J. Korean Math. Soc. 25(1988), No. 1, pp. 133–148

ASYMPTOTIC BEHAVIOR OF A GENERALIZED FOURIER-LAPLACE TRANSFORM

SUNG KI KIM*

We are interested in the behavior for large real numbers ξ and η of the integral

$$\mathcal{F}\mathcal{L}(u)(\xi,\eta)=\int_0^\infty e^{i\xi\phi(x)-\eta\psi(x)}u(x)\,dx$$

where ϕ and ψ are smooth functions, and u belongs to one of function spaces, distribution spaces or ultra-distribution spaces. The Fourier-Laplace transform of u is the special case $\phi(x)=\psi(x)=x$ of the above integral. We will call our integral $\mathcal{F}\mathcal{L}(u)$ *a generalized Fourier-Laplace transform of u.*

1. Generalized Fourier Transform

First consider the integral

$$\mathcal{F}(u)(\xi)=\int_{-\infty}^\infty e^{i\xi\phi(x)}u(x)\,dx$$

for u in the space $\mathcal{S}$ of rapidly decreasing functions. Note that this integral is absolutely convergent. Our transform F has the following properties:

1) $\mathcal{F}(i\phi u)(\xi)=\dfrac{d\mathcal{F}(u)(\xi)}{d\xi}$, $\mathcal{F}(i\phi^p u)=\dfrac{d^p\mathcal{F}(u)(\xi)}{d\xi^p}$.

2) $\mathcal{F}(du/dx)(\xi)=-i\xi\mathcal{F}(\phi' u)(\xi)$.

3) Let D denote the differential operator $Du=\dfrac{1}{i\phi'(x)}\dfrac{du}{dx}$ and let ${}^tD$ denote its transpose ${}^tDu=\dfrac{d}{idx}\left(\dfrac{u}{\phi'(x)}\right)$. Then we have

$$\mathcal{F}({}^tDu)(\xi)=\xi\mathcal{F}(u)(\xi),\quad \mathcal{F}(p({}^tD)u(\xi))=P(\xi)\mathcal{F}(u)(\xi)$$

where P is a polynomial.

Received October 30, 1987.
*This research is supported by the Ministry of Education and the KOSEF research grant, 1986–1987.

We will call our integral $\mathcal{F}$ *a generalized Fourier transform of u.*

THEOREM 1.1. *Assume ϕ belongs to the space O_M of slowly increasing functions at infinity and $|\phi'(x)|>c>0$ for some c. Then our transform $\mathcal{F}$ defines a continuous linear map from $\mathcal{S}$ into $\mathcal{S}$.*

Proof. Properties 1) and 3) above imply that

$$\xi^\alpha D^p \mathcal{F}(u)(\xi) = \int e^{i\xi\phi(x)} {}^tD^\alpha[\phi(x)^p u(x)]dx.$$

Since $\phi \in O_M$ and $|\phi'|>c>0$, ${}^tD^\alpha[\phi(x)^p u(x)]$ belongs to S and hence

$$(1+|x|^2)^k |{}^tD^\alpha[\phi(x)^p u(x)]|$$

is uniformly bounded for all integer $k \geq 0$.

By taking $k=1$,

$$|\xi^\alpha D^p \mathcal{F}(u)(\xi)| \leq \int \frac{1}{(1+|x|^2)} (1+|x|^2) |{}^tD^\alpha[\phi(x)^p u(x)]| dx$$
$$\leq C \sup (1+|x|^2) |{}^tD^\alpha[\phi(x)^p u(x)]|$$

which proves that for every α and for every p, $\xi^\alpha D^p \mathcal{F}(u)(\xi)$ is uniformly bounded in R, hence $u \in \mathcal{S}$. On the other hand, the same inequality shows that if $\{u_j\}$ is a sequence converging to zero in $\mathcal{S}$, then the sequence $\{\mathcal{F}(u_j)\}$ converges tozero in $\mathcal{S}$, therefore $\mathcal{F}$ is a continuous linear map.

We have seen that the generalized Fourier transform $\mathcal{F}$ maps $\mathcal{S}$ into $\mathcal{S}$. This allows us to define, by duality, the generalized Fourier transform of tempered distributions. Let $T \in \mathcal{S}'$. Its generalized Fourier transform $\mathcal{F}$ is the tempered distribution $\mathcal{F}T$ defined by

$$\langle \mathcal{F}(T), u \rangle = \langle T, \mathcal{F}(u) \rangle, \quad u \in \mathcal{S}.$$

Since $\mathcal{F}: \mathcal{S} \to \mathcal{S}$ is continuous, it follows that $\mathcal{F}: \mathcal{S}' \to \mathcal{S}'$ is also continuous in the sense of the strong topology of $\mathcal{S}'$.

THEOREM 1.2. *If $u, v \in \mathcal{S}$, we have*

$$\mathcal{F}(uv) = \frac{1}{\sqrt{2\pi} \int e^{-\phi(y)^2/2} dy} \mathcal{F}(u) * \mathcal{F}(v).$$

Proof. By theorem 1.1 and 1.2 and Fubini's theorem, we get

$$\mathcal{F}(uv)(\xi) = \int e^{i\xi\phi(x)} u(x) v(x) dx$$

$$=\int e^{i\xi\phi(x)}v(x)\left\{\frac{1}{\sqrt{2\pi}\int e^{-\phi(y)^2/2}dy}\int e^{-i\eta\phi(x)}\mathcal{F}(u)(\eta)d\eta\right\}dx$$

$$=\frac{1}{\sqrt{2\pi}\int e^{-\phi(y)^2/2}dy}\int\left\{\int e^{i\phi(x)(\xi-\eta)}v(x)dx\right\}\mathcal{F}(u)(\eta)d\eta$$

$$=\frac{1}{\sqrt{2\pi}\int e^{-\phi(y)^2/2}dy}\int\mathcal{F}(v)(\xi-\eta)\mathcal{F}(u)(\eta)d\eta$$

$$=\frac{1}{\sqrt{2\pi}\int e^{-\phi(y)^2/2}dy}(\mathcal{F}(u)*\mathcal{F}(v))(\xi).$$

2. Localization of $\mathcal{FL}$

There is a well known estimate (cf. [2], [6]): *Suppose* $u\in C_0^\infty(R^+)$ *and* $\phi'(x)\neq 0$ *on the support of* u. *Then* $\mathcal{F}(u)(\xi)=O(\xi^{-N})$ *as* $\xi\to\infty$ *for every* $N>0$.

Thus the asymptotic behavior of $\mathcal{F}(u)(\xi)$ is determined by these points where $\phi'(x)=0$. The following is a basic extension of the above result to the generalized Fourier-Laplace transform. Also we can see that the asymptotic behavior of $\mathcal{FL}(u)(\xi,\eta)$ is determined by these points where $\phi'(x)=0$ or $\psi'(x)=0$. In this section we assume $\psi(x)$ is a nonnegative smooth function.

THEOREM 2.1. *Let* $u\in C_0^\infty(R^+)$. *Suppose that* ϕ' *and* ψ' *are away from zero on the support of* u. *Then* $|\mathcal{FL}(u)(\xi,\eta)|\leq C|\xi+i\eta|^{-N}$ *for every* $N>0$.

Proof. Let D denote the differential operator $D=\frac{1}{i\xi\phi'(x)-\eta\psi'(x)}\frac{d}{dx}$, and let ${}^tD$ denote its transpose ${}^tDu$ $=\frac{d}{dx}\left(\frac{1}{i\xi\phi'(x)-\eta\psi'(x)}\right)$. Then clearly $D^N(e^{i\xi\phi(x)-\eta\psi(x)})=e^{i\xi\phi(x)-\eta\psi(x)}$ for every $N>0$, and integration by parts shows that

$$\mathcal{FL}(u)(\xi)=\int e^{i(\xi\phi(x)+i\eta\psi(x))}u(x)dx$$

$$=\int D^N(e^{i\xi\phi(x)-\eta\psi(x)})u(x)dx$$

$$=(-1)^N\int e^{i\xi\phi(x)-\eta\psi(x)}{}^tD^Nu(x)dx.$$

Since $|\phi'(x)|>c>0$ and $|\psi'(x)|>c>0$ on the support of u, we have $|{}^tD^N u(x)|\leq C'\frac{|f(x)|}{|\xi+i\eta|^N}$ for some $f\in C_0^\infty(R^+)$. Here C' and f depends on N, ϕ, ψ and u. Thus

$$|\mathcal{FL}(u)(\xi,\eta)|\leq C|\xi+i\eta|^{-N}.$$

COROLLARY 2.2. *Under the same assumption as in the above theorem, for* $I(\xi)=\int e^{i\xi\Phi(x)}u(x)dx$, $\Phi(x)=\phi(x)+i\psi(x)$ *we have*

$$|I(\xi)|\leq C|\xi|^{-N}.$$

The above result can be extened in higher dimension $n\geq 2$.

THEOREM 2.3. *Suppose* $\nabla\phi$ *and* $\nabla\psi$ *are away from zero in the support of* $u\in C_0^\infty(R^{+n})$. *Then*

$$|\mathcal{FL}(u)(\xi,\eta)|\leq C|\xi+i\eta|^{-N} \text{ for every } N\geq 0.$$

Proof. For each point in the support of u, take a unit vector $\mathfrak{n}$ and a small ball B centered at that point so that $(\mathfrak{n},\nabla)\phi(x)\geq c>0$ and $(\mathfrak{n},\nabla)\psi(x)\geq c>0$ for $x\in B$. Decompose the integral $\mathcal{FL}(u)(\xi,\eta)=\sum_k\int e^{i\xi\phi(x)-\eta\psi(x)}u_k(x)dx$ as a finite sum where each u_k is C^∞ and has compact support in one of these balls. It suffices to prove the corresponding estimate for each of these integrals. By choosing a coordinate system $x_1, \ldots, x_x$ so that x_1 lies along $\mathfrak{n}$, we have

$$\mathcal{FL}(u)(\xi,\eta)=\int\left(\int e^{i\xi\phi(x_1,\cdots,x_n)-\eta\psi(x_1,\cdots,x_n)}u_k(x_1,\ldots,x_n)dx_1\right)dx_2,\ldots,dx_n.$$

Since $\frac{\partial\phi(x_1,\ldots,x_n)}{\partial x_1}=(\mathfrak{n},\nabla)\phi(x)\geq c>0$ and $\frac{\partial\psi(x_1,\ldots,x_n)}{\partial x_1}=(\mathfrak{n},\nabla)\psi(x)\geq c>0$, the inner integral is equal or less than $c|\xi+i\eta|^{-N}$ by theorem 2.1, and hence our desired conclusion follows.

COROLLARY 2.4. $\left|\frac{\partial^m\mathcal{FL}(u)(\xi,\eta)}{\partial\xi^m}\right|$ *and* $\left|\frac{\partial^m\mathcal{FL}(u)(\xi,\eta)}{\partial\eta^m}\right|$ *is bounded by* $C|\xi+i\eta|^{-N}$ *for every* m *and* N.

If $u\in(R^+)$, then above results hold under the assumption ϕ and ψ are in O_M because $O_M\cdot\mathcal{S}\subset\mathcal{S}$.

REMARK. If $\phi=\psi$ alomost everywhere, $\mathcal{FL}(u)(\xi,\eta)$ define a holomorphic function on the upper half plane and conversely.

REMARK. The asymptotic behavior of the integral $\mathcal{L}(u)(\eta)$ $=\int e^{-\eta\phi(x)}u(x)\,dx$ has the same situation as $\mathcal{F}(u)(\xi)$.

THEOREM 2.5. *Let ϕ and ψ be smooth convex functions on $[0, \infty)$ with $\phi(0)=\phi'(0)=\psi(0)=\psi'(0)=0$. Let $u\in C_0^\infty(R)$. Then*

$$|\mathcal{F}\mathcal{L}(u)(\xi,\eta)|\leq\frac{5}{2}C[\phi^{-1}(1/|\xi|)+\psi^{-1}(1/\eta)] \text{ for } \eta>0$$

where C depends only on the L^1-norm of u'.

Proof. By integration by parts, we have

$$\mathcal{F}\mathcal{L}(u)(\xi,\eta)=\int_0^\infty e^{i\xi\phi(x)-\eta\psi(x)}u(x)\,dx$$
$$=-\int_0^\infty\Big[\int_a^x e^{i\xi\phi(\tau)-\eta\psi(\tau)}d\tau\Big]u'(x)\,dx.$$

Thus it suffices to show that for any positive numbers a and b,

$$\Big|\int_a^b e^{i\xi\phi(x)-\eta\psi(x)}dx\Big|\leq\frac{5}{2}[\phi^{-1}(1/|\xi|)+\psi^{-1}(1/|\eta|)].$$

For any $\varepsilon>0$ such that $\phi'(\varepsilon)$ and $\psi'(\varepsilon)$ are positive,

$$\int_a^b e^{i\xi\phi(x)-\eta\psi(x)}dx=\int_{x<\varepsilon}e^{i\xi\phi(x)-\eta\psi(x)}dx+\int_{x\geqq\varepsilon}e^{i\xi\phi(x)-\eta\psi(x)}dx$$
$$=I_1+I_2.$$

Now clearly $|I_1|<\varepsilon$ for $\eta>0$ and

$$|I_2|=\Big|\int_\varepsilon^b\frac{1}{i\xi\phi'(x)-\eta\psi'(x)}\frac{d}{dx}[e^{i\xi\phi(x)-\eta\psi(x)}]dx\Big|$$
$$=\Big|\Big[\frac{e^{i\xi\phi(x)-\eta\psi(x)}}{i\xi\phi'(x)-\eta\psi'(x)}\Big]_\varepsilon^b-\int_\varepsilon^b\frac{i\xi\phi''(x)-\eta\psi''(x)}{[i\xi\phi'(x)-\eta\psi'(x)]^2}e^{i\xi\phi(x)-\eta\psi(x)}dx\Big|$$
$$\leq\frac{2}{|i\xi\phi'(\varepsilon)-\eta\psi'(\varepsilon)|}+\int_\varepsilon^b\Big|\frac{i\xi\phi''(x)-\eta\psi''(x)}{[i\xi\phi'(x)-\eta\psi'(x)]^2}\Big|dx$$
$$\leq\frac{1}{|\xi|\phi'(\varepsilon)}+\frac{1}{\eta\psi'(\varepsilon)}+\int_\varepsilon^b\frac{\phi''(x)}{|\xi|\phi'(x)^2}dx+\int_\varepsilon^b\frac{\psi''(x)}{\eta\psi(x)^2}dx$$
$$\leq\frac{2}{|\xi|\phi'(\varepsilon)}+\frac{2}{\eta\psi'(\varepsilon)}.$$

Choose ε so that $\varepsilon\phi'(\varepsilon)<|\xi|^{-1}$ and $\varepsilon\psi'(\varepsilon)<\eta^{-1}$. Then we have

$$\Big|\int_a^b e^{i\xi\phi(x)-\eta\psi(x)}dx\Big|\leq\varepsilon+2\Big[\frac{1}{|\xi|\phi'(\varepsilon)}+\frac{1}{\eta\psi'(\varepsilon)}\Big]$$
$$\leq\frac{1}{2}\Big[\frac{1}{|\xi|\phi'(\varepsilon)}+\frac{1}{\eta\psi'(\varepsilon)}\Big]+2\Big[\frac{1}{|\xi|\phi'(\varepsilon)}+\frac{1}{\eta\psi'(\varepsilon)}\Big]$$

$$\leq \frac{5}{2}\left[\frac{1}{|\xi|\phi'(\varepsilon)}+\frac{1}{\eta\psi'(\varepsilon)}\right]$$
$$\leq \frac{5}{2}\left[\sigma^{-1}\left(\frac{1}{|\xi|}\right)+\tau^{-1}\left(\frac{1}{\eta}\right)\right]$$

where $\sigma(x)=x\phi'(x)$ and $\tau(x)=x\psi'(x)$. Since $\sigma(x)\geq\phi(x)$ and $\tau(x)\geq\psi(x)$ for any convex function ϕ and ψ, we have $\sigma^{-1}(x)\leq\phi^{-1}(x)$ and $\tau^{-1}(x)\leq\psi^{-1}(x)$ and thus the theorem is proved.

REMARK. Above result is an extension of

$$|\mathfrak{F}u(\xi)|\leq 4C\phi^{-1}(1/|\xi|)$$

where C is the same constant of our theorem [2].

LEMMA 2.6. *Let ϕ and ψ be convex with $\phi(0)=\phi'(0)=\psi(0)=\psi'(0)=0$. Assume $u(x)$ is in C_0^∞ and $u(x)=1$ in a neighborhood of 0. Then for ξ large,*

$$\int_0^\infty \cos(\xi\phi(x))e^{-\eta\psi(x)}u(x)\,dx$$
$$\geq\left[\frac{1}{2}\phi^{-1}(1/\xi)-2C\left(\frac{1}{\xi\phi'(\phi^{-1}(1/\xi))}+\frac{1}{\eta\psi'(\phi^{-1}(1/\xi))}\right)\right]e^{-\eta\psi(\phi^{-1}(1/\xi))}.$$

Proof. For ξ large,

$$\int_0^\infty \cos(\xi\phi(x))e^{-\eta\psi(x)}u(x)\,dx$$
$$=\int_0^{\phi^{-1}(1/\xi)}\cos(\xi\phi(x))e^{-\eta\psi(x)}dx+\int_{\phi^{-1}(1/\xi)}^\infty \cos(\xi\phi(x))e^{-\eta\psi(x)}u(x)\,dx$$
$$=I_1+I_2.$$

Since $\cos(1)>\frac{1}{2}$,

$$I_1>\frac{1}{2}\phi^{-1}(1/\xi)e^{-\eta\psi(\phi^{-1}(1/\xi))}$$

and

$$|I_2|=\left|\int_{\phi^{-1}(1/\xi)}^\infty\left(\int_{\phi^{-1}(1/\xi)}^x\cos(\xi\phi(t))e^{-\eta\psi(t)}dt\right)u'(x)\,dx\right.$$
$$\leq C\sup_x\left.\int_{\phi^{-1}(1/\xi)}^x\cos(\xi\phi(t))e^{-\eta\psi(t)}dt\right|$$
$$\leq C\sup_x\left|\int_{\phi^{-1}(1/\xi)}^x e^{i\xi\phi(t)-\eta\psi(t)}dt\right|$$

where C is the L^1-norm of u'.

But as in the proof of theorem 2.5, we have

$$\sup_x \left| \int_{\phi^{-1}(1/\xi)}^{x} e^{i\phi(t)-\eta\phi(t)} dt \right| \leq 2\left[\frac{1}{\xi\phi'(\phi^{-1}(1/\xi))}+\frac{1}{\eta\psi'(\phi^{-1}(1/\xi))}\right]e^{-\eta\phi(\phi^{-1}(1/\xi))}.$$

LEMMA 2.7. *If in addition to the hypothesis of lemma 2.6 we assume $\phi''(x)$ is monotone increasing, then*

$$\int_0^\infty \sin(\xi\phi(x))e^{-\eta\phi(x)}u(x)dx \geq \frac{1}{6}[\phi^{-1}(1/\xi)]^{-1}[\xi\phi'(\phi^{-1}(1/\xi))]^{-2}e^{-\eta\phi(\phi^{-1}(1/\xi))}.$$

Proof.

$$\int_0^\infty \sin(\xi\phi(x))e^{-\eta\phi(x)}u(x)dx$$

$$=\int_0^\infty \frac{e^{-\eta\phi(x)}u(x)}{\xi\phi'(x)}\frac{d}{dx}(1-\cos(\xi\phi(x)))dx$$

$$\geq \int_0^{\phi^{-1}(1/\xi)} \frac{e^{-\eta\phi(x)}}{\xi\phi'(x)}\frac{d}{dx}(1-\cos(\xi\phi(x)))dx \quad (\textit{for } \xi \textit{ large})$$

$$\geq \frac{1}{\xi}\int_0^{\phi^{-1}(1/\xi)} \frac{[\phi''(x)+\eta\psi'(x)\phi'(x)]e^{\eta\phi(x)}}{[\phi'(x)e^{\eta\phi(x)}]^2}[1-\cos(\xi\phi(x))]dx$$

$$\geq \frac{\xi}{2}\int_0^{\phi^{-1}(1/\xi)} \frac{[\phi''(x)+\eta\psi'(x)\phi'(x)]\phi(x)^2}{\phi'(x)^2e^{\eta\phi(x)}}dx$$

$$\geq \frac{\xi}{2}\int_0^{\phi^{-1}(1/\xi)} \frac{\left[\frac{\phi'(x)}{x}+\eta\psi'(x)\phi'(x)\right]\phi(x)^2}{\phi'(x)^2e^{\eta\phi(x)}}dx$$

(since $\phi''(x)$ is monotone increasing)

$$\geq \frac{\xi}{2}\frac{1}{\phi'(\phi^{-1}(1/\xi))^2e^{\eta\phi(\phi^{-1}(1/\xi))}} \times \int_0^{\phi^{-1}(1/\xi)}\left[\frac{\phi'(x)}{\phi^{-1}(1/\xi)}+\eta\psi'(x)\phi'(x)\right]\phi(x)^2dx$$

$$=\frac{\xi}{2}\frac{1}{\phi'(\phi^{-1}(1/\xi))^2e^{\eta\phi(\phi^{-1}(1/\xi))}\phi^{-1}(1/\xi)}\int_0^{\phi^{-1}(1/\xi)}\phi'(x)\phi(x)^2dx$$

$$\geq \frac{1}{6}[\phi^{-1}(1/\xi)]^{-1}[\xi\phi'(\phi^{-1}(1/\xi))]^{-2}e^{-\eta\phi(\phi^{-1}(1/\xi))}.$$

By the above two lemmas, we obtain the following

THEOREM 2.8. *Let ϕ and ψ be smooth convex functions on $[0, \infty)$ with $\phi(0)=\phi'(0)=\psi(0)=\psi'(0)=0$. For $u\in C_0^\infty$, we assume $u(x)=1$ in a neighborhood of 0, and assume $\phi''(x)$ is monotone increasing, then*

$$|\Im\mathcal{L}(u)(\xi,\eta)| \geq C[\phi^{-1}(1/\xi)-\frac{1}{\eta\psi'(\phi^{-1}(1/\xi))}]e^{-\eta\phi(\phi^{-1}(1/\xi))},$$

for some $C>0$.

Proof. Apply lemma 2.6 when $[\xi\phi'(\phi^{-1}(1/\xi))]^{-1}$ is less than a small multiple of $\phi^{-1}(1/\xi)$, and apply lemma 2.7 when it is greater than a small multiple of $\phi^{-1}(1/\xi)$.

3. Asymptotic expansion of $\mathcal{FL}(u)$

Now consider an expansion for integrals of the kind

$$\mathcal{FL}(u)(\xi, \eta) = \int e^{i\xi\phi(x) - \eta\psi(x)} u(x)\, dx, \quad u \in C_0^\infty,$$

where ϕ and ψ are real C^∞-functions.

We have seen that the behavior of $\mathcal{FL}(u)(\xi, \eta)$ is determined by those points x_0 where $\phi'(x_0)=0$ or $\psi'(x_0)=0$. When ϕ and ψ have no critical points on the support of u, $\mathcal{FL}(u)(\xi, \eta)$ is rapidly decreasing as $\zeta = \xi + i\eta \to \infty$ (cf. Theorem 2.1). In the proof of theorem 2.1, we can see that if ϕ (or ψ) is allowed to have critical points in the support of u, then $\mathcal{FL}(u)(\xi, \eta)$ is rapidly decreasing as $\eta \to \infty$ (or $\xi \to \infty$).

Now ϕ' and ψ' are allowed to vanish in the support of u, but instead one makes the hypothesis that ϕ'' and ψ'' do not vanish. This implies that ϕ' (and ψ') is a diffeomorphism of an open neighborhood of any critical point of ϕ (and ψ) onto an open neighborhood of 0 and therefore, such a critical point is necessarily isolated. Thus there are only finitely many critical points of ϕ and ψ on the support of u.

We may decompose u into a sum $u_1 + \dots + u_n$ such that $u_j \in C_0^\infty$ and the support of u_j contains only one critical point of ϕ and ψ. We obtain the asymptotic expansions of the integrals corresponding to each u_j and add them up. We are thus reduced to the case where ϕ and ψ has only one and the same critical point on the support of u, which we take to be the origin. Thus, we may assume that the support of u is contained in a neighborhood U of the origin as small as we wish.

Write $\phi(x) = \phi(0) + \frac{\phi''(0)}{2}x^2 + O(x^3)$ and $\psi(x) = \psi(0) + \frac{\psi''(0)}{2}x^2 + O(x^3)$.

Set $\phi(x) = \phi(0) + \frac{\phi''(0)}{2}x^2(1+\varepsilon_\phi(x))$ and $\psi(x) = \psi(0) + \frac{\psi''(0)}{2}x^2(1+\varepsilon_\psi(x))$

where ε_ϕ and ε_ψ are $O(x)$, and hence are less than 1 when x is sufficiently close to 0. Moreover $\phi'(x) \neq 0$, $\psi'(x) \neq 0$ when $x \neq 0$ but x lies in a sufficiently small neighborhood U of 0. Let $p(x) = x(1+\varepsilon_\phi(x))^{1/2}$

and $q(x)=x(1+\varepsilon_{\psi}(x))^{1/2}$. Then the mapping $x\to p$ and $x\to q$a re diffeomorphisms of the neighborhood U of 0 to a neighborhood of 0 and of course $\frac{\phi''(0)}{2}p^2=\phi(x)-\phi(0)$ and $\frac{\psi''(0)}{2}q^2=\psi(x)-\psi(0)$.

Let us now fix U sufficiently small such that we may assume $\varepsilon_{\phi}(x)=\varepsilon_{\psi}(x)$ on U. Then we have

$$\mathcal{FL}(u)(\xi,\eta)=e^{i\xi\phi(0)-\eta\psi(0)}\int e^{-[\psi''(0)\eta-i\phi''(0)\xi]p^2/2}u(x(p))\left|\frac{dx(p)}{dp}\right|dp$$

where we have denoted by $p\to x(p)$ the inverse of the map $x\to p(x)$. Note that $\frac{dx(0)}{dp}=0$ and the Fourier transform of the function $e^{-\alpha p^2/2}$ of $p\in R$ where $\alpha\neq 0$, $\operatorname{Re}\alpha\geq 0$ is $(2\pi/\alpha)^{1/2}e^{-t^2/2\alpha}$ (cf. [5]). In our case $\alpha=\psi'(0)\eta-i\phi''(0)$. By Parseval's formula we have

$$\mathcal{FL}(u)(\xi,\eta)$$

$$=e^{i\xi\phi(0)-\eta\psi(0)}\left(\frac{2\pi}{\psi''(0)\eta-i\phi''(0)\xi}\right)^{1/2}\left(\frac{1}{2\pi}\right)\int e^{-t^2/2(\psi''(0)-i\phi''(0)\xi)}\hat{v}(t)\,dt$$

where $v(p)=u(x(p))\left|\frac{dx(p)}{dp}\right|$

$$=e^{i\xi\phi(0)-\eta\psi(0)}\left(\frac{1}{2\pi[\psi''(0)\eta-i\phi''(0)\xi]}\right)^{1/2}\sum_{j=0}^{\infty}\frac{[2i(\phi''(0)\xi+i\psi''(0)\eta]^{-j}}{j!}$$

$$\times\int t^{2j}\hat{v}(t)\,dt$$

$$=e^{i\xi\phi(0)-\eta\psi(0)}\left(\frac{2\pi}{\psi''(0)-i\phi''(0)\xi}\right)^{1/2}$$

$$\times\sum_{j=0}^{\infty}\frac{[2i(\phi''(0)\xi+i\psi''(0)\eta]^{-j}}{j!}D^{2j}v(p)\,|_{p=0}$$

$$=e^{i\xi\phi(0)-\eta\psi(0)}\left(\frac{2\pi}{\psi''(0)\eta-i\phi''(0)\xi}\right)^{1/2}$$

$$\times\sum_{j=0}^{\infty}\frac{(2j)^j}{j!}D^{2j}u(0)[\phi''(0)\xi+i\psi''(0)\eta]^{-j}.$$

Thus we have proved the following

THEOREM 3.1. *Assume $\phi'(0)=\psi'(0)=0$, $\phi''(0)\neq 0$ and ψ $(0)>0$. Then*

$$\mathcal{FL}(u)(\xi,\eta)=e^{i\xi\phi(0)-\eta\psi(0)}\left(\frac{2\pi}{\psi''(0)\eta-i\phi''(0)\xi}\right)^{1/2}$$

$$\times\sum_{j=0}^{\infty}\frac{(2i)^j}{j!}D^{2j}u(0)[(\phi''(0)\xi+i\psi''(0)\eta])^{-j}.$$

REMARK. (cf. [3]) According to Laplace, the major contribution to the value of the integral $\mathcal{FL}(u)(\xi, \eta)$ arises from the immediate vicinity of those point of supp u at which $\psi(x)$ assumes its largest value. Assuming ψ has the only critical point at 0 with $\psi''(0)<0$, by the Laplace's result, we obtain

$$\mathcal{FL}(u)(\xi, \eta) \sim e^{i\xi\phi(0)-\eta\psi(0)}\left(\frac{\pi}{2\eta\psi''(0)}\right)^{1/2} u(0).$$

REMARK. (cf. [3]) According to Stokes and Kelvin, the major contribu tion to the value of the integral arises from the immediate vicinity of the end points of supp u and from the vicinity of critical points of $\phi(x)$, and in the first approximation the contribution of critical points is more important than the contribution of the end points. Assuming ϕ has the only critical point at 0 with $\phi''(0)>0$, by the Kelvin's result, we obtain

$$\mathcal{FL}(u)(\xi, \eta) \sim \left[\frac{2\pi}{\xi\phi''(0)}\right]^{1/2} e^{i\xi\phi(0)-\eta\psi(0)+i\pi/4} u(0).$$

4. Tempered functions and tempered distributions

We denote by $A(x)$ a function which satisfies the next conditions:

1) $A(X)$ is a C^∞-function on R
2) $A(x)$ is strictly convex
3) $A(x)/\psi(x)$ goes to ∞ when $|x|$ tends to ∞
4) For any integer $k\geq 1$, there exist some constant C_k and integer N_k with which we have

$$|A^{(k)}(x)| \leq C_k(1+|x|^2)^{N_k} \text{ for all } x$$

where $A^{(k)}(x)$ denotes the k-th derivatives of $A(x)$, i.e., $A(x)$ belongs to O_M the space of C^∞ functions slowly increasing at infinity. Put

$$_A\mathcal{S} = \{u \in C^\infty \mid e^{A(x)}u \in \mathcal{S}\},$$

and

$$_A\mathcal{S}' = \{u \in C^\infty \mid e^{A(x)}u \in \mathcal{S}'\}.$$

For a convex $\psi(x)$, the dual function $A_\psi^*(\eta) = (A \circ \psi^{-1})^*(\eta)$ of $A(x)$ with respect to $\psi(x)$ in the sense of Young is defined by $A_\psi^*(-\eta) = \operatorname{Max}_x(-A(x) - \eta\psi(x))$ (cf. [4]). By definition $e^{-A(x)-\eta\psi(x)}$ belongs to $S \subset O_M$. Thus for $u \in {}_A\mathcal{S}$, we have $e^{-\eta\psi(x)}u(x) \in \mathcal{S}$. Therefore we can consider the generalized Fourier-Laplace transform of an element of $_A\mathcal{S}$.

THEOREM 4.1. *Let $u \in {}_A\mathcal{S}$. Suppose ϕ' and ψ' are away from zero on*

$[0, \infty)$. *Then for any integers N and m, there exists a constant* $C_{N,m}$ *such that*

$$|\partial_\xi^m \mathcal{FL}(u)(\xi,\eta)| \leq C_{N,m}|\xi+i\eta|^{-N}\exp A_\phi^*(-\eta)$$

and

$$|\partial_\eta^m \mathcal{FL}(u)(\xi,\eta)| \leq C_{N,m}|\xi+i\eta|^{-N}\exp A_\phi^*(-\eta)$$

Proof. Let D donote the differential operator $D=\frac{1}{i\xi\phi'(x)-\eta\psi'(x)}\frac{d}{dx}$, and let ${}^tD$ denote its transpose ${}^tDu=\frac{d}{dx}\left(\frac{u(x)}{i\xi\phi'(x)-\eta\psi'(x)}\right)$. For any positive integers m and N, we have

$$\begin{aligned}\partial_\xi^m \mathcal{FL}(u)(\xi,\eta) &= \int_0^\infty e^{A(x)}u(x)e^{-A(x)}(i\phi(x))^m e^{i\xi\phi(x)-\eta\psi(x)}dx \\ &= \int_0^\infty e^{A(x)}u(x)e^{-A(x)}(i\phi(x))^m D^N(e^{i\xi\phi(x)-\eta\psi(x)})dx \\ &= (-1)^N\int_0^\infty {}^tD^N[e^{A(x)}u(x)e^{-A(x)}(i\phi(x))^m]e^{i\xi\phi(x)-\eta\psi(x)}dx.\end{aligned}$$

Using the Leibnitz formula, we have

$$|{}^tD^N[e^{A(x)}u(x)(i\phi(x))^m e^{-A(x)}]| \leq C'_{N,m}|\xi+i\eta|^{-N}f(x)e^{-A(x)}$$

for some $f \in \mathcal{S}$.

Thus

$$|\partial_\xi^m \mathcal{FL}(u)(\xi,\eta)| \leq C_{N,m}|\xi+i\eta|^{-N}\exp A_\phi^*(-\eta) \text{ for some } C_{N,m}.$$

Similarly we can prove the other inequality.

Put ${}_AO_M=\{u\,|\,e^{A(x)}u(x)\in O_M\}$. Since any element of O_M is a multiplier of $\mathcal{S}$ and $e^{-A(x)-\eta\psi(x)}$ belongs to $\mathcal{S}$, $e^{-\eta\psi(x)}u(x)$ belongs to $\mathcal{S}$ for every u in ${}_AO_M$. Thus we can consider the generalized Fourier-Laplace transform of an element of ${}_AO_M$.

THEOREM 4.2. *Let* $\phi, \psi \in O_M$ *and let* $u \in {}_AO_M$. *Suppose* ϕ' *and* ψ' *are away from zero on* $[0,\infty)$. *Then for any integers N, m and any positive* ε, *there exists a constant* $C_{N,m,\varepsilon}$ *such that*

$$|\partial_\xi^m \mathcal{FL}(u)(\xi,\eta)| \leq C_{N,m,\varepsilon}|\xi+i\eta|^{-N}\exp A_\phi^*(-\eta+\varepsilon)$$

and

$$|\partial_\eta^m \mathcal{FL}(u)(\xi,\eta)| \leq C_{N,m,\varepsilon}|\xi+i\eta|^{-N}\exp A_\phi^*(-\eta+\varepsilon)$$

Proof. Since there exists a constant $C'_{N,m}$ such that

$$|{}^tD^N(e^{A(x)}u(x)(i\phi(x))^m e^{-A(x)}| \leq C'_{N,m}|\xi+i\eta|^{-N}f(x)e^{-A(x)}$$

for some $f\in O_M$, we have

$$|\partial_\xi^m \mathcal{F}\mathcal{L}(u)(\xi,\eta)|=|(-1)^N\int_0^\infty {}^tD^N[e^{A(x)}u(x)e^{-A(x)}(i\phi(x))^m]e^{i\xi\phi(x)-\eta\phi(x)}dx|$$
$$\le C'_{N,m}|\xi+i\eta|^{-N}\int_0^\infty f(x)e^{-A(x)-\eta\phi(x)}dx$$
$$\le C'_{N,m}|\xi+i\eta|^{-N}\exp A_\phi^*(-\eta+\varepsilon)\int_0^\infty f(x)e^{-\varepsilon\phi(x)}dx$$
$$=C_{N,m,\varepsilon}|\xi+i\eta|^{-N}\exp A_\phi^*(-\eta+\varepsilon)$$

Similarly we can obtain the other inequality.

THEOREM 4.3. *Let $u\in {}_A\mathcal{S}'$. Suppose ϕ and ψ belong to O_M. Then for any positive ε and integer m, there exists an integer N and a constant $C_{\varepsilon,m}$ such that*

$$|\partial_\xi^m \mathcal{F}\mathcal{L}(u)(\xi,\eta)|\le C_{\varepsilon,m}(1+|\xi+i\eta|)^N\exp A_\phi^*(-\eta+\varepsilon)$$

and

$$|\partial_\eta^m \mathcal{F}\mathcal{L}(u)(\xi,\eta)|\le C_{\varepsilon,m}(1+\xi+i\eta|)^N\exp A_\phi^*(-\eta+\varepsilon).$$

Proof. Since $e^{-\eta\phi(x)-A(x)}$ belongs to $\mathcal{S}$, we have

$$\partial_\xi^m \mathcal{F}\mathcal{L}(u)(\xi,\eta)={}_{\mathcal{S}'}\langle e^{A(x)}u(x),\ (i\phi(x))^m e^{-A(x)+i\xi\phi(x)-\eta\phi(x)}\rangle_{\mathcal{S}}.$$

By definition, $e^{A(x)}u(x)$ lies in $\mathcal{S}'$, hence we have

$$e^{A(x)}u(x)=\frac{d^N}{dx^N}[(1+x^2)^k f(x)]$$

for some bounded continuous function f.

Thus we have

$$|\partial_\xi^m \mathcal{F}\mathcal{L}(u)(\xi,\eta)|$$
$$=|{}_{\mathcal{S}'}\langle(1+x^2)^k f(x),\ -\frac{d^N}{dx^N}(i\phi(x))^m e^{-A(x)+i\xi\phi(x)-\eta\phi(x)}\rangle_{\mathcal{S}}|$$
$$=|\int(1+x^2)^k f(x)\frac{d^N}{dx^N}(i\phi(x))^m e^{-A(x)+i\xi\phi(x)-\eta\phi(x)}dx|$$
$$\le C'_{\varepsilon,m}(1+|\xi+i\eta|)^N\int(1+x^2)^k f(x)h(x)e^{-A(x)-\eta\phi(x)+\varepsilon\phi(x)}e^{-\varepsilon\phi(x)}dx$$

(for some $h(x)\in O_M$)

$$\le C_{\varepsilon,m}(1+|\xi+i\eta|)^N\exp A_\phi^*(-\eta+\varepsilon)$$

The other inequality follows by the similar method.

5. ω-tempered functions and ω-tempered distributions

Let $\omega(x)=\omega(|x|)$ be an increasing function on $[0,\infty)$ with the following properties:

(α) $0=\omega(0)\le\omega(x+y)\le\omega(x)+\omega(y)$

(β) $\int\frac{\omega(x)}{(1+|x|)^2}dx<\infty$

(γ) $\omega(x)\ge a+b\log(1+|x|)$ for some real a and positive b.

Let $\mathcal{D}_{\omega,\phi}$ [cf. 1] be the space of all u in L^1 such that u has compact support with the seminorm

$$\|u\|_\lambda=\int|\mathcal{F}(u)(\xi)|e^{\lambda\omega(\xi)}d\xi<\infty$$

for all $\lambda>0$.

Theorem 5. 1. *Under pointwise multiplication, $\mathcal{D}_{\omega,\phi}$ is an algebra, and for each $\lambda>0$ we have*

$$\|uv\|_\lambda\le\frac{1}{\sqrt{2\pi}\int e^{-\frac{\phi(y)^2}{2}}dy}\|u\|_\lambda\|v\|_\lambda,\quad u,v\in\mathcal{D}_{\omega,\phi}.$$

Proof. By theorem 1. 3, $F(uv)=\frac{1}{c}\,F(u)*F(v)$ where

$c=\sqrt{2\pi}\int e^{-\frac{\phi(y)^2}{2}}dy$. Thus we have

$$\begin{aligned}\|uv\|_\lambda&=\int|\mathcal{F}(uv)(\xi)|e^{\lambda\omega(\xi)}d\xi\\&=\frac{1}{c}\int e^{\lambda\omega(\xi)}\left|\int\mathcal{F}(u)(\xi-\eta)\mathcal{F}(v)(\eta)d\eta\right|d\xi\\&\le\frac{1}{c}\int e^{\lambda\omega(\eta)}\left\{\int e^{\lambda\omega(\xi-\eta)}|\mathcal{F}(u)(\xi-\eta)|d\xi\right\}|\mathcal{F}(v)(\eta)|d\eta\\&=\frac{1}{c}\|u\|_\lambda\|v\|_\lambda\end{aligned}$$

Theorem 5. 2. *We have ${}^tD^\alpha u\in\mathcal{D}_{\omega,\phi}$ and the mapping $u\to{}^tD^\alpha u$ is continuous on $\mathcal{D}_{\omega,\phi}$.*

Proof. Since $\mathcal{F}({}^tD^\alpha u)=\xi^\alpha\mathcal{F}(u)(\xi)$, we get

$$\begin{aligned}\|{}^tD^\alpha u\|_\lambda&=\int|\xi^\alpha\mathcal{F}(u)(\xi)|e^{\lambda\omega(\xi)}d\xi\\&\le\int|\mathcal{F}(u)(\xi)e^{(\lambda+\alpha/b)\omega(\xi)}d\xi\\&\le c\|u\|_{\lambda+\alpha/b}\end{aligned}$$

where b is the constant of condition (γ).

REMARK. $\mathcal{F}(u*v)=\mathcal{F}(u)\mathcal{F}(v)$ holds if and only if $\phi(x)=ax$ for some a.

We denote by $\mathcal{S}_{\omega,\phi}$ (cf. [1]) the set of all functions u in L^1 with the property (u and $\mathcal{F}u\in C^\infty$) for each α and each nonnegative number λ we have

$$p_{\alpha,\lambda}(u)=\sup e^{\lambda\omega(x)}|D^\alpha u(x)|<\infty$$

and

$$\pi_{\alpha,\lambda}(u)=\sup e^{\lambda\omega(\xi)}|D^\alpha\mathcal{F}(u)(\xi)|<\infty.$$

The topology of $\mathcal{S}_{\omega,\phi}$ is defined by semi-norms $p_{\alpha,\lambda}$ and $\pi_{\alpha,\lambda}$. The symmetry of the definition of $\mathcal{S}_{\omega,\phi}$ implies $\mathcal{F}$ is a continuous automorphism of $\mathcal{S}_{\omega,\phi}$ and applying (r) we get $\mathcal{S}_{\omega,\phi}\subset\mathcal{S}$ (if $\phi(x)=x$, then $\mathcal{S}_{\omega,\phi}=\mathcal{S}_\omega$, and if $\phi(x)=x, \omega(x)=\log(1+|x|)$, then $\mathcal{S}_{\omega,\phi}=\mathcal{S}$). Let $\mathcal{S}'_{\omega,\phi}$ be the topological dual of $\mathcal{S}_{\omega,\phi}$.

THEOREM 5.3. *$\mathcal{S}_{\omega,\phi}$ is a topological algebra under pointwise multiplication.*

Proof. Let $u, v\in\mathcal{S}_{\omega,\phi}$. The Leibniz' formula proves that $(p_{\alpha,\lambda}(uv)<\infty$. Also

$$D^\alpha\mathcal{F}(uv)(\xi)=\frac{1}{c}D^\alpha\mathcal{F}(u)*\mathcal{F}(v) \text{ where } c=\sqrt{2\pi}\int e^{-\frac{\phi(y)^2}{2}}dy.$$

Thus

$$\pi_{\alpha,\lambda}(uv)\leq\frac{1}{c}\sup\int|D^\alpha u(\xi-\eta)|e^{\lambda\omega(\xi-\eta)}|\mathcal{F}(v)(\eta)e^{\lambda\omega(\eta)}d\eta$$
$$\leq\frac{1}{c}\pi_{\alpha,\lambda}(u)\pi_{0,l}(v)\int e^{(\lambda-l)\omega(\eta)}d\eta$$

which is finite by choosing l sufficiently large.

In dealing with the analyticity of the generalized Fourier-Lapalce transform, we must put $\phi(x)=\psi(x)$ a. e..

THEOREM 5.4. *Let K be a compact interval $[-a, a]$. An entire function $F(\xi)$ is the generalized Fourier-Laplace transform of a function $u\in\mathcal{D}_{\omega,\phi}$ if and only if for each $\lambda, \varepsilon\geq 0$ there exists a constant $C_{\lambda,\varepsilon}$ such that*

$$|F(\xi+i\eta)|\leq C_{\lambda,\varepsilon}e^{(a+\varepsilon)|\eta|-\lambda\omega(\xi)}.$$

Proof. By theorem 1.4.1 in [1], if the above inequality holds, $F=\hat{v}$ for some $v\in\mathcal{D}_\omega$ with $\operatorname{supp} v\subset[-a, a]$ where $\wedge$ is the usual

Fourier transform. Now putting $u(x)=v(\phi(x))\phi'(x)$, we have $u\in\mathcal{D}_{\omega,\phi}$ and F is the generalized Fourier–Laplace transform of u. The converse follows from theorem 1.4.1 [1].

If $u\in\mathcal{S}'_{\omega,\phi}$ we define the generalized Fourier transform $\mathcal{F}(u)\in\mathcal{S}'_{\omega,\phi}$ by

$$\mathcal{F}u(h)=u(\mathcal{F}h),\quad h\in\mathcal{S}_{\omega,\phi}.$$

It follows that the generalized Fourier transform is a continuous automorphism of $\mathcal{S}'_{\omega,\phi}$.

Assume that $e^{-A(x)-\eta\phi(x)}$ belongs to $\mathcal{S}_{\omega,\phi}$ for all η. We define spaces ${}_A\mathcal{S}_{\omega,\phi}$ and ${}_A\mathcal{S}'_{\omega,\phi}$ as follows;

$$ {}_A\mathcal{S}_{\omega,\phi}=\{u\in C^\infty\,|\,e^{A(x)}u(x)\in\mathcal{S}_{\omega,\phi}\}$$
$$ {}_A\mathcal{S}'_{\omega,\phi}=\{u\in\mathcal{D}'_{\omega,\phi}\,|\,e^{A(x)}u(x)\in\mathcal{S}'_{\omega,\phi}\}.$$

THEOREM 5.5. *For* $u\in{}_A\mathcal{S}_{\omega,\phi}$, we have

$$|\mathcal{FL}(u)(\xi,\eta)|\leq C\exp A_\phi^*(-\eta)\ \text{for some } C.$$

Proof. Since $\mathcal{S}_{\omega,\phi}$ is an algebra under multiplication $e^{-\eta\phi(x)}u(x)$ belongs to $\mathcal{S}_{\omega,\phi}$ and so we can consider the generalized Fourier–Laplace transform of u.

$$\begin{aligned}|\mathcal{FL}(u)(\xi,\eta)| &= \left|\int e^{-A(x)+i\xi\phi(x)-\eta\phi(x)}e^{A(x)}u(x)\,dx\right|\\ &\leq\int e^{-A(x)-\eta\phi(x)}\,|e^{A(x)}u(x)e^{\lambda\omega(x)}|\,e^{-\lambda\omega(x)}dx\\ &\leq C\exp A_\phi^*(-\eta).\end{aligned}$$

THEOREM 5.6. *For* $u\in{}_A\mathcal{S}'_{\omega,\phi}$, *we have*

$$|\mathcal{FL}(u)(\xi,\eta)|\leq C\exp A_\phi^*(-\eta+\varepsilon)\ \text{for some } \varepsilon>0.$$

Proof.
$$\begin{aligned}|\mathcal{FL}(u)(\xi,\eta)| &= |{}_{\mathcal{S}'\omega,\phi}\langle e^{A(x)}u(x),\, e^{-A(x)+i\xi\phi(x)-\eta\phi(x)}\rangle_{\mathcal{S}\omega,\phi}|\\ &\leq C'P_{0,\lambda}(e^{-A(x)+i\xi\phi(x)-\eta\phi(x)})\\ &\leq C'\sup e^{\lambda\omega(x)-A(x)-\eta\phi(x)}\\ &\leq C\sup e^{-A(x)-(\eta-\varepsilon)\phi(x)}\\ &\leq C\exp A_\phi^*(-\eta+\varepsilon)\ \text{for some } \varepsilon.\end{aligned}$$

References

1. G. Björk, *Linear partial differential operators and generalized distributions*, Ark. Math. **6**(1966), 351–407.
2. J. Bruna, A. Nagel and S. Wainger, *Convexity hypersurfaces and Fourier transforms*, Preprint, 1985.

3. A. Erdélyi, *Asymptotic Expansions*, Dover publication, 1956.
4. K. Hayakawa, *Some results on the Fourier–Laplace image of special classes of distributions*, Osaka J. Math. **9**(1972), 499–512.
5. L. Hörmander, *The analysis of linear partial differential operators I*, Springer–Verlag, 1983.
6. E. M. Stein, *Oscillatory integrals in Fourier Analysis*, Annals of Math. Studies No. 112, Princeton Univ. Press, Princeton, 1986.

Seoul National University
Seoul 151, Korea