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TRANSVERSE FIELDS ON FOLIATED RIEMANNIAN MANIFOLDS

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0. On a foliated Riemannian manifold, the transverse Killing field was defined by Molino [6], and was discussed by Kamber and Tondeur ([4]), Molino ([6], [7]), and others. In this paper, we will discuss geometric transverse fields, that is, transverse affine, projective, conformal, and Killing fields, on foliated Riemannian manifolds. If the foliation is one by points, then transverse affine, projective, conformal, and Killing fields are usual affine, projective, conformal, and Killing vector fields on Riemannian manifolds respectively. Thus, if the foliation is one by points then our results reduce to well-known results that were shown in [1], [2], [5] and [10]. That is, our results are generalizations to foliated cases.

We shall be in C^∞-category and deal only with connected and oriented manifolds without boundary. We use the following convention on the range of indices: $1\leq i, j\leq p$ and $p+1\leq a, b, c, d\leq p+q$. The Einstein summation convention will be used.

1. Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundlelike metric g_M with respect to $\mathcal{F}$([8]). Let ∇ be the Levi-Civita connection with respect to g_M. Let TM be the tangent bundle of M and E the integrable subbundle of TM given by $\mathcal{F}$. The normal bundle Q of $\mathcal{F}$ is given by $Q=TM/E$. The metric g_M defines a splitting σ of the exact sequence

$$0\longrightarrow E\longrightarrow TM\underset{\sigma}{\overset{\pi}{\rightleftarrows}}Q\longrightarrow 0$$

with $\sigma(Q)=E^\perp$ (the orthogonal complement bundle of E in TM) ([3]). Then g_M induces a metric g_Q on Q:

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(1.1) $$g_Q(s,t)=g_M(\sigma(s),\sigma(t)) \text{ for any } s,t\in\Gamma(Q).$$

In a flat chart $U(x^i, x^a)$ with respect to $\mathcal{F}$ ([8]), a local frame $\{X_i, X_a\} = \{\partial/\partial x^i, \partial/\partial x^a - A^j{}_a\partial/\partial x^j\}$ is called the basic adapted frame to $\mathcal{F}$([8], [9], [11]). Here $A^j{}_a$ are functions on U with $g_M(X_i, X_a)=0$. It is trivial that $\{X_i\}$ (resp. $\{X_a\}$) spans $\Gamma(E|_U)$ (resp. $\Gamma(E^\perp|_U)$). We omit "$|_U$" for simplicity. We set

(1.2) $$\begin{aligned} &g_{ij}=g_M(X_i, X_j) \quad g_{ab}=g_M(X_a, X_b) \\ &(g^{ij})=(g_{ij})^{-1} \quad (g^{ab})=(g_{ab})^{-1} \end{aligned}$$

A connection D in Q is defined by

(1.3) $$\begin{aligned} D_Xs&=\pi([X,Y]) && X\in\Gamma(E) \\ & && s\in\Gamma(Q) \quad \text{with } \pi(Y)=s \\ D_Xs&=\pi(\nabla_XY_s) && X\in\Gamma(E^\perp) \\ & && s\in\Gamma(Q) \quad \text{with } Y_s=\sigma(s) \end{aligned}$$

([3]). Then we have

PROPOSITION 1.1 ([3]). *The connection D in Q is torsion-free and metrical with respect to g_Q.*

PROPOSITION 1.2 ([3]). *It holds that*

$$\begin{aligned} 2g_Q(D_Xs,t)=&X(g_Q(s,t))+Y(g_Q(\pi(X),t)) \\ &-Z(g_Q(\pi(X),s))+g_Q(\pi([X,Y]),t) \\ &+g_Q(\pi([Z,X]),s)-g_Q(\pi([Y,Z]),\pi(X)) \end{aligned}$$

for any $X\in\Gamma(TM)$ and $s,t\in\Gamma(Q)$ with $\pi(Y)=s$, $\pi(Z)=t$.

The curvature R_D of D is defined by

(1.4) $$R_D(X,Y)s=D_XD_Ys-D_YD_Xs-D_{[X,Y]}s$$

for any $X, Y\in\Gamma(TM)$ and $s\in\Gamma(Q)$. Since $i(X)R_D=0$ for any $X\in\Gamma(E)$ ([3]), we can define the Ricci operator $\rho_D:\Gamma(Q)\longrightarrow\Gamma(Q)$ of $\mathcal{F}$ by

(1.5) $$\rho_D(s)=g^{ab}R_D(\sigma(s),\pi(X_a))\pi(X_b)$$

([4]).

Let $V(\mathcal{F})$ be the space of all vector fields Y on M satisfying

(1.6) $$[Y,Z]\in\Gamma(E)$$

for any $Z\in\Gamma(E)$. An element of $V(\mathcal{F})$ is called an infinitesimal automorphism of $\mathcal{F}$([4], [7]). We set

(1.7) $$\bar{V}(\mathcal{F})=\{s\in\Gamma(Q)\mid s=\pi(Y),\ \ Y\in V(\mathcal{F})\}.$$

The $s\in\bar{V}(\mathcal{F})$ satisfies

(1.8) $$D_X s=0$$

for any $X\in\Gamma(E)$ ([4], [7]).

The transverse Lie derivative $\theta(Y)$ with respect to $Y\in V(\mathcal{F})$ is defined by

(1.9) $$\theta(Y)s=\pi([Y, Y_s])$$

for any $s\in\Gamma(Q)$ with $\pi(Y_s)=s$.

Definition 1.3 ([4], [6], [7]). If $Y\in V(\mathcal{F})$ satisfies $\theta(Y)g_Q=0$, then $s=\pi(Y)$ is called a transverse Killing field (t. K. f.) of $\mathcal{F}$.

Definition 1.4. If $Y\in V(\mathcal{F})$ satisfies $\theta(Y)g_Q=2f_Y\cdot g_Q$ where f_Y is a functionon M, then $s=\pi(Y)$ is called a transverse conformal field (t. c. f.) of $\mathcal{F}$ and f_Y is called the characteristic function of s.

Definition 1.5 ([4]). If $Y\in V(\mathcal{F})$ satisfies $\theta(Y)D=0$, then $s=\pi(Y)$ is called a transverse affine field (t. a. f.) of $\mathcal{F}$.

Definition 1.6. If $Y\in V(\mathcal{F})$ satisfies $(\theta(Y)D)_X t=\varphi_Y(X)t+\varphi_Y(\sigma(t))\pi(X)$ for any $X\in\Gamma(TM)$ and $t\in\Gamma(Q)$, where φ_Y is a 1-form on M, then $s=\pi(Y)$ is called a transverse projective field (t. p. f.) of $\mathcal{F}$ and φ_Y is called the characteristic form of s.

For $Y\in V(\mathcal{F})$, we define an operator $A_D(Y):\Gamma(Q)\to\Gamma(Q)$ by

(1.10) $$A_D(Y)t=\theta(Y)t-D_Y t.$$

Then we have

(1.11) $$A_D(Y)t=-D_{Y_t}\pi(Y)$$

where $t=\pi(Y_t)$. This shows that

(i) $A_D(Y)$ depends only on $s=\pi(Y)$,

(ii) $A_D(Y)$ is a linear operator of $\Gamma(Q)$.

Thus we can use $A_D(s)$ instead of $A_D(Y)$ ([4]).

Proposition 1.7 ([4], [5]). *For $Y\in V(\mathcal{F})$, it holds that*

$$(\theta(Y)D)_{Y_t}u=R_D(\pi(Y),\ t)u-(D_{Y_t}A_D(\pi(Y)))u$$

for any $t, u\in\Gamma(Q)$ with $Y_t=\sigma(t)$.

Let $\Omega^r(M, Q)$ be the space of all Q-valued r-forms on M. When M is compact, the inner product $\langle\langle\, ,\rangle\rangle$ on $\Omega^r(M, Q)$ is defined by

$$(1.12)\qquad \langle\!\langle t, u\rangle\!\rangle = \int_M g_Q(t\wedge {}^*u)$$

([3]). Let $d_D : \Omega^r(M, Q)\rightarrow\Omega^{r+1}(M, Q)$ be the exterior defferential operator and the operator $d_D{}^* : \Omega^r(N, Q)\longrightarrow\Omega^{r-1}(M, Q)$ is defined ([3]). If M is compact, then $d_D{}^*$ is the adjoint operator of d_D with respect to $\langle\!\langle,\rangle\!\rangle$ ([3]). The Laplacian Δ_D acting on $\Omega^r(M, Q)$ is defined by

$$(1.13)\qquad \Delta_D = d_D d_D{}^* + d_D{}^* d_D.$$

REMARK 1.8 ([3]). An element of $\Gamma(Q)$ is regarded as an element of $\Omega^0(M, Q)$. The bundle map $\pi : TM\longrightarrow Q$ is an element of $\Omega^1(M, Q)$.

The Q-valued bilinear form α on M is defined by

$$(1.14)\qquad \alpha(X, Y) = -(D_X\pi)(Y)$$

for any $X, Y\in\Gamma(TM)$ ([3]). Since $\alpha(X, Y) = \pi(\nabla_X Y)$ for any $X, Y\in\Gamma(E)$, we call α the second fundamental form of $\mathcal{F}$([3]). The tension field τ of $\mathcal{F}$ is defined by

$$(1.15)\qquad \tau = g^{ij}\alpha(X_i, X_j)$$

([3]). We remark that $\tau = d_D{}^*\pi\in\Gamma(Q)$.

DEFINITION 1.9 ([3]). The foliation $\mathcal{F}$ is minimal if $\tau=0$.

Let $C^\infty(M)$ be the space of all function on M. We define an operator $\mathrm{div}_D : \Gamma(Q)\longrightarrow C^\infty(M)$ by

$$(1.16)\qquad \mathrm{div}_D s = g^{ab} g_Q(D_{X_a}s, \pi(X_b)).$$

We call $\mathrm{div}_D s$ the transvers divergence of s with respect to D ([12]). We define the transverse gradient $\mathrm{grad}_D f$ of a function f with respect to D by

$$(1.17)\qquad \mathrm{grad}_D f = g^{ab} X_a(f)\pi(X_b).$$

Let $\Lambda^r(M)$ be the space of all r-forms on M and $\Lambda^{r_1, r_2}(M)$ the space of all (r_1, r_2)-forms on M ([8], [9], [11]). If $\varphi\in\Lambda^1(M)$ satisfies $i(X)\varphi=0$ for any $X\in\Gamma(E)$, then $\varphi\in\Lambda^{0,1}(M)$. We have $\Lambda^r(M) = \sum_{r_1+r_2=r}\Lambda^{r_1, r_2}(M)$. The exterior differential operator $d : \Lambda^r(M)\longrightarrow\Lambda^{r+1}(M)$ is decomposed into the form $d=d'+d''+d'''$ ([8], [9], [11]). For a function f on M, we have $df=d'f+d''f$ where $d'f\in\Lambda^{1,0}(M)$ and $d''f\in\Lambda^{0,1}(M)$. When M is compact, the inner product $(\!(,)\!)$ on $\Lambda^r(M)$ is defined by

$$(1.18)\qquad (\!(\varphi_1, \varphi_2)\!) = \int_M \varphi_1\wedge{}^*\varphi_2.$$

The operator $\delta : \Lambda^r(M)$ is also defined, and, if M is compact then δ

is the adjoint operator of d with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$. The Laplacian operator Δ acting on $\Lambda^r(M)$ is defined by $\Delta=d\delta+\delta d$. We remark that δ is decomposed into the form $\delta=\delta'+\delta''+\delta'''$ ([8], [9], [11]).

Definition 1.10 ([8], [9]). A function f on M is a foliated function if $d'f=0$.

Proposition 1.11. *If $s\in\bar{V}(\mathcal{F})$, then $div_D s$ is a foliated function on M, and $d(div_D s)=d''(div_D s)$.*

2. We have the following propositions:

Proposition 2.1 ([4], [12]). *If $s\in\bar{V}(\mathcal{F})$, then it holds that*

$$\Delta_D s=d_D{}^*d_D s=D_{\sigma(\tau)}s+g^{ab}(D_{X_a}A_D(s))\pi(X_b).$$

Proposition 2.2 ([4]). *Let $s\in\bar{V}(\mathcal{F})$. The following conditions are equivalent:*

(i) s is a t.K.f. of $\mathcal{F}$.

(ii) $A_D(s)g_Q=0$.

(iii) $g_Q(A_D(s)t, u)+g_Q(t, A_D(s)u=0$ for any $t, u\in\Gamma(Q)$.

Proposition 2.3 ([4]). *Let $s=\pi(Y)\in\bar{V}(\mathcal{F})$. Then s is a t.a.f. of $\mathcal{F}$ if and only if*

$$D_{\sigma(t)}A_D(s)=R_D(\sigma(t), Y)$$

for any $t\in\Gamma(Q)$.

Proposition 2.4 ([4]). *Every t.K.f. of $\mathcal{F}$ is a t.a.f. of $\mathcal{F}$.*

Proposition 2.5. *If $s=\pi(Y)\in\bar{V}(\mathcal{F})$ is a t.c.f. of $\mathcal{F}$, then it holds that*

$$(\Theta(Y)D)_{X_a}\pi(X_b)=\{X_a(f_Y)\delta_b{}^c+X_b(f_Y)\delta_a{}^c-X_d(f_Y)g_{ab}g^{dc}\}\pi(X_c)$$

where $\delta_b{}^c$ denotes the Kronecker's delta.

Proposition 2.6. *Let $s=\pi(Y)\in\bar{V}(\mathcal{F})$.*

(i) *If s is a t.a.f. of $\mathcal{F}$, then $d''(div_D s)=0$.*

(ii) *If s is a t.K.f. of $\mathcal{F}$, then $div_D s=0$.*

(iii) *If s is a t.c.f. of $\mathcal{F}$ with characteristic function f_Y, then $div_D s=q\cdot f_Y$.*

(iv) *If s is a t.p.f. of $\mathcal{F}$ with characteristic 1-form φ_Y, then $d''(div_D s)=(q+1)\cdot\varphi_Y$.*

Thus we have

THEOREM 2.7. *Let* $s\in \overline{V}(\mathcal{F})$

(i) *If* s *is a t.a.f. of* $\mathcal{F}$, *then*

$$\Delta_D s=D_{\sigma(\tau)}s+\rho_D(s) \ \text{and} \ d''(div_D s)=0.$$

(ii) *If* s *is a t.K.f. of* $\mathcal{F}$, *then*

$$\Delta_D s=D_{\sigma(\tau)}s+\rho_D(s) \ \text{and} \ div_D s=0.$$

(iii) *If* s *a t.c.f. of* $\mathcal{F}$, *then*

$$\Delta_D s=D_{\sigma(\tau)}s+\rho_D(s)+\left(1-\frac{2}{q}\right) grad_D div_D s$$

(iv) *If* s *is a t.p.f. of* $\mathcal{F}$, *then*

$$\Delta_D s=D_{\sigma(\tau)}s+\rho_D(s)-\frac{1}{q+1} grad_D div_D s.$$

3. We assume that M is compact. Then we have

THEOREM 3.1 ([12]). *Suppose that* M *is compact. It holds that*

$$\int_M div_D t \ dM=\ll\tau, t\gg.$$

for any $t\in\Gamma(Q)$.

THEOREM 3.2 ([12]). *Suppose that* M *is compact. It holds that*

$$\ll\Delta_D t, u\gg=\ll Dt, Du\gg$$

for any $t, u\in\overline{V}(\mathcal{F})$, *where* $\ll Dt, Du\gg=\int_M g^{ab}g_Q(D_{X_a}t, D_{X_b}u)dM$.

As corollary of Theorem 3.1, we have

COROLLARY 3.3 ([12]). *Suppose that* M *is compact. If* $\mathcal{F}$ *is minimal, then*

$$\int_M div_D t \ dM=0$$

for any $t\in\Gamma(Q)$.

The Ricci operator ρ_D of $\mathcal{F}$ is non-positive (resp. negative) at $x\in M$ if $g_Q(\rho_D(t), t)|_x\leq 0$ (resp. <0) for any $t\in\Gamma(Q)$ (resp. $t(x)\neq 0$). If ρ_D is non-positive everywhere on M, then we have $\ll\rho_D(t), t\gg\leq 0$ for any $t\in\Gamma(Q)$.

If $t\in\Gamma(Q)$ satisfies $Dt=0$, that is, $D_X t=0$ for any $X\in\Gamma(TM)$, then t is called D-parallel.

By Theorems 2.7 and 3.2, we have

THEOREM 3.4 ([4], [12]). *Suppose that M is compact and $\mathcal{F}$ is minimal, and let s be a t.K.f. of $\mathcal{F}$. If ρ_D is non-positive everywhere on M, then s is D-parallel. If ρ_D is non-positive everywhere and negative for at least one point of M, then $s=0$.*

By the direct calculation, we have

$$g_Q(\mathrm{grad}_D \mathrm{div}_D t, t) = \sigma(t)(\mathrm{div}_D t) \tag{3.1}$$

$$\mathrm{div}_D((\mathrm{div}_D t)\cdot t) = \sigma(t)(\mathrm{div}_D t) + (\mathrm{div}_D t)^2 \tag{3.2}$$

for any $t \in \Gamma(Q)$.

THEOREM 3.5. *Suppose that M is compact and $\mathcal{F}$ is of codimension $q \geq 2$ and minimal. Let s be a t.c.f. of $\mathcal{F}$. If ρ_D is non-positive everywhere on M, then s is D-parallel. If ρ_D is non-positive everywhere and negative for at least one point of M, then $s=0$.*

Proof. We have, by (3.1), (3.2) and Corollary 3.3,

$$\ll \mathrm{grad}_D \mathrm{div}_D s, s \gg = -\int_M (\mathrm{div}_D s)^2 \, dM. \tag{3.3}$$

Thus we have, by Theorems 2.7 and 3.2, and (3.3),

$$\begin{aligned} 0 \leq \ll Ds, Ds \gg &= \langle\!\langle \Delta_D s, s \rangle\!\rangle \\ &= \langle\!\langle \rho_D(s), s \rangle\!\rangle + \left(1 - \frac{2}{q}\right) \langle\!\langle \mathrm{grad}_D \mathrm{div}_D s, s \rangle\!\rangle \\ &= \langle\!\langle \rho_D(s), s \rangle\!\rangle - \left(1 - \frac{2}{q}\right) \int_M (\mathrm{div}_D s)^2 \, dM \leq 0. \end{aligned}$$

4. We assume that M is compact. By the direct calculation, we have

$$\mathrm{div}_D(A_D(t)t) = -g^{ab} g_Q(D_{X_a} D_{\sigma(t)} t, \pi(X_b)) \tag{4.1}$$

$$\mathrm{Tr}(A_D(t) A_D(t)) = g^{ab} g_Q(D_{\sigma(\nabla_{X_a} \sigma(t))} t, \pi(X_b)) \tag{4.2}$$

for any $t \in \Gamma(Q)$ ([12]), where Tr denotes the trace operator. By the above equalities, Corollary 3.3 and (3.2), we have

THEOREM 4.1 ([12]). *Suppose that M is compact and $\mathcal{F}$ is minimal. Then it holds that*

$$\int_M \{Ric_D(s) + Tr(A_D(s) A_D(s)) - (div_D s)^2\} \, dM = 0$$

for any $s \in \overline{V}(\mathcal{F})$, where $Ric_D(s) = g_Q(\rho_D(s), s)$.

Let ${}^t A_D(s)$ be the transpose of $A_D(s)$, that is, ${}^t A_D(s)$ satisfies the following equality:

$$g_Q(A_D(s)t, u) = g_Q(t, {}^tA_D(s)u)$$

for any $t, u \in \Gamma(Q)$.

THEOREM 4.2 ([12]). *Suppose that M is compact and $\mathcal{F}$ is minimal. Then it holds that*

$$\int_M \{Ric_D(s) - Tr({}^tA_D(s)A_D(s)) + \frac{1}{2}Tr(A_D(s) + {}^tA_D(s)) - (div_D s)^2\}\, dM = 0$$

for any $s \in \bar{V}(\mathcal{F})$.

Theorem 4.2 implies the following theorem

THEOREM 4.3 ([12]). *Suppose that M is compact and $\mathcal{F}$ is minimal. If $s \in \bar{V}(\mathcal{F})$ satisfies*

$$\Delta_D s = \rho_D(s) \text{ and } div_D s = 0$$

then s is a t.K.f. of $\mathcal{F}$.

For $s \in \bar{V}(\mathcal{F})$, let $B_D(s) : \Gamma(Q) \longrightarrow \Gamma(Q)$ be an operator defined by

$$B_D(s) = A_D(s) + {}^tA_D(s) + \frac{2}{q}(\mathrm{div}_D s) \cdot I \tag{4.3}$$

where $I : \Gamma(Q) \longrightarrow \Gamma(Q)$ denotes the identity map. The operator $B_D(s)$ is symmetric, that is,

$$g_Q(B_D(s)t, u) = g_Q(t, B_D(s)u)$$

for any $t, u \in \Gamma(Q)$.

PROPOSITION 4.4. *Let $s \in \bar{V}(\mathcal{F})$. If $B_D(s) = 0$, then s is a t.c.f. of $\mathcal{F}$.*

THEOREM 4.5. *Suppose that M is compact and $\mathcal{F}$ is minimal. If $s \in \bar{V}(\mathcal{F})$ satisfies*

$$\Delta_D s = \rho_D(s) + \left(1 - \frac{2}{q}\right) grad_D div_D s,$$

then s is a t.c.f. of $\mathcal{F}$.

Proof. We have

$$\mathrm{Tr}((B_D(s))^2) = \mathrm{Tr}((A_D(s) + {}^tA_D(s))^2) + \frac{4}{q}(\mathrm{div}_D s) \cdot \mathrm{Tr}(A_D(s) + {}^tA_D(s)) + \frac{4}{q}(\mathrm{div}_D s)^2$$

$$\mathrm{Tr}(A_D(s)+{}^tA_D(s))=-2\cdot(\mathrm{div}_D s),$$

so that we have

$$\mathrm{Tr}((B_D(s))^2)=\mathrm{Tr}((A_D(s)+{}^tA_D(s))^2)-\frac{4}{q}(\mathrm{div}_D s)^2.$$

By Theorem 4.2 and the equality: $\int_M \mathrm{Tr}({}^tA_D(s)A_D(s))dM=\langle\!\langle Ds, Ds\rangle\!\rangle$, we have

$$\langle\!\langle \rho_D(s), s\rangle\!\rangle-\langle\!\langle Ds, Ds\rangle\!\rangle+\frac{1}{2}\int_M \mathrm{Tr}((A_D(s)+{}^tA_D(s))^2)dM$$
$$-\int_M (\mathrm{div}_D s)^2 dM=0.$$

By the above equalities, we have

$$\int_M \mathrm{Tr}((B_D(s))^2)dM=0.$$

Since $B_D(s)$ is symmetric, we have $B_D(s)=0$.

Let $s\in\bar{V}(\mathcal{F})$ be a t.a.f. of $\mathcal{F}$. Then, by Proposition 2.6, $\mathrm{div}_D s$ is a constant function on M. If $\mathcal{F}$ is minimal, then $\mathrm{div}_D\ s=0$. This is proved by Corollary 3.3. Thus we have

THEOREM 4.6. *Suppose that M is compact and $\mathcal{F}$ is minimal. Then every t.a.f. of $\mathcal{F}$ is a t.K.f. of $\mathcal{F}$.*

Let $s=\pi(Y)\in\bar{V}(\mathcal{F})$ be a t.c.f. of $\mathcal{F}$ with characteristic function f_Y. If $\mathcal{F}$ is minimal and $f_Y\geq 0$ (or ≤ 0), then, by Proposition 2.6 and Corollary 3.3, $f_Y=0$. Thus we have

THEOREM 4.8. *Suppose that M is compact and $\mathcal{F}$ is minimal. Then every t.c.f. of $\mathcal{F}$ with non-positive (or non-negative) valued characteristic function is a t.K.f. of $\mathcal{F}$.*

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