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CONDITIONAL WIENER INTEGRALS AND AN INTEGRAL EQUATION

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1. Introduction

Let $(C[0, t], \beta^*, m_w)$ be the standard Wiener space; that is, $C[0, t]$ is the space of all real valued continuous functions x on $[0, t]$ with $x(0)=0$, m_w is the standard Wiener measure and β^* is the completion of the Borel σ-algebra of $C[0, t]$. Here, $t>0$ is arbitrary but fixed. Let $\boldsymbol{R}$ and $\boldsymbol{C}$ denote the real and complex numbers respectively.

J. Yeh [8] introduced the concept of conditional Wiener integral which means the conditional expectation $E^w[Y|X]$ of a real or $\boldsymbol{C}$-valued Wiener integrable function Y conditioned by a Wiener measurable function X on $C[0, t]$ which is given as a function on the value space of X (precise definition will be given in the next section) and then derived inversion formulae for conditional Wiener integrals.

Let $\theta(\cdot, \cdot)$ be a $\boldsymbol{C}$-valued Borel measurable function on $[0, t_0]\times\boldsymbol{R}$ that is bounded by the constant $M>0$ where t_0 is any positive real number. For $t\in[0, t_0]$ and $\xi\in\boldsymbol{R}$, let Y_t and X_t be Wiener measurable functions on $C[0, t]$, respectively, defined by

$$(1.1)\qquad Y_t(x)=\exp\left\{\int_0^t \theta(s, x(s)+\xi)\,ds\right\},$$
$$X_t(x)=x(t)+\xi.$$

Let us define a function on $(0, t_0]\times\boldsymbol{R}\times\boldsymbol{R}$ by

$$(1.2)\qquad U(t\,;\,\xi, \eta)=E^w[Y_t|X_t](\eta)\,(1/\sqrt{2\pi t})\exp\left\{-\frac{(\eta-\xi)^2}{2t}\right\}.$$

In Section 3, we show that by using an inversion formula (Proposition 2.2) for $E^w[Y_t|X_t]$ obtained by Yeh [7, 8], the function U given by (1.2) satisfies the integral equation

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$$(1.3)\quad U(t\,;\xi,\eta)=\frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{(\eta-\xi)^2}{2t}\right\}+\int_0^t\int_{\boldsymbol{R}}\theta(s,\zeta)\,U(s\,;\xi,\zeta)\cdot \frac{1}{\sqrt{2\pi(t-s)}}\exp\left\{-\frac{(\eta-\zeta)^2}{2(t-s)}\right\}d\zeta ds$$

for $(t,\xi,\eta)\in(0,t_0]\times\boldsymbol{R}\times\boldsymbol{R}$ and the initial condition

$$(1.4)\quad \lim_{t\to0+}U(t\,;\xi,\eta)=\delta(\eta-\xi)$$

where δ is the Dirac delta function.

In [8], J. Yeh proved this result as an alternate way of deriving the Kac–Feynman formula [5] in the case where $\xi=0$ and $\theta(\cdot,\cdot)$ is given by a real valued, non-negative continuous function V on $\boldsymbol{R}$ that is independent of s and that satisfies the condition

$$\int_{\boldsymbol{R}}V(u)\exp\left\{-\frac{u^2}{2t}\right\}du<\infty \text{ for every } t>0.$$

In Section 4, we obtain a series expansion for the solution $U(t\,;\xi,\eta)$ of the integral equation (1.3) which is absolutely and uniformly convergent on $(0,t_0]\times\boldsymbol{R}\times\boldsymbol{R}$. We also show that if θ is sufficiently smooth, then the fundamental solution of a partial differential equation is given in terms of conditional Wiener integral.

In Section 5, we apply our results to the case where $\theta(s,\cdot)$'s are given by Fourier-Stieltjes transforms of elements σ_s of $M(\boldsymbol{R})$, the space of $\boldsymbol{C}$-valued countably additive (and hence bounded) Borel measures on $\boldsymbol{R}$.

2. Definitions and preliminaries.

A real valued function on the Wiener space $(C[0,t],\beta^*,m_w)$ is called Wiener measurable if it is β^*-measurable. For a Wiener measurable function F, we write

$$E^w[F]=\int_{C[0,t]}F(x)\,dm_w(x)$$

whenever the integral exists. We say that F is Wiener integrable when the Wiener integral of F exists and is finite. The Wiener measurability and Wiener integrability of a $\boldsymbol{C}$-valued function on $C[0,t]$ are defined in terms of its real and imaginary parts.

Let X be a real valued Wiener measurable function and Y a real-valued Wiener integrable function on $C[0,t]$. The conditional Wiener integral of Y given X, written $E^w[Y|X]$, is defined as the equivalence class of real valued Borel measurable and P_X-integrable function ψ modulo null functions on $(\boldsymbol{R}, \beta(\boldsymbol{R}), P_X)$ such that

$$\int_{X^{-1}(B)} Y(x)\,dm_w(x) = \int_B \psi(\zeta)\,dP_X(\zeta)$$

for every $B \in \beta(\boldsymbol{R})$, where $\beta(\boldsymbol{R})$ denotes the Borel σ-algebra of $\boldsymbol{R}$ and P_X is the probability distribution of X defined by $P_X(B) = m_w(X^{-1}(B))$ for $B \in \beta(\boldsymbol{R})$. By the Radon-Nikodym theorem such a function ψ exists and is determined up to a null function on $(\boldsymbol{R}, \beta(\boldsymbol{R}), P_X)$. We will let $E^w[Y|X]$ denote a representative of equivalence class and so for all $B \in \beta(\boldsymbol{R})$, we have

$$\int_{X^{-1}(B)} Y(x)\,dm_w(x) = \int_B E^w[Y|X](\zeta)\,dP_X(\zeta).$$

If Z is a $\boldsymbol{C}$-valued Wiener integrable function on $C[0, t]$, i.e. $Z = Z_1 + iZ_2$, where Z_1 and Z_2 are real-valued Wiener integrable functions, then $E^w[Z|X]$ is defined as

$$E^w[Z|X] = E^w[Z_1|X] + iE^w[Z_2|X].$$

Next we state three propositions which will be used in the sequel. The first two propositions can be proved, respectively, by using Proposition 1 and Theorem 3 of [8] and the fact that every $\boldsymbol{C}$-valued function can be expressed in terms of its real and imaginary parts. The last proposition is a well-known integral formula.

PROPOSITION 2.1. *Let X be a real-valued Wiener measurable functional on $C[0, t]$ and Z a $\boldsymbol{C}$-valued Wiener integrable functional on $C[0, t]$. Let g be a $\boldsymbol{C}$-valued measurable function on $(\boldsymbol{R}, \beta(\boldsymbol{R}))$. Then*

$$E^w[(g \cdot X)Z] = \int_{\boldsymbol{R}} g(\zeta)\,E^w[Z|X](\zeta)\,dP_X(\zeta)$$

in the sense that existence of one side implies that of the other as well as the equality of two.

PROPOSITION 2.2. *Let X and Z be as in Proposition 2.1. Assume that P_X is absolutely continuous with respect to Lebesgue measure $d\zeta$ on $\boldsymbol{R}$ and that $E^w[e^{iuX}Z]$ is an integrable function of u on $\boldsymbol{R}$. Then there exists a version of $E^w[Z|X]$ such that for $\zeta \in \boldsymbol{R}$,*

$$E^w[Z|X](\zeta)\frac{dP_X}{d\zeta}(\zeta) = \frac{1}{2\pi}\int_{\boldsymbol{R}} e^{-iu\zeta} E^w[e^{iuX}Z]\,du.$$

REMARK 2.1. It is worth to note that Proposition 2.2 shows that we can always choose a version of $E^w[Z|X]$ that is a continuous function of ζ on $\boldsymbol{R}$.

PROPOSITION 2.3. *If* $\mathrm{Re}\, b>0$ *and* c *is real,*

$$\frac{1}{\sqrt{2\pi}}\int_{\boldsymbol{R}} e^{-\frac{b}{2}y^2+icy}dy=(1/b^{\frac{1}{2}})e^{-\frac{c^2}{2b}}.$$

3. The integral equation.

In this section, by applying a Yeh's inversion formula for the conditional Wienr ingtegral, we derive the integral equation given by (1.3).

THEOREM 3.1. *For* $t\in[0,t_0]$, *let* Y_t *and* X_t *be as in* (1.1). *Then there exists a version of* $E^w[Y_t|\ X_t]$ *such that the function* U *given by* (1.2) *satisfies the integral equation* (1.3) *and the initial condition* (1.4).

Proof. It is clear that Y_t is Wiener integrable for every $t\in[0,t_0]$ since

$$|Y_t(x)|\le\exp\int_0^t|\theta(r,x(r)+\xi)|dr\le e^{Mt_0}.$$

Hence $E^w[Y_t|X_t]$ exists. Since $dY_s/ds=Y_s\cdot\theta(s,x(s)+\xi)$ for a.e. $s\in[0,t]$, we have by integrating with respect to s on $[0,t]$

$Y_t=1+\int_0^t Y_s(x)\cdot\theta(s,x(s)+\xi)ds$. After multiplying this equation by e^{iuX_t} and integrating over $C[0,t]$ we obtain

$$(3.1)\quad I(u)\equiv E^w[e^{iuX_t}Y_t]=E^w[e^{iu(x(t)+\xi)}] + E^w[e^{iu(x(t)+\xi)}\int_0^t Y_s(x)\cdot\theta(s,x(s)+\xi)ds].$$

Now in order to derive the integral equation (1.3) by applying Proposition 2.2, we first need to show that $E^w[e^{iuX_t}Y_t]$ is a Lebesgue integrable function of u in $\boldsymbol{R}$.

First note that the function

$$(3.2)\qquad I_1(u)\equiv E^w[e^{iu(x(t)+\xi)}]=e^{iu\xi-\frac{t}{2}u^2}$$

is clearly a Lebesgue integrable function of u on $\boldsymbol{R}$. Now let

$$(3.3)\qquad I_2(u)\equiv E^w[e^{iu(x(t)+\xi)}\int_0^t Y_s(x)\theta(s,x(s)+\xi)ds]$$

and we will show that $I_2(u)$ is a Lebesgue integrable function of u in

$\boldsymbol{R}$. To interchange the order of Wiener integral and the integral with respect to s on $[0, t]$ in (3. 3), note that

$$|\exp\{iu(x(t)+\xi)\}\, Y_s(x)\theta(s, x(s)+\xi)\}| \leq Me^{Mt_0}$$

and that

$$\int_0^t E^w[Me^{Mt_0}]ds \leq Mt_0 e^{Mt_0} < \infty.$$

Thus the Fubini Theorem is applicable and hence we have

$$I_2(u) = \int_0^t E^w[e^{iu(x(t)+\xi)}\, Y_s(x)\theta(s, x(s)+\xi)]ds.$$

Let's write $e^{iux(t)} = e^{iu(x(t)-x(s))}e^{iux(s)}$. By the facts that $\{x(t)-x(s), x(r)\}$ is independent system of random variables on $(C[0, t], \beta^*, m_w)$ for $r \in (0, s]$ and that θ is Borel measurable on $[0, t_0] \times \boldsymbol{R}$, it can be shown [or see [6]] that $e^{iu(x(t)-x(s))}$ and $e^{iux(s))}Y_s(x)\theta(s, x(s)+\xi)$ are independent random variables on $C[0, t]$. Hence it follows that

$$I_2(u) = \int_0^t e^{-\frac{t-s}{2}u^2} E^w[e^{iu(x(s)+\xi)}\, Y_s(x)\theta(s,\ x(s)+\xi)]ds. \tag{3.4}$$

Let g be the measurable function on $(\boldsymbol{R}, \beta(\boldsymbol{R}))$ defined by $g(v) = e^{iuv}\theta(s, v)$ for fixed $u \in \boldsymbol{R}$ and $s \in (0, t_0]$. Then by Proposition 2. 1, we have

$$\begin{aligned} E^w[e^{iu(x(s)+\xi)}\, Y_s(x)\theta(s, x(s)+\xi)] &= E^w[(g \cdot X_s)\, Y_s] \\ &= \int_{\boldsymbol{R}} e^{iu\zeta}\theta(s, \zeta)\, E^w[Y_s | X_s](\zeta) \frac{dP_{X_s}}{d\zeta}(\zeta)\, d\zeta \\ &= \int_{\boldsymbol{R}} e^{iu\zeta}\theta(s, \zeta)\, U(s\ ;\ \xi, \zeta)\, d\zeta \end{aligned}$$

and so from (3. 4)

$$I_2(u) = \int_0^t e^{-\frac{t-s}{2}u^2} \int_{\boldsymbol{R}} e^{iu\zeta}\theta(s, \zeta)\, U(s\ ;\ \xi, \zeta)\, d\zeta ds. \tag{3.5}$$

To show that $I_2(u)$ is Lebesgue integrable, observe that

$$|e^{-\frac{t-s}{2}u^2} e^{iu\zeta}\theta(s, \zeta)\, U(s\ ;\ \xi, \zeta)| \leq Me^{-\frac{t-s}{2}u^2} |U(s\ ;\ \xi, \zeta)|$$

for $(s, \xi, \zeta, u) \in (0, t_0] \times \boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R}$, and that

$$\begin{aligned} \int_{\boldsymbol{R}} |U(s\ ;\ \xi, \zeta)|\, d\zeta &= \int_{\boldsymbol{R}} |E^w[Y_s | X_s](\zeta)| \frac{dP_{X_s}}{d\zeta}(\zeta)\, d\zeta \\ &\leq \int_{\boldsymbol{R}} E^w[|Y_s|\, | X_s](\zeta)\, dP_{X_s}(\zeta) \\ &= E^w[|Y_s|]. \end{aligned} \tag{3.6}$$

Now using (3.5) with (3.6) and applying the Fubini Theorem, we obtain

$$(3.7)\quad \int_{\mathbf{R}} |I_2(u)|\,du \leq \int_0^t \int_{\mathbf{R}} M e^{Mt_0} e^{-\frac{t-s}{2}u^2}\,du\,ds \leq 2\sqrt{2\pi t_0}\,M e^{Mt_0}.$$

From (3.2) and (3.7), the functions $I_1(u)$ and $I_2(u)$ are Lebesgue integrable. Hence by (3.1), $E^w[e^{iuX_t}Y_t]$ is a Lebesgue integrable function of u on $\mathbf{R}$. It follows from Proposition 2.2 and the fact $dP_{X_t}/d\eta = (1/\sqrt{2\pi t})\exp\{-(\eta-\xi)^2/2t\}$ that there exists a version of $E^w[Y_t|X_t]$ such that for $\eta \in \mathbf{R}$,

$$(3.8)\quad U(t\,;\,\xi,\eta) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{-iu\eta} E^w[e^{iuX_t}Y_t]\,du$$

$$= \frac{1}{2\pi}\int_{\mathbf{R}} e^{-iu\eta}(I_1(u)+I_2(u))\,du.$$

By using (3.2) and Proposition 2.3, we have

$$(3.9)\quad \frac{1}{2\pi}\int_{\mathbf{R}} e^{-iu\eta} I_1(u)\,du = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(\eta-\xi)^2}{2t}}$$

By using (3.5) and Fubini Theorem, we have

$$(3.10)\quad \frac{1}{2\pi}\int e^{-iu\eta} I_2(u)\,du$$

$$= \frac{1}{2\pi}\int_{\mathbf{R}} e^{-iu\eta}\int_0^t e^{-\frac{t-s}{2}u^2}\int_{\mathbf{R}} e^{iu\zeta}\theta(s,\zeta)\,U(s\,;\,\xi,\zeta)\,d\zeta\,ds\,du$$

$$= \frac{1}{2\pi}\int_0^t\int_{\mathbf{R}}\left[\int_{\mathbf{R}} e^{-iu(\eta-\zeta)} e^{-\frac{t-s}{2}u^2}\,du\right]\theta(s,\zeta)\,U(s\,;\,\xi,\zeta)\,d\zeta\,ds$$

$$= \int_0^t \frac{1}{\sqrt{2\pi(t-s)}}\int_{\mathbf{R}} e^{-\frac{(\eta-\zeta)^2}{2(t-s)}}\theta(s,\zeta)\,U(s\,;\,\xi,\zeta)\,d\zeta\,ds$$

Hence by substituting (3.9) and (3.10) into (3.8) we obtain the integral equation (1.3). The initial condition (1.4) is verified by noting that

$$\lim_{t\to 0+}\int_0^t \frac{1}{\sqrt{2\pi(t-s)}}\int_{\mathbf{R}} \theta(s,\zeta)\,U(s\,;\,\xi,\zeta)\exp\left\{-\frac{(\eta-\zeta)^2}{2(t-s)}\right\}d\zeta\;ds=0$$

and that

$$\lim_{t\to 0+}\frac{1}{\sqrt{2\pi t}} e^{-\frac{(\eta-\xi)^2}{2t}} = \delta(\eta-\xi).$$

This completes the proof of the theorem.

4. A series expansion for the solution of the integral equation.

In this section we will obtain a useful series expansion in terms of integrals over finite dimensional spaces for the solution of the integral equation (1. 3). We also show that the fundamental solution of a partial differential equation is given in terms of the conditional Wiener integral.

THEOREM 4. 1. *Let X_t and Y_t be as in* (1. 1). *The solution of the integral equation* (1. 3) *can be expressed by a series*

$$(4.1)\quad U(t\,;\,\xi,\eta)=\sum_{n=0}^{\infty}U_n(t\,;\,\xi,\eta)\quad for\ \ (t,\xi,\eta)\in(0,t_0]\times\boldsymbol{R}\times\boldsymbol{R}$$

where

$$(4.2)\qquad U_0(t\,;\,\xi,\eta)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{(\eta-\xi)^2}{2t}}$$

and for $n\geq 1$.

$$(4.3)\quad U_n(t\,;\,\xi,\eta)=\int_{\Delta_n(t)}\left[\prod_{j=1}^{n+1}2\pi(s_j-s_{j-1})^2\right]^{-\frac{1}{2}}\int_{\boldsymbol{R}^n}\prod_{j=1}^{n}\theta(s_j,v_j)\cdot \exp\left\{-\frac{1}{2}\sum_{j=1}^{n+1}\frac{(v_j-v_{j-1})^2}{s_j-s_{j-1}}\right\}d\vec{v}\,d\vec{s},$$

where $\vec{v}=(v_1,v_2,\cdots,v_n)$, $v_{n+1}=\eta$, $v_0=\xi$ *and*

$$\Delta_n(t)=\{\vec{s}=(s_1,s_2,\cdots,s_n)\in[0,t]^n;0=s_0<s_1<\cdots<s_n\leq s_{n+1}=t\}.$$

Furthermore, the series (4. 1) *converges absolutely and uniformly on* $(0,t_0]\times\boldsymbol{R}\times\boldsymbol{R}$.

Proof. We have shown that in the proof of Theorem 3. 1 there exists a version of $E^w[Y_t|X_t]$ such that for $\eta\in\boldsymbol{R}$,

$$(4.4)\quad U(t\,;\,\xi,\eta)=\frac{1}{2\pi}\int_{\boldsymbol{R}}e^{-iu\eta}E^w[e^{iu(x(t)+\xi)}\exp\int_0^t\theta(s,x(s)+\xi)ds\}]du.$$

By expanding the last exponential in the right side of (4. 4), we obtain a series expansion $U(t\,;\,\xi,\eta)=U_0(t\,;\,\xi,\eta)+U'(t\,;\,\xi,\eta)$ where

$$(4.5)\quad U_0(t\,;\,\xi,\eta)=\frac{1}{2\pi}\int_{\boldsymbol{R}}e^{-iu\eta}E^w[e^{iu(x(t)+\xi)}]du=\frac{1}{\sqrt{2\pi t}}e^{-\frac{(\eta-\xi)^2}{2t}}$$

and

$$(4.6)\qquad U'(t\,;\,\xi,\eta)=\frac{1}{2\pi}\int_{\boldsymbol{R}}e^{-iu\eta}E^w[e^{iu(x(t)+\xi)}\sum_{n=1}^{\infty}\frac{f^n(x)}{n!}]du$$

with $f^n(x)=\{\int_0^t\theta(s,x(s)+\xi)ds\}^n$. Thus to get our desired series expansion, we will work with $U'(t;\xi,\eta)$. Note that for all $k\geq 1$,

$$\left|\sum_{n=1}^{k}\frac{1}{n!}\left[\int_0^t\theta(s,x(s)+\xi)ds\right]^n\right|\leq\sum_{n=1}^{k}\frac{(Mt)^n}{n!}\leq\sum_{n=1}^{\infty}\frac{(Mt_0)^n}{n!}.$$

Hence by the dominated convergence theorem, (4.6) equals

$$(4.7)\quad \frac{1}{2\pi}\int_R e^{-iu\eta}\sum_{n=1}^{\infty}\frac{1}{n!}E^w\left[e^{iu(x(t)+\xi)}\left[\int_0^t\theta(s,x(s)+\xi)ds\right]^n\right]du$$
$$=\frac{1}{2\pi}\int_R e^{-iu\eta}\sum_{n=1}^{\infty}\frac{1}{n!}\int_{[0,t]^n}E^w\left[e^{iu(x(t)+\xi)}\prod_{j=1}^{n}\theta(s_j,x(s_j)+\xi)\right]d\vec{s}\ du$$
$$=\frac{1}{2\pi}\int_R e^{-iu\eta}\sum_{n=1}^{\infty}\int_{\Delta_n(t)}E^w\left[e^{iu(x(t)+\xi)}\prod_{j=1}^{n}\theta(s_j,x(s_j)+\xi)\right]d\vec{s}\ du$$

where the first equality is justified by the Fubini Theorem. Now applying a basic Wiener integral formula we see that

$$(4.8)\quad E^w[e^{iu(x(t)+\xi)}\prod_{j=1}^{n}\theta(s_j,x(s_j)+\xi)]$$
$$=\int_{R^{n+1}}e^{iu(w_{n+1}+\xi)}\prod_{j=1}^{n}\theta(s_j,w_j+\xi)\left[\prod_{j=1}^{n+1}2\pi(s_j-s_{j-1})\right]^{-\frac{1}{2}}\cdot$$
$$\exp\left\{-\frac{1}{2}\sum_{j=1}^{n+1}\frac{(w_j-w_{j-1})^2}{s_j-s_{j-1}}\right\}dw_{n+1}\ d\vec{w}$$

where $\vec{w}=(w_1,w_2,\cdots,w_n)$, $s_{n+1}=t$ and $w_0=s_0=0$. Next by applying Proposition 2.3, we integrate first with respect to w_{n+1}. Then we obtain

$$(4.9)\quad \int_{R^n}\left[\prod_{j=1}^{n}2\pi(s_j-s_{j-1})\right]^{-\frac{1}{2}}\prod_{j=1}^{n}\theta(s_j,w_j+\xi)\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(w_j-w_{j-1})^2}{s_j-s_{j-1}}\right.$$
$$\left.+iu(w_n+\xi)-\frac{t-s_n}{2}u^2\right\}d\vec{w}.$$

Now making substitution $v_j=w_j+\xi$ for $j=1,2,\cdots,n$ in (4.9), we see that (4.9) equals

$$(4.10)\quad \int_{R^n}\left[\prod_{j=1}^{n}2\pi(s_j-s_{j-1})\right]^{-\frac{1}{2}}\prod_{j=1}^{n}\theta(s_j,v_j)\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(v_j-v_{j-1})^2}{s_j-s_{j-1}}\right.$$
$$\left.+iv_nu-\frac{t-s_n}{2}u^2\right\}d\vec{v},$$

where $v_0=\xi$. Next we want to interchange the order of the integral with respect to du and the summation in (4.7). We can justify doing this by finding a Lebesgue integrable function of u on $\boldsymbol{R}$ which dominates the series

$$(4.11)\quad \sum_{n=1}^{\infty}\left|\int_{\Delta_n(t)}\left[\prod_{j=1}^{n}2\pi(s_j-s_{j-1})\right]^{-\frac{1}{2}}\int_{\mathbf{R}^n}\prod_{j=1}^{n}\theta(s_j,v_j)\cdot \exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(v_j-v_{j-1})^2}{s_j-s_{j-1}}+iv_nu-\frac{t-s_n}{2}u^2\right\}d\vec{v}\ d\vec{s}\right|.$$

But it can be shown that the series (4.11) is dominated by the function

$$(4.12)\quad \left[\sum_{n=1}^{\infty}\frac{(Mt_0)^n}{n!}\cdot\frac{1}{t_0}\right]\int_0^t e^{-\frac{t-s}{2}u^2}ds,$$

which is clearly a Lebesgue integrable function of u in $\mathbf{R}$. Hence by the dominated convergence theorem, we have, from (4.7),

$U'(t\,;\,\xi,\eta)=\sum_{j=1}^{\infty}U_n(t\,;\,\xi,\eta)$ where

$$(4.13)\quad U_n(t\,;\,\xi,\eta)=\frac{1}{2\pi}\int_{\mathbf{R}}e^{-iu\eta}\int_{\Delta_n(t)}\left[\prod_{j=1}^{n}2\pi(s_j-s_{j-1})\right]^{-\frac{1}{2}}\int_{\mathbf{R}^n}\prod_{j=1}^{n}\theta(s_j,v_j)\cdot \exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(v_j-v_{j-1})^2}{s_j-s_{j-1}}+iv_nu-\frac{t-s_n}{2}u^2\right\}d\vec{v}\ d\vec{s}\ du$$

with $v_0=\xi$. Now by applying Proposition 2.3, we integrate the right hand side of (4.13) with respect to u to obtain

$$(4.14)\quad U_n(t\,;\,\xi,\eta)=\int_{\Delta_n(t)}\left[\prod_{j=1}^{n+1}2\pi(s_j-s_{j-1})\right]^{-\frac{1}{2}}\int_{\mathbf{R}^n}\prod_{j=1}^{n}\theta(s_j,v_j)\cdot \exp\left\{-\frac{1}{2}\sum_{j=1}^{n+1}\frac{(v_j-{}_{j-1})^2}{s_j-s_{j-1}}\right\}d\vec{v}\ d\vec{s}$$

where $v_{n+1}=\eta$, $v_0=\xi$, $s_{n+1}=t$ and $s_0=0$. Hence by making use of (4.5) and (4.14) we obtain the series expansion in the theorem. Furthermore, observe that for $n\geq 1$,

$$|U_n(t\,;\,\xi,\eta)|\leq\frac{(Mt_0)^n}{n!}2(\sqrt{2\pi t_0})^{-1}$$

and so the series $U(t\,;\,\xi,\eta)=\sum_{n=0}^{\infty}U_n(t\,;\,\xi,\eta)$ converges absolutely and uniformly on $(0,t_0]\times\mathbf{R}\times\mathbf{R}$.

The following corollary shows that the sequence $\{U_n\}$ for the solution of the integral equation (1.3) can be obtained recursively.

COROLLARY 4.2. *For* $n=0,1,2,\cdots$ *and* $(t,\xi,\eta)\in(0,t_0]\times\mathbf{R}\times\mathbf{R}$,

$$(4.15)\quad U_{n+1}(t\,;\,\xi,\eta)=\int_0^t\frac{1}{\sqrt{2\pi(t-s)}}\int_{\mathbf{R}}\exp\left\{-\frac{(\eta-\zeta)^2}{2(t-s)}\right\}\theta(s,\zeta)\cdot U_n(s\,;\,\xi,\zeta)d\zeta\,ds$$

where $U_n(t\,;\,\xi,\eta)$*'s are given by* (4.2) *and* (4.3).

Proof. The proof can be easily done by using the induction on n and the Fubini Theorem.

Let $\theta(\cdot,\cdot)$ be as in Section 1. For $t\in[0,t_0]$ and $\xi\in\boldsymbol{R}$, let X_t and F_t be the functionals on $(C[0,t_0],\beta^*,m_w)$, respectively, defined by

(4.16) $$X_t(x)=x(t)+\xi,$$
$$F_t(x)=\exp\left\{\int_0^t\theta(t-s,x(s)+\xi)\,ds\right\}.$$

If θ satisfies the following smoothness condition:

(4.17) the first derivatives θ_t and θ_ξ in $(0,t_0)\times\boldsymbol{R}$ are continuous and have the properties, $|\theta_t(t,\xi)|\le B_1\exp\{B_2|\xi|\}$ and $|\theta_\xi(t,\xi)|\le B_1\exp\{B_2|\xi|\}$ for some B_1, B_2,

then from the Theorem 4 of [1] the function G on $(0,t_0]\times\boldsymbol{R}$ given by

$$G(t,\xi)=\int_{C[0,t]}F_t(x)\psi(x(t)+\xi)\,dm_w(x)$$

is a solution of a partial differential equation (modified heat equation):

(4.18) $$\frac{\partial G}{\partial t}=\frac{1}{2}\frac{\partial^2 G}{\partial\xi^2}+\theta(t,\xi)G \text{ in } (0,t_0)\times\boldsymbol{R}$$

with the initial condition

(4.19) $$\lim_{t\to 0+}G(t,\xi)=\psi(\xi)$$

for every $\xi\in\boldsymbol{R}$ where ψ is a $\boldsymbol{C}$-valued continuous function such that $\psi(\eta)\exp\{-\alpha\eta^2\}$ is Lebesgue integrable on $\boldsymbol{R}$ for each α satisfying $\alpha>\dfrac{1}{4t_0}$.

Remark 4.1. The condition on ψ implies that $G(t,\xi)$ exists and is finite for each $(t,\xi)\in(0,t_0]\times\boldsymbol{R}$: Indeed, note that

$$\exp\left\{-\frac{(\eta-\xi)^2}{2t}\right\}=\exp\left\{-\frac{(\eta/2-2\xi)^2}{2t}-\frac{3\eta^2}{8t}+\frac{3\xi^2}{2t}\right\}$$
$$\le\exp\left\{-\frac{3\eta^2}{8t}+\frac{3\xi^2}{2t}\right\}.$$

Hence for each $(t,\xi)\in(0,t_0]\times\boldsymbol{R}$, we have

$$|G(t,\xi)|\le e^{Mt_0}\frac{1}{\sqrt{2\pi t}}\int_{\boldsymbol{R}}|\psi(\eta)|e^{-\frac{(\eta-\xi)^2}{2t}}\,d\eta<\infty$$

since $\frac{3}{8t} > \frac{1}{4t_0}$.

The following theorem shows that if θ satisfies the smoothness condition (4.17), then the function Γ on $(0, t_0] \times \boldsymbol{R} \times \boldsymbol{R}$ defined by

$$(4.20) \quad \Gamma(t\,;\,\eta, \xi) = E^w[F_t | X_t](\eta)\,(1/\sqrt{2\pi t})\,\exp\left\{-\frac{(\eta-\xi)^2}{2t}\right\}$$

is the fundamental solution of the partial differential equation (4.18) with (4.19).

THEOREM 4.3. *Let X_t and F_t be as in (4.16) and let $\phi(\eta)$ be a $\boldsymbol{C}$-valued function such that $\phi(\eta)\exp\{-\alpha\eta^2\}$ is Lebesgue integrable on $\boldsymbol{R}$ for all $\alpha > \frac{1}{4t_0}$. Then*

$$(4.21) \quad \int_{C[0,t]} F_t(x)\phi(x(t)+\xi)\,dm_w(x) = \int_{\boldsymbol{R}} \phi(\eta)\,\Gamma(t\,;\,\eta, \xi)\,d\eta$$

and integrals on both sides exist. Furthermore $\Gamma(t\,;\,\eta, \xi)$ has a series expansion given by the series (4.1) with $v_{n+1} = \xi$ and $v_0 = \eta$.

Proof. Let $g(\eta) = \phi(\eta)$. Then by Proposition 2.1,

$$\begin{aligned}
&\int_{C[0,t]} \exp\left\{\int_0^t \theta(t-s,\, x(s)+\xi)\,ds\right\}\phi(x(t)+\xi)\,dm_w(x) \\
&= E^w[(\phi\cdot X_t)F_t] \\
&= \int_{\boldsymbol{R}} \phi(\eta)\,E^w[F_t|X_t](\eta)\frac{1}{\sqrt{2\pi t}}e^{-\frac{(\eta-\xi)^2}{2t}}\,d\eta \\
&= \int_{\boldsymbol{R}} \phi(\eta)\,\Gamma(t\,;\,\eta, \xi)\,d\eta.
\end{aligned}$$

By Remark 4.1, integrals of both sides in (4.21) exist. By noting the fact that

$$\int_0^t \theta(t-s,\, x(s)+\xi)\,ds = \int_0^t \theta(s,\, x(t-s)+\xi)\,ds,$$

we can obtain a series expansion for $\Gamma(t\,;\,\eta, \xi)$ from the series (4.1) by interchanging the starting point ξ at $s=0$ and the ending poing η at $s=t$.

REMARK 4.2. We may say that $E^w[F_t|X_t](\eta)$ means the expectation of F_t over the spaces of continuous functions $y(s) = x(t-s)+\xi$ starting at η at time $s=0$ and ending at ξ at time $s=t$, where x is in $C[0, t]$.

5. An application.

Let $\mathcal{Q}$ be the set of all $\boldsymbol{C}$-valued functions on $[0, t_0]\times\boldsymbol{R}$ of the form

$$\theta(s, u)=\int_{\boldsymbol{R}}\exp(iuv)\,d\sigma_s(v) \tag{5.1}$$

where $\{\sigma_s\,;\,0\leq s\leq t_0\}$ is a family from $M(\boldsymbol{R})$ satisfying the following two conditions:

(5.1a) For every $B\in\beta(\boldsymbol{R})$, $\sigma_s(B)$ is Borel measurable in s.

(5.1b) There exists $K>0$ such that $\|\sigma_s\|\leq K$ for all s in $[0, t_0]$, where $\|\sigma_s\|$ is the total variation of σ_s.

REMARK 5.1. (1) Any function θ in $\mathcal{Q}$ is $\boldsymbol{C}$-valued, Borel measurable and bounded on $[0, t_0]\times\boldsymbol{R}$ (see [4]). We note that $\mathcal{Q}$ is an important class of functions in the theory of Feynman integral (see, for examples, [2], [4]).

(2) Let $\mu\in M(\boldsymbol{R})$ and define $\sigma_s\equiv T_{1+s}\mu$ where $(T_{1+s}\mu)(B)=\mu\left(\frac{1}{1+s}B\right)$ for $B\in\beta(\boldsymbol{R})$. Then it can be shown (or see Lemma 2.3 [3]) that for every $B\in\beta(\boldsymbol{R})$, $\sigma_s(B)$ is a Borel measurable function of s in $[0, t_0]$. Since $\|\sigma_s\|=\|\mu\|<\infty$ for all $s\in[0, t_0]$, $\{\sigma_s\,;\,0\leq s\leq t_0\}$ is a family in $M(\boldsymbol{R})$ satisfying the conditions (5.1a, b).

THEOREM 5.1. *Let $\theta\in\mathcal{Q}$ be given by* (5.1) *and X_t and Y_t be as in* (1.1). *Then the function U given by* (1.2) *is a solution of the integral equation* (1.3) *with* (1.4). *A series expansion for the solution $U(t\,;\,\xi, \eta)$ can be expressed by $U(t\,;\,\xi, \eta)=\sum_{n=0}^{\infty}U_n(t\,;\,\xi, \eta)$ where*

$$U_0(t\,;\,\xi, \eta)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{(\eta-\xi)^2}{2t}} \tag{5.2}$$

and for $n\geq 1$

$$\begin{aligned} U_n(t, \xi, \eta)=\frac{1}{\sqrt{2\pi t}}\int_{\Delta_n(t)}\int_{\boldsymbol{R}^n}\exp\Big\{&-\frac{1}{2}\sum_{l=1}^{n}\sum_{j=1}^{l}(2-\delta_{j,l})\,y_jy_ls_j \\ &+i\xi\sum_{j=1}^{n}y_j+\frac{1}{2t}\Big[\sum_{j=1}^{n}y_js_j+i(\eta-\xi)\Big]^2\Big\} \\ &d\sigma_{s_1}(y_1)\,d\sigma_{s_2}(y_2)\cdots d\sigma_{s_n}(y_n)\,d\vec{s} \end{aligned} \tag{5.3}$$

where $\delta_{j,l}$ denotes the Kronecker delta.

Proof. Since θ is $\boldsymbol{C}$-valued, bounded Borel measurable on $[0, t_0]\times\boldsymbol{R}$, by Theorem 3.1 $U(t\,;\,\xi, \eta)$ is a solution of integral equation (1.3) with (1.4). By Theoem 4.1 the solution $U(t\,;\,\xi,\eta)$ has a series expansion $U(t\,;\,\xi,\eta)=\sum_{n=0}^{\infty}U_n(t\,;\,\xi,\eta)$, where $U_0(t\,;\,\xi,\eta)$ and $U_n(t\,;\,\xi,\eta)$ are given as in (4.2) and (4.3), respectively.

To get our desired series expansion, we will work with $U_n(t\,;\,\xi,\eta)$ given in (4.3). Now by noting that

$$\frac{1}{2\pi}\int_{R}\exp\left\{-iu\eta+iuv_n-\frac{t-s_n}{2}u^2\right\}du=[2\pi(t-s_n)]^{-\frac{1}{2}}e^{-\frac{(\eta-v_n)^2}{2(t-s_n)}},$$

and that $\prod_{j=1}^{n}\theta(s_j,v_j)=\int_{R^n}\exp\left\{i\sum_{j=1}^{n}y_jv_j\right\}d\sigma_{s_1}(y_1)\cdots d\sigma_{s_n}(y_n)$, we can rewrite $U_n(t\,;\,\xi,\eta)$ as

$$\text{(5.4)}\quad \int_{\Delta_n(t)}\left[\prod_{j=1}^{n}2\pi(s_j-s_{j-1})\right]^{-\frac{1}{2}}\int_{R^n}\int_{R}\frac{1}{2\pi}\int_{R^n}e^{-iu\eta-\frac{t-s_n}{2}u^2}\cdot \exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(v_j-v_{j-1})^2}{s_j-s_{j-1}}+i\sum_{j=1}^{n}y_jv_j+iv_nu\right\} d\sigma_{s_1}(y_1)\,d\sigma_{s_2}(y_2)\cdots d\sigma_{s_n}(y_n)\,du\,d\vec{v}\,d\vec{s}.$$

Now one can easily justify interchanging the order of integrations with respect to $d\sigma_{s_1}(y_1)d\sigma_{s_2}(y_2)\cdots d\sigma_{s_n}(y_n)$ and $d\vec{v}$. By doing this and carrying out the integrations with respect to v_n, then with respect to $v_{n-1},\cdots$, and finally with respect to v_1, we see that (5.4) equals

$$\text{(5.5)}\quad \int_{\Delta_n(t)}\int_{R^n}\frac{1}{2\pi}\int_{R}e^{-iu\eta}\exp\Big[-\frac{1}{2}\{(t-s_n)u^2+(s_n-s_{n-1})(u+y_n)^2 +\cdots+(s_1-s_0)(u+y_1+\cdots+y_n)^2\}+i\xi(u+y_1+\cdots+y_n)\Big] du\,d\sigma_{s_1}(y_1)\,d\sigma_{s_2}(y_2)\cdots d\sigma_{s_n}(y_n)\,ds_1ds_2\cdots ds_n$$
$$=\int_{\Delta_n(t)}\int_{R^n}\frac{1}{2\pi}\int_{R}e^{-iu\eta}\exp\Big[-\frac{1}{2}\sum_{l=1}^{n}\sum_{j=1}^{l}(2-\delta_{j,l})y_jy_ls_j +i\xi(y_1+\cdots+y_n)\Big\}e^{i\xi u-\frac{t}{2}u^2+\left(\sum_{j=1}^{n}y_js_j\right)u}\,du\,d\sigma_{s_1}(y_1)\cdots d\sigma_{s_n}(y_n)\,d\vec{s},$$

where $s_0=0$.

Now carrying out integration with respect to du, we see that

$$\text{(5.6)}\quad U_n(t\,;\,\xi,\eta)=\frac{1}{\sqrt{2\pi t}}\int_{\Delta_n(t)}\int_{R^n}\exp\left\{-\frac{1}{2}\sum_{l=1}^{n}\sum_{j=1}^{l}(2-\delta_{j,l})y_jy_ls_j +\frac{1}{2t}\left[\sum_{j=1}^{n}s_jy_j\right]^2-\frac{(\eta-\xi)^2}{2t}+i\frac{1}{t}(\eta-\xi)\sum_{j=1}^{n}s_jy_j+i\xi\sum_{j=1}^{n}y_j\right\} d\sigma_{s_1}(y_1)\,d\sigma_{s_2}(y_2)\cdots d\sigma_{s_n}(y_n)\,d\vec{s}.$$

Since (5.6) equals (5.3), we obtain the series in the theorem.

COROLLARY 5.2. *Let $\theta \in \mathcal{G}$ be given by* (5.1) *and Y_t be as in* (1.1). *Then $E^W[Y_t]$ is given by the series*

$$1+\sum_{n=1}^{\infty}\int_{\Delta_n(t)}\int_{R^n}\exp\left\{-\frac{1}{2}\sum_{l=1}^{n}\sum_{j=1}^{l}(2-\delta_{j,l})y_jy_ls_j+i\xi\sum_{j=1}^{n}y_j\right\}$$
$$d\sigma_{s_1}(y_1)d\sigma_{s_2}(y_2)\cdots d\sigma_{s_n}(y_n)d\vec{s}.$$

Proof. In the expression (5.6), one can show that

$$\frac{1}{2}\left[\sum_{l=1}^{n}\sum_{j=1}^{l}(2-\delta_{j,l})y_jy_ls_j-\frac{1}{t}\left[\sum_{j=1}^{n}s_jy_j\right]^2\right]>0 \text{ for all } n\geq 1.$$

So we have

$$\sum_{n=0}^{\infty}|U_n(t\,;\,\xi,\eta)|\leq\left[\sum_{n=0}^{\infty}\frac{(Mt_0)^n}{n!}\right]\frac{1}{\sqrt{2\pi t}}e^{-\frac{(\eta-\xi)^2}{2t}}, \tag{5.7}$$

so that $\sum_{n=0}^{\infty}|U_n(t\,;\,\xi,\eta)|$ is dominated by a Lebesgue integrable function of η. Hence

$$\begin{aligned}E^w[Y_t]&=\int_R E^w[Y_t|X_t](\eta)\frac{dP_{X_t}}{d\eta}(\eta)d\eta\\&=\int_R\sum_{n=0}^{\infty}U_n(t\,;\,\xi,\eta)d\eta\\&=\sum_{n=0}^{\infty}\int_R U_n(t\,;\,\xi,\eta)d\eta\\&=\int_R\frac{1}{\sqrt{2\pi t}}e^{-\frac{(\eta-\xi)^2}{2t}}d\eta+\end{aligned}$$

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{2\pi t}}\int_{\Delta_n(t)}\int_{R^n}\int_R\exp\left\{-\frac{1}{2}\sum_{l=1}^{n}\sum_{j=1}^{l}(2-\delta_{j,l})y_jy_ls_j+i\xi\sum_{j=1}^{n}y_j\right.$$
$$\left.+\frac{1}{2t}\left[\sum_{j=1}^{n}s_jy_j\right]^2-\frac{1}{2t}(\eta-\xi)^2+i\frac{1}{t}(\eta-\xi)\sum_{j=1}^{n}y_js_j\right\}d\eta d\sigma_{s_1}(y_1)\cdots d\sigma_{s_n}(y_n)d\vec{s}$$
$$=1+\sum_{n=1}^{\infty}\int_{\Delta_n(t)}\int_{R^n}\exp\left\{-\frac{1}{2}\sum_{l=1}^{n}\sum_{j=1}^{l}(2-\delta_{j,l})y_jy_ls_j+i\xi\sum_{j=1}^{n}y_j\right\}$$
$$d\sigma_{s_1}(y_1)d\sigma_{s_2}(y_2)\cdots d\sigma_{s_n}(y_n)d\vec{s},$$

where the third equality is justified by (5.7) and the fourth equality by the Fubini Theorem.

COROLLARY 5.3. *Let $\theta \in \mathcal{G}$ be given by* (5.1) *and F_t and X_t as in* (4.16). *Then the function $\Gamma(t\,;\,\eta,\xi)$ given by* (4.20) *has a series expansion given by $\sum_{n=0}^{\infty}\Gamma_n(t\,;\,\eta,\xi)$ where*

(5.8) $$\Gamma_0(t\ ;\ \eta, \xi)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{(\eta-\xi)^2}{2t}}$$

and

(5.9) $$\Gamma_n(t\ ;\ \eta, \xi)=\frac{1}{\sqrt{2\pi t}}\int_{\Delta_n(t)}\int_{R^n}\exp\Big\{-\frac{1}{2}\sum_{l=1}^{n}\sum_{j=1}^{l}(2-\delta_{j,l})y_jy_l(t-s_l) + \frac{1}{2t}\Big[\sum_{j=1}^{n}y_j(t-s_j)+i(\eta-\xi)\Big]^2+i\xi\sum_{j=1}^{n}y_j\Big\} d\sigma_{s_1}(y_1)d\sigma_{s_2}(y_2)\cdots d\sigma_{s_n}(y_n)d\vec{s}.$$

Proof. For a fixed $t\in[0, t_0]$, let $\sigma_s{}^*=\sigma_{t-s}$ and $\theta^*(s, \cdot)=\theta(t-s, \cdot)$ for $s\in[0, t_0]$. Then $\theta^*(s, u)=\int_R \exp\{iuv\}d\sigma_s{}^*(v)$. Since $\{\sigma_s{}^* : 0\leq s\leq t_0\}$ satisfies the conditions (5.1a, b) [Prop. 3.2 of [4]] and $F_t(x)=\exp\Big\{\int_0^t\theta^*(s, x(s)+\xi)ds\Big\}$, it follows from Theorem 5.1 that $\Gamma(t\ ;\ \eta, \xi)$ has a series expansion $\sum_{n=0}^{\infty}\Gamma_n(t\ ;\ \eta, \xi)$ where $\Gamma_0(t\ ;\ \eta, \xi)$ is given as (5.2) and $\Gamma_n(t\ ;\ \eta, \xi)$ is given as

(5.10) $$\frac{1}{\sqrt{2\pi t}}\int_{\Delta_n(t)}\int_{R^n}\exp\Big\{-\frac{1}{2}\sum_{l=1}^{n}\sum_{j=1}^{l}(2-\delta_{j,l})y_jy_ls_j+i\xi\sum_{j=1}^{n}y_j+ \frac{1}{2t}\Big[\sum_{j=1}^{n}y_js_j+i(\eta-\xi)\Big]^2\Big\}d\sigma_{t-s_1}(y_1)d\sigma_{t-s_2}(y_2)\cdots d\sigma_{t-s_n}(y_n)d\vec{s}.$$

If we set $s_j{}'=t-s_{n+1-j}$ and $y_j{}'=y_{n+1-j}$, we obtain

$$\Gamma_n(t\ ;\ \eta, \xi)=\frac{1}{\sqrt{2\pi t}}\int_{\Delta_n(t)}\int_{R^n}\exp\Big\{-\frac{1}{2}\sum_{l=1}^{n}\sum_{j=1}^{l}(2-\delta_{j,l})y'_{n+1-j}y'_{n+1-l}(t-s'_{n+1-j}) +i\xi\sum_{j=1}^{n}y'_{n+1-j}+\frac{1}{2t}\Big[\sum_{j=1}^{n}y'_{n+1-j}(t-s'_{n+1-j})+i(\eta-\xi)\Big]^2 d\sigma_{s_n'}(y_n')d\sigma_{s'_{n-1}}(y'_{n-1})\cdots d\sigma_{s_1'}(y_1')ds_1'ds_2'\cdots ds_n'$$
$$=\frac{1}{\sqrt{2\pi t}}\int_{\Delta_n(t)}\int_{R^n}\exp\Big\{-\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{i}(2-\delta_{i,k})y_i'y_k'(t-s_i') +i\xi\sum_{k=1}^{n}y_k'+\frac{1}{2t}\Big[\sum_{k=1}^{n}y_k'(t-s_k')+i(\eta-\xi)\Big]^2 d\sigma_{s_1'}(y_1')d\sigma_{s_2'}(y_2')\cdots d\sigma_{s_n'}(y_n')ds_1'ds_2'\cdots ds_n'.$$

Hence this completes the proof of the corollary.

REMARK 5.2 Let $\{\sigma_s : 0\leq s\leq t_0\}$ be a family in $M(\boldsymbol{R})$ such that the corresponding $\boldsymbol{C}$-valued Borel measurable function θ on $[0, t_0]\times\boldsymbol{R}$ satisfies the smoothness condition (4.17). Then Theorem 4.3 shows that $\Gamma(t\,;\eta,\xi)$ is the fundamental solution of the partial differential equation (4.18) with (4.19). We note that such a family $\{\sigma_s : 0\le s\le t_0\}$ exists; for example, $\{\sigma_s = T_{1+s}\mu : 0\le s\le t_0\}$, where μ is the Gaussian measure on $\boldsymbol{R}$ of which Fourier transform is given by $e^{-\frac{u^2}{2}}$.

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