J. Korean Math. Soc. 25(1988), No. 1, pp. 115-123

THE DUALITY BETWEEN FRAMES AND ORDERED TOPOLOGICAL SPACES

T. H. CHOE* AND S. S. HONG**

0. Introduction

Since Priestley has established the duality between the category of distributive lattices and the category of compact 0-dimensional ordered spaces, ordered topological spaces are also known to play an important role to establish dualities.

Usually, to get dualities between topological structures and algebraic structures, one considers some algebraic structures on the set of continuous maps on a topological space into a chosen character space. As Priestley has done, one considers some algebraic structures on the set of continuous isotones on an ordered topological space into a character space. In this direction, we refer to [1, 2, 6, 7, 8].

Recently, the frame (=locale=complete Heyting algebra) is known to be very appropriate for the study of pointless topology. Using two point chain as a character object, one establishes the duality between the category of spatial frames and the category of sober spaces. For this duality and some more interesting development in the theory of frames, we refer to [4].

The purpose to write this paper is that using frames, we establish a duality between the category of spatial frames and that of ordered sober spaces.

For the terminology of ordered topological spaces, we refer to [5] and for that of category thoery, to [2].

1. Preliminaries

A frame A is a complete lattice such that for any subset S of A and

Received October. 16, 1987.
This research was supported by the NSERC, under grant A4809* and a grant from the Korea Science and Engineering Foundation**.

$x \in A$, the infinite distributive law $x \wedge (\bigvee S) = \bigvee \{x \wedge s \mid s \in S\}$ holds in A. For any topological space X, the lattice $O(X)$ of all open sets on X is clearly a frame. Since for a continuous map $f: X \to Y, f^{-1}: O(Y) \to O(X)$ preserves arbitrary joins and finite meets, frame homomorphisms are defined to be those maps between frames preserving arbitrary joins and finite meets. The category of frames and frame homomorphisms will be denoted by **Frm** and its opposite category $\mathbf{Frm}^{op}$ by **Loc**.

The category of T_0-spaces and continuous maps will be denoted by $\mathbf{Top}_0$. For any $X \in \mathbf{Top}_0$, let $\Omega(X)$ denote the set of all continuous maps on X to the Sierpinski space $K = \{0, 1\}$ with the non-trivial open set $\{1\}$. Then $\Omega(X)$ is a subframe of the power frame K^X. In fact, $\Omega(X)$ is isomorphic with $O(X)$ in **Frm**. Furthermore, for any f in $\mathbf{Top}_0$, $\Omega(f): \Omega(Y) \to \Omega(X)$ $(\Omega(f)(u) = u \circ f)$ is a frame homomorphism. Hence $\Omega: \mathbf{Top}_0 \to \mathbf{Loc}$ is a functor. For any $A \in \mathbf{Loc}$, let $\hom(A)$ denote the space of all homomorphisms on A to the frame $\{0, 1\}$ endowed with topology $\{p_x^{-1}(1) = \{h \in \hom(A) \mid h(x) = 1\} \mid x \in A\}$. For a morphism $f: A \to B$ in **Loc**, $\hom(f): \hom(A) \to \hom(B)$ $(\hom(f)(h) = h \circ f)$ is continuous. Thus $\hom: \mathbf{Loc} \to \mathbf{Top}_0$ is a functor. Moreover it is known [4] that Ω is a left adjoint of hom, that for any $X \in \mathbf{Top}_0$, the front adjunction of X is a homeomorphism iff X is sober, and that for any $A \in \mathbf{Loc}$, the back adjunction of A is an isomorphism iff A is a spatial locale. Hence one has the duality between the category of sober spaces and the category of spatial frames. For the detail, see [4].

2. Adjoint situations

The following definition is due to Ward Jr. [9]. (See also [5]).

2.1 Definition. 1) A triple $(X, \leq, \mathscr{T})$ is said to be an *ordered topological space* if $(X, \leq)$ is a poset and $(X, \mathscr{T})$ is a topological space.

2) An ordered topological space $(X, \leq, \mathscr{T})$ is said to be *upper continuous* if for $x \not\leq y$ in X, there is an upper open set V with $x \in V$ and $y \notin V$.

3) A map $f: (X, \leq, \mathscr{T}) \to (Y, \leq', \mathscr{T}')$ between ordered topological spaces is said to be a *continuous isotone* if $f: (X, \leq) \to (Y, \leq')$ is an isotone and $f: (X, \mathscr{T}) \to (Y, \mathscr{T}')$ is continuous.

2.2 Remark. 1) Every upper continuous ordered topological space is a T_0-space.

2) An ordered topological space $(X, \leq, \mathscr{T})$ is upper continuous iff for any $x \in X$, $\downarrow x = \{y \in X \mid y \leq x\}$ is closed.

NOTATION. 1) The category of all upper continuous ordered topological spaces and continuous isotones will be denoted by **UOTop**.

2) For any $(X, \leq, \mathscr{T}) \in$ **UOPop**, $\uparrow \mathscr{T}$ denotes the set of all upper open sets, i.e., $\uparrow \mathscr{T} = \{V \in \mathscr{T} \mid V = \uparrow V\}$.

It is immediate that for any $(X, \leq, \mathscr{T}) \in$ **UOTop**, $\uparrow \mathscr{T}$ is a T_0-topology on X and that for any $f: (X, \leq, \mathscr{T}) \to (Y, \leq', \mathscr{T}')$ in **UOTop**, $f: (X, \uparrow \mathscr{T}) \to (Y, \uparrow \mathscr{T}')$ is again continuous. Thus one has a functor U : **UOTop**$\to$**Top**$_0$, where $U((X, \leq, \mathscr{T})) = (X, \uparrow \mathscr{T})$ $((X, \leq, \mathscr{T}) \in$ **UOTop**$)$ and $U(f) = f$ $(f \in$ **UOTop**$)$.

2.3. DEFINITION. For $(X, \mathscr{T}) \in$ **Top**$_0$, we define a relation $\leq_s$ on X as follows: $x \leq_s y$ iff $x \in \overline{\{y\}}$. Then $\leq_s$ is said to be the *specialization order* of $(X, \mathscr{T})$.

2.4 REMARK. Let $\{X, \mathscr{T}\}$ be a T_0-space, then one has,

1) $\leq_s$ is a partial order, and $x \leq_s y$ iff the neighborhood filter $\mathscr{U}(x)$ of x is contained in the neighborhood filter $\mathscr{U}(y)$ of y.

2) Every open set V in $(X, \mathscr{T})$ is an upper set in $(X, \leq_s)$ and for any $x \in X$, $\overline{\{x\}} = \downarrow x$ in $(X, \leq_s)$.

It is obvious that for any $f: (X, \mathscr{T}) \to (Y, \mathscr{T}')$ in **Top**$_0$, $f: (X, \leq_s) \to (Y, \leq_s)$ is an isotone.

Furthermore, for any $(X, \mathscr{T}) \in$ **Top**$_0$, $(X, \leq_s, \mathscr{T})$ belongs to **UOTop**. Indeed, suppose $x \not\leq_s y$, then $x \notin \overline{\{y\}}$; hence there is an open neighborhood V of x with $y \notin V$. Since V is open, V is an upper set.

Hence one has a functor Σ : **Top**$_0$$\to$**UOTop**, where $\Sigma((X, \mathscr{T})) = (X, \leq_s, \mathscr{T})$ $((X, \mathscr{T}) \in$ **Top**$_0)$ and $\Sigma(f) = f$ $(f \in$ **Top**$_0)$.

Using U and Σ, we have an adjoint situation between **Top**$_0$ and **UOTop**.

2.5 THEOREM. *The functor* U : **UOTop**$\to$**Top**$_0$ *is a left adjoint of the functor* Σ.

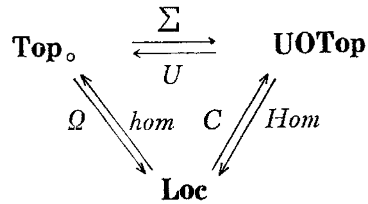
Proof. For any $(X, \leq, \mathscr{T}) \in$ **UOTop**, let $\eta: (X, \leq, \mathscr{T}) \to \Sigma U((X, \leq, \mathscr{T}))$ be the identity map. Then η is clearly continuous, for $\uparrow \mathscr{T} \subseteq \mathscr{T}$. Suppose $x \leq y$ on $(X, \leq)$ and $V \in \uparrow \mathscr{T}$

is an open neighborhood of x, then $y\in\uparrow V=V$ and hence $x\leq_s y$ on $(X, \uparrow\mathcal{T})$. Thus η is a continuous isotone. Take any $f:(X,\leq,\mathcal{T})\to\sum((Y,\mathcal{T}'))$ in **UOTop**. For any $W\in\mathcal{T}', f^{-1}(W)\in\mathcal{T}$, for $f:(X,\leq,\mathcal{T})\to(Y,\leq_s,\mathcal{T}')$ is a continuous isotone. For $x\in f^{-1}(W)$ and $x\leq y$, one has $f(x)\leq_s f(y)$ and $f(x)\in W$, so that $f(y)\in W$, i.e., $y\in f^{-1}(W)$. Hence $f^{-1}(W)=\uparrow f^{-1}(W)\in\uparrow\mathcal{T}$. In all, $f:U((X,\leq,\mathcal{T}))\to(Y,\mathcal{T}')$ is a continuous map and $\sum(f)\circ\eta=f$.
This completes the proof.

Let K denote the two point chain $\{0,1\}$ endowed with the topology with the only non-trivial open set $\{1\}$. Then it is clear that K is a topological frame, i.e., every join and finite meet are both continuous. For any $X\in$ **UOTop**, let $C(X)=\{f\mid f:X\to K$ is a continuous isotone$\}$, then $C(X)$ is a subframe of the power K^X of the two point frame K, because $C(X)$ is closed under arbitrary joins and finite meets in K^X. We note also that $C(X)$ is isomorphic with the open set lattice $\Omega(U(X))=\uparrow\mathcal{T}$, where $X=(X,\leq,\mathcal{T})$. For any $f:X\to Y$ in **UOTop**, $C(f):C(Y)\to C(X)$ defined by $C(f)(u)=u\circ f$, is clearly a frame homomorphism. Hence one has a functor $C:$ **UOTop**$\to$**Loc** described by the above.

For any $A\in$ **Loc**, let $\mathrm{Hom}(A)=\{f\mid f:A\to K$ is a frame homomorphism$\}$ endowed with the subspace topology of the product space K^A and the suborder of K^A, i.e., $f\leq g$ iff for all $x\in A, f(x)\leq g(x)$. It is known [4] that $\{p_x^{-1}(1)=\{f\in\mathrm{Hom}(A)\mid f(x)=1\}\mid x\in A\}$ is the topology on $\mathrm{Hom}(A)$. If $f\not\leq g$ in $\mathrm{Hom}(A)$, then $f(x)\not\leq g(x)$ for some $x\in A$. So $f(x)=1$ and $g(x)=0$. Hence $f\in p_x^{-1}(1)=\uparrow p_x^{-1}(1)$ and $g\notin p_x^{-1}(1)$. Thus $\mathrm{Hom}(A)$ belongs to **UOTop**. For any $f:A\to B$ in **Loc**, let $\mathrm{Hom}(f):\mathrm{Hom}(A)\to\mathrm{Hom}(B)$ be the map defined by $\mathrm{Hom}(f)(u)=u\circ f$. We note that $\mathrm{Hom}(A)\xrightarrow{\mathrm{Hom}(f)}\mathrm{Hom}(B)\hookrightarrow K^B\xrightarrow{p_x}K=\mathrm{Hom}(A)\hookrightarrow K^A\xrightarrow{p_{f(x)}}K$ for all $x\in B$, where $\hookrightarrow$ denotes the inclusion map and p_x the projection. Thus $\mathrm{Hom}(f):\mathrm{Hom}(A)\to\mathrm{Hom}(B)$ is a continuous isotone, for $(\mathrm{Hom}(B)\hookrightarrow K^B\xrightarrow{p_x}K)_{x\in B}$ is an initial source in **UOTop**.
Hence one has a functor $\mathrm{Hom}:$ **Loc**$\to$**UOTop**.

2.6 THEOREM *One has $Hom=\sum\circ hom$, $C=\Omega\circ U$, $hom=U\circ Hom$ and $\Omega=C\circ\sum$.*



Proof. Since $\sum(f)=f$ and $U(g)=g$, it is enough to show that the identities hold for objects.

For any $A \in$ **Loc**, the topology of $\sum(\text{hom}(A))$ is that of $\text{hom}(A)$, which is also the topology of $\text{Hom}(A)$. Suppose $f \leq g$ in $\text{Hom}(A)$ and $f \in p_x^{-1}(1) = \{h \in \text{hom}(A) \mid h(x)=1\}$, then $1=f(x) \leq g(x)$; hence $g \in p_x^{-1}(1)$. Thus $f \leq g$ in $\sum(\text{hom}(A))$. Conversely, if $f \leq g$ in $\sum(\text{hom}(A))$ and $f(x)=1$, then $f \in p_x^{-1}(1)$, so that $g \in p_x^{-1}(1)$, i.e., $g(x)=1$. Hence $f \leq g$ in $\text{Hom}(A)$. In all, $\sum(\text{hom}(A))=\text{Hom}(A)$.

For $X \in$ **UOTop**, a map $f: X \to K$ is a continuous isotone iff $f^{-1}(1)$ is an upper open set in X. Thus $f \in C(X)$ iff $f: U(X) \to K$ is continuous i.e., $f \in \Omega(U(X))$. Hence $C(X)=\Omega(U(X))$. We note that for any $A \in$ **Loc**, the underlying set (topology) of $\text{Hom}(A)$ is the underlying set (topology, respectively) of $\text{hom}(A)$. Moreover, every open set $p_x^{-1}(1)$ in $\text{Hom}(A)$ is an upper set and hence the topologies of U(Hom and $\text{Hom}(A)$ are the same. Thus one can conclude that $U(\text{Hom}(A))=$ $(A))\text{hom}(A)$ for all $A \in$ **Loc**. Finally, for $X \in$ **Top**$_\circ$, a map $f: X \to K$ is continuous iff $f: \sum(X) \to K$ is a continuous isotone. Hence one has $\Omega(X)=C(\sum(X))$. This completes the proof.

As already mentioned, Ω : **Top**$_\circ$→**Loc** is a left adjoint of hom : **Loc**→**Top**$_\circ$ and U : **Top**$_\circ$→**UOTop** a left adjoint of $\sum$: **UOTop**→**Top**$_\circ$. Thus using the above theorem, one has immediately the following:

2.7 Theorem. *The functor* C : **UOTop**→**Loc** *is a left adjoint of Hom* : **Loc**→**UOTop**.

2.8 Remark. 1) For the adjunction $C \dashv$ Hom : **Loc**→**UOTop**, the front adjunction is given by $\eta_X : X \to \text{Hom}(C(X))$ $(\eta_X(x)(f)=f(x),\ x \in X,\ f \in C(X))$ for $X \in$ **UOTop** and the back adjunction by $\varepsilon_A : A \to C(\text{Hom}(A))$ $(\varepsilon_A(a)(h)=h(a),\ a \in A, h \in \text{Hom}(A))$ for $A \in$ **Loc**, because the front and back adjunctions for $U \dashv \sum$ are given by identity maps. (See 27.8 in [2]).

2) Using the above $(\eta_X)_{X\in\mathrm{UOTop}}$ and $(\varepsilon_A)_{A\in\mathrm{Loc}}$, one can also directly prove the above Theorem 2.7.

3. Duality

It is well known that for any adjoint situation $(\eta, \varepsilon) : F\dashv G : \boldsymbol{A}\to\boldsymbol{B}$, it induces an equivalence between the subcategory **Fix** η of $\boldsymbol{B}$ determined by all those objects $\boldsymbol{B}$ whose η_B is an isomorphism and the subcategory **Fix** ε of $\boldsymbol{A}$ determined by all those objets A whose ε_A is an isomorphism. In this section, we characterize **Fix** η and **Fix** ε for the adjoint sitation $(\eta, \varepsilon) : C\dashv\mathrm{Hom} : \mathbf{Loc}\to\mathbf{UOTop}$ so that we get the corresponding equivalence.

3.1 Proposition. *For* $A\in\mathbf{Loc}$, ε_A *is an isomorphism iff* A *is a spatial locale.*

Proof. Since $C(\mathrm{Hom}(A))=\Omega(U(\mathrm{Hom}(A)))=\Omega(\mathrm{hom}(A))$, $\varepsilon_A : A\to C(\mathrm{Hom}(A))$ is precisely the back adjunction $A\to\Omega(\mathrm{hom}(A))$ for $\Omega\dashv \mathrm{hom} : \mathbf{Loc}\to\mathbf{Top}_0$. It is known [4] that the back adjunction $A\to\Omega(\mathrm{hom}(A))$ is an isomorphism iff A is a spatial locale. This completes the proof.

3.2 Lemma. *For any* $X\in\mathbf{UOTop}$, $C(X)$ *is an initial mono-source in the category* **POSet** *of posets and isotones. In other words,* $x\le y$ *in* X *iff for any* $f\in C(X), f(x)\le f(y)$.

Proof. If $x\le y$ in X, then clearly $f(x)\le f(y)$. Suppose $x\not\le y$ in X, then there is an upper open set V in X with $x\in V$ but $y\notin V$. Thus the characteristic map χ_V of V belongs to $C(X)$ and $\chi_V(x)\not\le\chi_V(y)$.

3.3 Remark. In general, η_X need not be an embedding. Indeed, let X be the real line endowed with the usual topology and the usual order, then $C(X)$ is not an initial source in the category **Top** of topological spaces and continuous maps, for the initial topology on the real line R with respect to $C(X)$ is $\{(x, \to) \mid x\in R\}\cup\{\phi, R\}$. Hence η_X is not initial in the category **Top**.

3.4 Definition. An object X of **UOTop** is said to be an *upper space* if $C(X)$ is an initial source in **UOTop**, equivalently X is isomorphic with a subspace of a power of K in **UOTop**.

3.5 Proposition. *For* $(X, \le, \mathscr{T})\in\mathbf{UOTop}$, *the following are equivalent:*

1) *X is an upper space.*
2) *Every open set of X is an upper set, i.e.,* $(X, \mathscr{T}) = U((X, \leq, \mathscr{T}))$.
3) *For any* $x \in X$, $\overline{\{x\}} = \downarrow x$, *i.e.,* $(X, \leq, \mathscr{T}) = \Sigma((X, \mathscr{T}))$.

Proof. 1)⇒2). Since X is an upper space, $C(X)$ is an initial source. Hence the topology $\mathscr{T}$ on X is the initial topology with respect to $C(X)$, which is generated by $\{f^{-1}(1) \mid f \in C(X)\}$. In fact, $\{f^{-1}(1) \mid f \in C(X)\} = \uparrow \mathscr{T}$ is itself a topology on X, Thus $\mathscr{T} = \uparrow \mathscr{T}$.

2)⇒3). Since $X \in$ **UOTop**, for any $x \in X$ $\downarrow x$ is closed in X. Hence the implication is immediate from Proposition 1.8 in [4], for $\phi(X, \leq) \subseteq \mathscr{T} \subseteq \gamma(X, \leq)$.

3)⇒1). Take any $Y \in$ **UOTop** and $f : Y \to X$ such that for all $u \in C(X)$, $u \circ f$ is a continuous isotone. By Lemma 3.2, f is an isotone. Take any open set V on X, then by the assumption, V is an upper set; hence $\chi_V \in C(X)$. Since $\chi_V \circ f$ is continuous, $(\chi_V \circ f)^{-1}(1) = f^{-1}(V)$ is again open in Y. Thus f is continuous. Henc $C(X)$ is an initial source in **UOTop**, i.e., X is an upper space.

We observe that for any $X \in$ **UOTop**, the source $C(X)$ is factorized through η_X via $X \xrightarrow{f} K = X \xrightarrow{\eta_X} \mathrm{Hom}C(X) \longrightarrow K^{C(X)} \xrightarrow{p_f} K$. Hence η_X is an isomorphism iff $C(X)$ is an initial mono-source and η_X is onto. In other words, η_X is an isomorphism iff X is an upper space and η_X is onto.

3.6 Definition. An upper continuous ordered topological space X is said to be *ordered-sober* if for any irreducible closed subset A of X, there is an $x_0 \in A$ with $A = \downarrow x_0$.

Suppose the order on $X \in$ **UOTop** is discrete, then X is ordered-sober iff X is a T_1 sober space, because the topology on X is a T_1-topology.

Now using ordered-sober spaces, we characterize **Fix** η.

3.7 Theorem. *For* $X \in$ **UOTop**, $\eta_X : X \to \mathrm{Hom}C(X)$ *is an isomorphism iff X is an ordered-sober space.*

Proof. Suppose η_X is an isomorphism, then by the previous remark, X is an upper space. Take any irreducible closed subset A on X, then CA is an upper open set, so that $\chi_{CD} \in C(X)$, where CA denotes the complement of A. We define $h : C(X) \to K$ as follows:

$h(u) = 0$ iff $u \leq \chi_{CA}$. Since χ_{CA} is a prime element of $C(X)$, h is a frame homomorphism. Since η_X is onto, there is an $x_o \in X$ with $h=\eta_X(x_o)$. Thus one has, $h(u)=\eta_X(x_o)(u)=u(x_o)=0$ $(u\in C(X))$ iff $u\leq \chi_{CA}$ iff $A\subseteq u^{-1}(0)$. Since $h(\chi_{C\downarrow x_o})=\eta_X(x_o)(\chi_{C\downarrow x_o})=\chi_{C\downarrow x_o}(x_o)=0$, $A\subseteq \chi_{C\downarrow x_o}{}^{-1}(0)=\downarrow x_o$. Moreover, $\chi_{CA}(x_o)=\eta_X(x_o)(\chi_{CA})=h(\chi_{CA})=0$; therefore $x_o\in A$. Thus one has $\downarrow x_o\subseteq A$. In all, $A=\downarrow x_o$.

Conversely suppose X is an ordered-sober space. For any $x\in X$, $\overline{\{x\}}$ is an irreducible closed subset, so that there is an $a\in X$ with $\overline{\{x\}}=\downarrow a$. Since $\downarrow x$ is closed, $x\in \downarrow a=\overline{\{x\}}\subseteq \downarrow x$, so that $a=x$. Hence $\overline{\{x\}}=\downarrow x$; therefore X is an upper space. It remains to show that η_X is onto. Take any $h\in \mathrm{Hom}(C(X))$. Let $u_h=\bigvee\{u\in C(X)\mid h(u)=0\}$, then $h(u)=0$ iff $u\leq u_h$. Let $A=u_h{}^{-1}(0)$, then A is a non-empty closed subset of X, for u_h can not be the constant map 1 with the value 1. Furthermore A is irreducible. Indeed, suppose $A=F\cup G$ for some closed F, G in X. Since X is an upper space, CF and CG are both upper open sets; hence $\chi_{CF}, \chi_{CG}\in C(X)$ and $u_h=\chi_{CF}\wedge\chi_{CG}$. Thus $0=h(u_h)=h(\chi_{CF}\wedge\chi_{CG})=h(\chi_{CF})\wedge h(\chi_{CG})$, which implies that either $\chi_{CF}\leq u_h$ or $\chi_{CG}\leq u_h$. Hence one has $A=F$ or $A=G$. By the assumption, there is an $x_o\in X$ with $A=\downarrow x_o$. For any $u\in C(X)$, $\eta_X(x_o)(u)=0$ iff $u(x_o)=0$ iff $A=\downarrow x_o\subseteq u^{-1}(0)$ iff $u\leq u_h$ iff $h(u)=0$. Thus $\eta_X(x_o)=h$. This completes the proof.

3.8 Proposition. *For any $A\in$ **Loc**, $Hom(A)$ is ordered-sober and for any $X\in$ **UOTop**, $C(X)$ is a spatial locale.*

Proof. We note that for any $A\in$ **Loc**, $\varepsilon_A: A\to C(\mathrm{Hom}(A))$ is an onto map and hence $\mathrm{Hom}(\varepsilon_A)$ is 1-1. Hence for any $A\in$ **Loc**, the adjunction equation $\mathrm{Hom}(\varepsilon_A)\circ\eta_{\mathrm{Hom}(A)}=1_{\mathrm{Hom}(A)}$ implies that $\eta_{\mathrm{Hom}(A)}$ is an isomorphism. Furthermore, for any $X\in$ **UOTop**, $C(X)=\Omega(U(X))$ and hence it is spatial (See 1.5 in [4]). Directly using $C(\eta_X)\circ\varepsilon_{C(X)}=1_{C(X)}$, one can also conclude that $C(X)$ is a spatial locale.

Collecting above theorems and propositions, one can conclude the following:

3.9 Theorem. *The adjoint situation $C\dashv Hom$: **Loc**$\to$**UOTop** induces the eqivalence between the category **SLoc** of spatial locales and the category **OSob** of ordered-sober spaces. Furthermore, Hom(**Loc**) = **OSob** and C(**UOTop**) = **SLoc**, and **OSob** and the category **Sob** of sober spaces and continuous maps are equivalent.*

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McMaster University
Hamilton, Canada
and
Sogang University
Seoul 121, Korea