrightarrow{\partial}\pi_n(A, x_0)\xrightarrow{i_*}\pi_n(X, x_0)\xrightarrow{j_*}\pi_n(X, A, x_0)\longrightarrow$$
$$\cdots\xrightarrow{j_*}\pi_2(X, A, x_0)\xleftarrow{\partial}\pi_1(A, x_0)\xrightarrow{i_*}\pi_1(X, x_0)$$

which is called *the homotopy exact sequence of a triplet* $(X, A, x_0)$.

Consider the subgroups $G_n(A, x_0)$ $(=G_n(A, A, x_0))$ which is defined by Gottlieb [3], $G_n(X, A, x_0)$ and $G_n^{\mathrm{Rel}}(X, A, x_0)$ of $\pi_n(A, x_0)$, $\pi_n(X, x_0)$ and $\pi_n(X, A, x_0)$ respectively. Then we have the followings.

LEMMA 9. *$\partial$ carries $G_n^{\mathrm{Rel}}(X, A, x_0)$ into $G_{n-1}(A, x_0)$ for $n>1$.*

*proof*. Let $[f]\in G_n^{\mathrm{Rel}}(X, A, x_0)$. Then there is a weakly cyclic homotopy $H : (X\times I^n,\ A\times\partial I^n,\ x_0\times J^{n-1})\to(X, A, x_0)$ with the trace $f$. Consider $F=H|_{A\times I^{n-1}} : A\times I^{n-1}\to A$. Then

$$F(x_0, u)=f|_{I^{n-1}}(u),\ \text{where } u\in I^{n-1},$$
$$F(x, v)=x \text{ for } v\in\partial I^{n-1}.$$

Thus $F$ is an affiliated map to $[f|_{I^{n-1}}]$. Therefore $\partial[f]=[f|_{I^{n-1}}]\in G_{n-1}(A, x_0)$.

LEMMA 10. [11]. $i_*(G_n(A, x_0)) \subset G_n(X, A, x_0)$. *Moveover, if* $(X, A)$ *is a CW-pair and the inclusion* $i: A \to X$ *has a left homotogy inverse, then* $i_*(G_n(A, x_0)) = i_*(\pi_n(A, x_0)) \cap G_n(X, A, x_0)$.

LEMMA 11. *Let* $(X, A)$ *be a CW-pair. Then*

$$j_*(G_n(X, A, x_0)) \subset G_n^{\mathrm{Rel}}(X, A, x_0).$$

*proof*. Let $[g] \in G_n(X, A, x_0)$. Then there is a map

$$H: (A \times I^n,\ x_0 \times \partial I^n) \to (X, x_0)$$

such that $H(a, u) = a$ for every $u \in \partial I^n$, $H(x_0, u) = g(u)$ for every $u \in I^n$. Let $h: (I^n, I^{n-1} \times 0) \equiv (I^n, J^{n-1})$ be a homeomorphism such that $hh = 1_{(I^n, I^{n-1} \times 0)}$ and $h(\partial I^n) \subset \partial I^n$ [5. p81]. Let $H' = H \circ (1 \times h): A \times I^n \to X$.

Then $\quad H'(a, u) = H(a, h(u)) = a$ for every $u \in I^{n-1} \times 0$,

$\quad\quad\quad H'(x_0, v) = H(x_0, h(v)) = g(h(v))$ for every $v \in I^n$.

Define $\quad F: X \times I^{n-1} \times 0 \cup A \times I^{n-1} \times I \to X$

by $\quad F(x, u, 0) = x$ and $F(a, u, s) = H'(a, u, s)$.

Then $F$ is well-defined and continuous. Since the pair $(X \times I^{n-1}, A \times I^{n-1})$ is a $CW$-pair, there is an extension $\bar{F}: X \times I^n \to X$ of $F$ by the absolute homotopy extension property. Let $F' = \bar{F}(1 \times h^{-1}): X \times I^n \to X$.

Then $F'$ is a weakly cyclic homotopy with the trace $g$. Therefore $j_*[g] = [jg] \in G_n^{\mathrm{Rel}}(X, A, x_0)$.

Consequently, if $(X, A)$ is a $CW$-pair, we obtain a sequence

$$\longrightarrow G_{n+1}^{\mathrm{Rel}}(X, A, x_0) \xrightarrow{\partial} G_n(A, x_0) \xrightarrow{i_*} G_n(X, A, x_0) \xrightarrow{j_*} G_n^{\mathrm{Rel}}(X, A, x_0)$$
$$\cdots \longrightarrow G_2^{\mathrm{Rel}}(X, A, x_0) \xrightarrow{\partial} G_1(A, x_0) \xrightarrow{i_*} G_1(X, A, x_0).$$

This sequence will be called the *G-sequence for the pair* $(X, A)$. Is the $G$-sequence exact? It may not be true in general. But the $G$-sequence is exact for some pairs. Let $(X, A)$ be a $CW$-pair. Then the natural map $p: X^X \to X^A$ given by $p(f) = f|A$ is a fiber map [6]. Since $A(X^X) = p^{-1}(A^A)$,

$$p_*: \pi_n(X^X, A(X^X), 1_X) \to \pi_n(X^A, A^A, 1_A)$$

is one-one fashion [6].

THEOREM 12. *Let* $(X, A)$ *be a CW-pair and the inclusion* $i: A \to X$ *has a left homotopy inverse. Then the G-sequence for* $(X, A)$ *is exact.*

*proof*. Consider the homotopy exact sequence for $(X, A)$;

$$\cdots\longrightarrow\pi_{n+1}(X, A, x_0)\xrightarrow{\partial}\pi_n(A, x_0)\xrightarrow{i_*}\pi_n(X, x_0)\xrightarrow{j_*}\cdots$$
$$\cdots\longrightarrow\pi_2(X, A, x_0)\xrightarrow{\partial}\pi_1(A, x_0)\xrightarrow{i_*}\pi_1(X, x_0).$$

Since $i$ has a left homotopy inverse, $i_*$'s are monomorphisms and $\partial$'s are 0-homomorphisms.

Thus in the $G$-sequence for $(X, A)$;

$$\longrightarrow G_{n+1}^{\mathrm{Rel}}(X, A, x_0)\xrightarrow{\partial}G_n(A, x_0)\xrightarrow{i_*}G_n(X, A, x_0)\xrightarrow{j_*}$$
$$G_n^{\mathrm{Rel}}(X, A, x_0)\longrightarrow\cdots,$$

we have $\ker\, i_*=\operatorname{Im}\,\partial$. Consequently, the $G$-sequence is exact at $G_n(A, x_0)$ for $n\geqslant 1$. By Lemma 10,

$$i_*(G_n(A, x_0))=i_*(\pi_n(A, x_0))\cap G_n(X, A, x_0)=\ker\, j_*\cap G_n(X, A, x_0)$$
$$=\ker\,(j_*|G_n(X, A, x_0)).$$

So the $G$-sequence is exact at $G_n(X, A, x_0)$ for $n\geqslant 2$.

We will show that the $G$-sequence is exact at $G_n^{\mathrm{Rel}}(X, A, x_0)$ for $n\geqslant 2$. Since $\partial$ is 0-homomorphism, it is sufficient to show that $j_*(G_n(X, A, x_0))=G_n^{\mathrm{Rel}}(X, A, x_0)$ for $n\geqslant 1$.

The diagram

$$\begin{array}{ccc}(X^X, A(X^X), 1_X) & \xrightarrow{p} & (X^A, A^A, 1_A)\\ & {}_{w}\searrow \quad \swarrow_{w} & \\ & (X, A, x_0) & \end{array}$$

is commutative, where $w$ is the evaluation map.

So, $G_n^{\mathrm{Rel}}(X, A, x_0)=w_*(\pi_n(X^X, A(X^X), 1_X))=w_*p_*(\pi_n(X^X, A(X^X), 1_X))$
$$=w_*(\pi_n(X^A, A^A, 1_A)).$$

Since the inclusion $i: A\to X$ has a left homotopy inverse, the inclusion $\bar{i}: A^A\to X^A$ given by $\bar{i}(f)=f$ has a left homotopy inverse. In fact, if $H: A\times I\to A$ is a homotopy between $1_A$ and $r\cdot i$, where $r$ is a left homotopy inverse of $i$, then the map $\bar{H}: A^A\times I\to A^A$ given by $\bar{H}(f, t)(a)=H(f(a), t)$ is a homotopy between $1_{A^A}$ and $\bar{r}\circ\bar{i}$, where $\bar{r}: X^A\to A^A$ is the map given by $\bar{r}(f)=r\circ f$.

Thus in the following exact sequence;

$$\longrightarrow\pi_n(A^A, 1_A)\xrightarrow{i_*}\pi_n(X^A, 1_A)\xrightarrow{j_*}\pi_n(X^A, A^A, 1_A)\xrightarrow{\partial}\pi_{n-1}(A^A, 1_A)\longrightarrow$$

for $n>1$, $i_*$ is a monomorphism and $\partial$ is an 0-homomorphism.

Therefore $j_*$ is an epimorphism. The diagram

$$\begin{array}{ccc} \pi_n(X^A, 1_A) & \xrightarrow{j_*} & \pi_n(X^A, A^A, 1_A) \\ \downarrow w_* & & w_* \downarrow \\ G_n(X, A, x_0) & \xrightarrow{j_*} & G_n^{\text{Rel}}(X, A, x_0) \end{array}$$

is commutative. So we have

$$\begin{aligned} G_n^{\text{Rel}}(X, A, x_0) &= w_*(\pi_n(X^A, A^A, 1_A)) = w_* j_*(\pi_n(X^A, 1_A)) \\ &= j_* w_*(\pi_n(X^A, 1_A)) = j_*(G_n(X, A, x_0)). \end{aligned}$$

COROLLARY 13. *If* $(X, A)$ *is a CW-pair and the inclusion* $i : A \to X$ *has a left homotopy inverse* $r$, *then*

$$G_n(X, A, x_0) = G_n(A, x_0) \oplus G_n^{\text{Rel}}(X, A, x_0) \quad \textit{for } n > 1.$$

*proof*. It follows from the split short exact sequence;

$$0 \to G_n(A, x_0) \underset{r_*}{\overset{i_*}{\rightleftarrows}} G_n(X, A, x_0) \xrightarrow{j_*} G_n^{\text{Rel}}(X, A, x_0) \to 0.$$

Let $(X, A)$ and $(Y, B)$ be topological pairs. Then the homomorphism

$$h : \pi_n(X \times Y, A \times B, x_0 \times y_0) \to \pi_n(X, A, x_0) \times \pi_n(Y, B, y_0),$$

for $n > 1$, given by $h(\alpha) = (p_{1*}(\alpha), p_{2*}(\alpha))$ is an isomorphism, where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are the natural projections. The inverse homomorphism of $h$ is $h^{-1}(\alpha, \beta) = i_{1*}(\alpha) + i_{2*}(\beta)$, where $i_1 : X \to X \times Y$ is defined by $i_1(x) = (x, y_0)$ and $i_2 : Y \to X \times Y$ is given by $i_2(y) = (x_0, y)$.

THEOREM 14. *If* $(X, A)$ *and* $(Y, B)$ *are CW-pairs such that* $A$ *and* $B$ *are pathwise connected, then we have*

$$G_n^{\text{Rel}}(X \times Y, A \times B, x_0 \times y_0) = G_n^{\text{Rel}}(X, A, x_0) \oplus G_n^{\text{Rel}}(Y, B, y_0)$$

*for* $n > 1$.

*proof*. We first show that $i_{1*}(G_n^{\text{Rel}}(X, A, x_0)) \subset G_n^{\text{Rel}}(X \times Y, A \times B, x_0 \times y_0)$. Let $\alpha \in G_n^{\text{Rel}}(X, A, x_0)$. Then there is a map

$$H : (X \times I^n, A \times \partial I^n, x_0 \times J^{n-1}) \to (X, A, x_0)$$

such that $H(x, u) = x$ for $x \in X$, $u \in J^{n-1}$ and $[H(x_0, \cdot)] = \alpha$. Define

$$F : (X \times Y \times I^n, A \times B \times \partial I^n, x_0 \times y_0 \times J^{n-1}) \to$$

$(X \times Y, A \times B, x_0 \times y_0)$ given by

$$F(x, y, u) = (H(x, u), y) \quad \text{for } x \in X, \ y \in Y \text{ and } u \in I^n.$$

Then $F$ is a weakly cyclic homotopy with the trace $i_1 f$.

Therefore

$$i_{1*}[f]=i_{1*}\alpha\in G_n^{\mathrm{Rel}}(X\times Y,\ A\times B,\ x_0\times y_0).$$

Similarly, $i_{2*}(G_n^{\mathrm{Rel}}(Y, B, y_0))\subset G_n^{\mathrm{Rel}}(X\times Y, A\times B, x_0\times y_0)$. Since $p_1i_1 =1_X$, $p_2i_2=1_Y$ and by Theorem 6,

$$p_{1*}(G_n^{\mathrm{Rel}}(X\times Y, A\times B, x_0\times y_0))\subset G_n^{\mathrm{Rel}}(X, A, x_0),$$
$$P_{2*}(G_n^{\mathrm{Rel}}(X\times Y, A\times B, x_0\times y_0))\subset G_n^{\mathrm{Rel}}(Y, B, y_0)$$

and $i_{1*}, i_{2*}$ are monomorphisms and $p_{1*}, p_{2*}$ are epimorphisms. Thus we have the following split short exact sequence;

$$0\to G_n^{\mathrm{Rel}}(X, A, x_0)\underset{p_{1*}}{\overset{i_*}{\rightleftarrows}}G_n^{\mathrm{Rel}}(X\times Y, A\times B, x_0\times y_0)$$

$$\xrightarrow{p_{2*}}G_n^{\mathrm{Rel}}(Y, B, y_0)\to 0.$$

Since $G_n^{\mathrm{Rel}}$ is an abelian group for $n>1$, we have $G_n^{\mathrm{Rel}}(X\times Y, A\times B, x_0\times y_0)=G_n^{\mathrm{Rel}}(X, A, x_0)\oplus G_n^{\mathrm{Rel}}(Y, B, y_0)$.

COROLLARY 15. $$G_m^{\mathrm{Rel}}(S^n\times S^n, S^n)=G_m^{\mathrm{Rel}}(S^n, S^n)\oplus G_m^{\mathrm{Rel}}(S^n, *)=0\oplus\pi_m(S^n)=\begin{cases}Z \ \mathit{if}\ m=n\\ 0\ \mathit{if}\ 1<m<n.\end{cases}$$

CORROLLARY 16. $$G_n(S^n\times S^n, S^n)=G_n(S^n, S^n)\oplus G_n(S^n, *, *)=G_n(S_n)\oplus\pi_n(S^n, *)=\begin{cases}Z & \mathit{if}\ n=\mathit{even}\\ 2Z\oplus Z & \mathit{if}\ n\neq 1, 3, 7\\ Z\oplus Z & \mathit{if}\ n=1, 3, 7\end{cases}$$

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