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FUBINI PRODUCTS OF LIMINAL C^*-ALGEBRAS WITH HAUSDORFF SPECTRA

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1. Introduction.

Let A and B be C^*-algebras and $A\otimes B$ denote the minimal tensor product of A and B. After Tomiyama [11] introduced the notion of Fubini product $A\otimes_F B$ of A and B, it has been proved to be useful to study some pathological properties of minimal tensor products [1, 4, 8, 12, 13]. Tomiyama [11] also proved that if A is subhomogeneous, that is, every irreducible representation of A is finite dimensional with bounded dimension, then A isa C^*-algebra *with trivial Fubini products*, i.e., $A\otimes_F B=A\otimes B$ for all C^*-algebra B.

By using the techniques of Wassermann [12, 13], Huruya [5] gave an example of C^*-algebra whose irreducible representations are all finite-dimensional but which has a nontrivial Fubini product. So, it is only natural to consider the converse of Tomiyama's result. In this vein, the author [9] showed that the converse is true for AF C^*-algebras. The purpose of this note is to prove the converse of the Tomiyama's result for the class of liminal C^*-algebras with Hausdorff spectra.

The structures of liminal C^*-algebras with Hausdorff spectra are well understood through the theory of continuous fields of C^*-algebras, and we follow Dixmier's book [3] for the related notations and terminologies. Also, $\mathcal{B}(\mathcal{H})$ (respectively $\mathcal{K}(\mathcal{H})$) denotes the C^*-algebra of all bounded linear (respectively compact) operators on the seperable Hilbert space $\mathcal{H}$, throughout this note.

In Section 2, we review definitions and some basic properties of Fubini products. Section 3 is devoted to an extension of Kirchberg's result [7] on exact C^*-algebras, which will be useful for the proof of the main theorem in the last section.

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2. Preliminaries.

To begin with, let us recall the definition of Fubini product. Let A, B, C and D be C^*-algebras with $A\subseteq C$ and $B\subseteq D$. For $\phi\in C^*$ (respectively $\psi\in D^*$), there exists a unique bounded linear map (called the slice map) $R_\phi : C\otimes D\to D$ (respectively $L_\psi : C\otimes D\to C$) such that $R_\phi(c\otimes d)=\phi(c)d$ (respectively $L_\psi(c\otimes d)=\psi(d)c$) for $c\in C$ and $d\in D$. The *Fubini product* $F(A, B, C\otimes D)$ *of* A *and* B *with respect to* $C\otimes D$ is defined by

$$F(A, B, C\otimes D)=\{x\in C\otimes D\ ;\ R_\phi(x)\in B, L_\psi(x)\in A \text{ for all } \phi\in C^*, \psi\in D^*\}.$$

Let A and B be fixed C^*-algebras. Although the Fubini products $F(A, B, C\otimes D)$ of A and B depend on $C\otimes D$, they are all isomorphic and constitute the largest among them if C^*-algebras C and D are injective [5]. We denote by $A\otimes_F B$ any one of these isomorphic Fubini products of A and B.

Let M_n denote the C^*-algebra of all $n\times n$ matrices over complex field, and M (respectively M_0) be the l^∞*-sum* (respectively c_0*-sum*) of the family $\{M_n\}$ of C^*-algebras. That is,

$$M=\{(x_n)\in \textstyle\prod_{n=1}^{\infty} M_n\ ;\ \sup\|x_x\|<\infty\}$$
$$M_0=\{(x_n)\in \textstyle\prod_{n=1}^{\infty} M_n\ ;\ \lim\|x_n\|=0\}.$$

Then M_o is just the example of Huruya [5] mentioned in the introduction:

$$\mathcal{B}(\mathcal{H})\otimes M_o \subsetneq F(\mathcal{B}(\mathcal{H}), M_o, \mathcal{B}(\mathcal{H})\otimes M).$$

The following two lemmas deal with C^*-subalgebras and C^*-quotients for Fubini products in some special cases, which are useful for our main theorem.

LEMMA 2.1. *Let A and C be nuclear C^*-algebras with $A\subseteq C$. If C is a C^*-algebra with trivial Fubini products, then so is A.*

Proof. See [9, Lemma 3.1].

LEMMA 2.2. *Let $\alpha : D\to E$ be a surjective $*$-homomorphism between C^*-algebras and B be a C^*-subalgebra of D. If A is a C^*-algebra then we have*

$$(2.1)\qquad (1_A\otimes\alpha)(F(A, B, A\otimes D))\subseteq F(A, \alpha(B), A\otimes E).$$

Furthermore, if $Ker\,\alpha\subseteq B$ *then the equality holds in* (2.1).

Proof. If $z\in F(A, B, A\otimes D)$ then $R_\phi((1_A\otimes\alpha)(z))=\alpha(R_\phi(z))\in\alpha(B)$ for all $\phi\in A^*$. So, we have $(1_A\otimes\alpha)(z)\in F(A,\alpha(B), A\otimes E)$. For the converse, let $w\in F(A,\alpha(B), A\otimes E)$. Then we can find $z\in A\otimes D$ such that $(1_A\otimes\alpha)(z)=w$. Now,

$$\alpha(R_\phi(z))=R_\phi((1_A\otimes\alpha)(z))=R_\phi(w)\in\alpha(B)$$

for all $\phi\in A^*$. So, $R_\phi(z)\in B+\mathrm{Ker}\,\alpha=B$ for all $\phi\in A^*$, and $z\in F(A, B, A\otimes D)$. Hence, $w=(1_A\otimes\alpha)(z)\in(1_A\otimes\alpha)(F(A, B, A\otimes D))$.

3. Exact C^*-algebras.

Now, we recall that a C^*-algebra A is *C^*-exact* if

$$A\otimes J=F(A, J, A\otimes B)$$

for every C^*-algebra B and its two-sided norm-closed ideal J. Note that the Fubini product $F(A, J, A\otimes B)$ of the right side is just the kernel of the $*$-homomorphism $A\otimes B\to A\otimes(B/J)$ in general. In the literature [2, 7], one can find several conditions which are equivalent to C^*-exactness in terms of Fubini product. Especially, Kirchberg [7, Theorem 1.1] showed that if

$$A\otimes\mathcal{K}(\mathcal{H})=F(A, \mathcal{K}(\mathcal{H}), A\otimes\mathcal{B}(\mathcal{H}))$$

then A is C^*-exact.

Let E_n be a fixed C^*-algebra which admits a finite-dimensional irreducible representation with dimension larger than or equal to n, for $n=1, 2, \ldots$, and denote by E (respectively E_0) the l^∞-sum (respectively c_0-sum) of $\{E_n\ ;\ n=1, 2, \ldots\}$, throughout this section. We show that $\mathcal{B}(\mathcal{H})$ (respectively $\mathcal{K}(\mathcal{H})$) can be replaced by E (respectively E_0) in the above mentioned Kirchberg's result.

LEMMA 3.1. *Let A be a C^*-algebra and $x\in A\otimes M_n$. Then for any $\varepsilon>0$, there exist a completely positive contraction $W: M_n\to E_n$ such that*

$$\|(1_A\otimes W)(x)\|>\|x\|-\varepsilon.$$

Proof. By [6, Lemma 2] (see also [10, Lemma 2.7]), there exist completely positive contractions $W: M_n\to E_n$ and $V\ ;\ E_n\to M_n$ such that

$$\|VW-id\|_{cb}<\frac{\varepsilon}{\|x\|}.$$

Then, we have

$$\|(1_A\otimes V)(1_A\otimes W)(x)-x\|<\varepsilon$$

and it follows that

$$\|x\|-\varepsilon<\|(1_A\otimes V)(1_A\otimes W)(x)\|\leq\|(1_A\otimes W)(x)\|.$$

LEMMA 3.2. *Let A and B be C^*-algebras and $s\in A\otimes B$. Then, we have*

$$(3.1)\qquad \|s\|=\sup\{\|(1_A\otimes V)(s)\|\ ;\ V \text{ is a completely positive contraction from } B \text{ to } E_n,\ n=1,2,\ldots\}.$$

Proof. Let $\varepsilon>0$ be given. For a completely positive contraction $V: B\to M_n$ we can choose, by Lemma 3.1, a completely positive contraction $W: M\to E_n$ such that

$$\|(1_A\otimes W)(1_A\otimes V)(s)\|\geq\|(1_A\otimes V)(s)\|-\varepsilon.$$

Now, $WV: B\to E_n$ is a completely positive contraction and we have

$$\|(1_A\otimes WV)(s)\|\geq\|(1_A\otimes V)(s)\|-\varepsilon.$$

Hence, the right side of (3.1) is larger than or equal to

$$\|s\|=\sup\{\|(1_A\otimes V)(s)\|\ ;\ V \text{ is a completely positive contraction from } B \text{ to } M_n,\ n=1,2,\ldots\},$$

which is the equality proved in [7, Lemma 2.3].

In order to follow the proof of [7, Theorem 1.1], we adopt all notations in [7] such as $m(B)$, $c_0(b)$ and p_n. Let S be the set of all completely positive contractions V from $m(B)$ into E such that $V(c_0(B))\subseteq E_0$.

THEOREM 3.3. *Let A be a C^*-algebra. Then, A is C^*-exact if and only if*

$$F(A, E_0, A\otimes E)=A\otimes E_0.$$

Proof. The arguments of [7] go well except that of [7, Lemma 2.3], for which we will give the following subsitute:

LEMMA 3.4. *Let $t\in A\otimes m(B)$. Then we have*

i) *$t\in A\otimes c_0(B)$ if and only if $(1_A\otimes V)(t)\in A\otimes E_0$ for every $V\in S$.*

ii) *$t\in F(A, c_0(B), A\otimes m(B))$ if and only if $(1_A\otimes V)(t)\in F(A, E_0, A\otimes E)$ for every $V\in S$.*

Proof. Assume that $t\in A\otimes m(B)\setminus A\otimes c_0(B)$. Then, there exists a strictly increasing sequence $\{\nu(n)\ ;\ n=1,2,\ldots\}$ such that $\|(1_A\otimes p_{\nu(n)})(t)\|>2\varepsilon$, for all $n=1,2,\ldots$ So, by Lemma 3.2, there exists completely positive contractions $V_n: B\to E_{\nu(n)}$ such that

$$\|(1_A \otimes V_n)(1_A \otimes p_{\nu(n)})(t)\| > \varepsilon$$

for all $n=1, 2, \ldots$ If we define $V : m(B) \to E$ by

$$V(b_1, b_2, \ldots) = (0, \ldots, 0, V_1(b_{\nu(1)}), 0, \ldots, 0, V_2(b_{\nu(2)}), \ldots),$$

where each $V_n(b_{\nu(n)})$ is in the $\nu(n)$-th position, then $V \in S$.

Let $\pi_n ; E \to E_n$ denote the projection onto the n-th component. Then, every $s \in A \otimes E_0$ satisfies $\lim_{n\to 0}\|(1_A \otimes \pi_n)(s)\| = 0$. But, we have

$$\|(1_A \otimes \pi_{\nu(n)})(1_A \otimes V)(t)\| = \|(1_A \otimes V_n)(1_A \otimes p_{\nu(n)})(t)\| \geq \varepsilon$$

for all $n=1, 2, \ldots$, which shows that $(1_A \otimes V)(t) \in A \otimes E \backslash A \otimes E_0$. The remaining statements are easy.

4. Main Result.

Throughout the remainder of this note, let A denote a liminal C^*-algebra with Hausdorff spectrum T, and $\mathcal{A} = ((A(t))_{t\in T}, \Gamma)$ be the continuous field of C^*-algebras defined by A. Then, by [3, Theorem 10.5.4], we have

$$A \cong \{x \in \Gamma ; \lim_{t\to\infty}\|x(t)\| = 0\}.$$

THEOREM 4.1. *Let A be a liminal C^*-algebra with Hausdorff spectrum. Then, A is a C^*-algebra with trivial Fubini products if and only if A is subhomogeneous.*

Proof. It suffices to prove the necessity. To do this, we assume that A is not subhomogeneous and show that A has a nontrival Fubini product. We consider the following two cases:

Case I: *A has an infinite-dimensional irreducible representation.*

In this case, we have $A(t_0) = \mathcal{K}(\mathcal{H})$ for some $t_0 \in T$. Define $\tilde{A}(t) = A(t)$ for $t \neq t_0$ and $\tilde{A}(t_0) = \mathcal{B}(\mathcal{H})$. Let Λ be the set of $y \in \prod_{t\in T}\tilde{A}(t)$ such that $t \mapsto \|y(t)\|$ is continuous on T and y coincides to some $x \in \Gamma$ on the set $T \backslash \{t_0\}$. Then, there exists a unique subset $\tilde{\Gamma}$ of $\prod_{t\in T}\tilde{A}(t)$ containing Λ such that $\tilde{\mathcal{A}} = (\tilde{A}(t), \tilde{\Gamma})$ is a continuous field of C^*-algebras on T. Put

$$\tilde{A} = \{x \in \tilde{\Gamma} ; \lim_{t\to\infty}\|x(t)\| = 0\}.$$

Then, A is naturally embedded in $\tilde{A}$. We define $\alpha : \tilde{A} \to \mathcal{B}(\mathcal{H})$ by $\alpha(y) = y(t_0)$ for $y \in \tilde{A}$. Then α is surjective and $\alpha(A) = \mathcal{K}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$. Furthermore $\operatorname{Ker}\alpha \subseteq A$. So, we have

$$(1_A\otimes\alpha)(F(\mathcal{K}(\mathcal{H}), A, \mathcal{K}(\mathcal{H})\otimes\tilde{A}) = F(\mathcal{K}(\mathcal{H}), \alpha(A), \mathcal{K}(\mathcal{H})\otimes\mathcal{K}(\mathcal{H})).$$

by Lemma 2.2. But, we know that the Fubini product of the right side is nontrivial [13, Theorem 8], which implies that A also has a nontrivial Fubini product in the left side.

Case II: *Every irreducible representation of A is finite-dimensional.*

We can write $T=\bigcup_{n=1}^{\infty}T(n)$, where $T(n)$ is the set of all irreducible representations whose dimensions are less than or equal to n. Then $T(n)$ is closed [3, Proposition 3.6.3] and T is of second category because T is locally compact [3, Corollary 3.3.8]. Now, it is easy to see that there exist a strictly increasing sequence $\{n_i\}$ of natural numbers and sequences $\{V_i\}$ of open subsets in T such that $V_i\subset T(n_i)\setminus T(n_i-1)$.

Put $S=\bigcup_{i=1}^{\infty}V_i$, and let $\mathcal{A}|_{V_i}=((A(t))_{t\in V_i}, \Gamma_{V_i})$ and $\mathcal{A}|_S=(A(t)_{t\in S}, \Gamma_S)$ be the continuous fields induced on $V_i (i=1, 2, \dots)$ and S, respectively. Define

$$E_i=\{x\in\Gamma_{V_i}\ ;\ \lim_{t\to\infty}\|x(t)\|=0$$
$$E_0=\{x\in\Gamma_S\ ;\ \lim_{t\to\infty}\|x(t)\|=0\}.$$

Then, since each E_i is n_i-homogeneous and E_0 is the c_0-sum of $\{E_i\ ;\ i=1, 2, \dots\}$, it follows that E_0 has a nontrivial Fubini product by Theorem 3.3. Also, there exists an embedding $E_0\hookrightarrow A$ defined by $x\mapsto\tilde{x}$, where

$$\tilde{x}(t)=\begin{cases} x(t), & \text{for } t\in S \\ 0, & \text{for } t\notin S.\end{cases}$$

Now, both A and E_0 are nuclear C^*-algebras and we see that A is a C^*-algebra with a nontrivial Fubini product by Lemma 2.1. This completes the proof.

Added in proof: Professor Huruya and the author showed that Theorem 4.1 holds for general C^*-algebras in their paper "Fubini products of C^*-algebras and applications to C^*-exactness".

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