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New Signature Invariant of Higher Dimensional Links

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ABSTRACT. We develope a signature invariant for odd higher dimensional links. This signature has an advantage that it is defined as a G-signature for a non-abelian group G so that it can distinguish two links whose different were not detected by other invariants defined on commutative set-ups.

We obtain a signature link cobordism invariant for some higher dimensional links. One of authors proposed an example of the boundary link that is not boundary slice in [**K**]. Dute to its construction, all signature invariants of the link over abelian group rings should vanish, and hence it has not yet been known whether the link is slice.

Therefore one needs to have a signature invariant over non abelian group rings. After the fact that Orr's invariants vanish for all higher demensional links is known [**O**, **C**], the obvious condidates of the non abelian groups considered are nilpotent groups. And for the pratical purpose of calculations, we choose to use finite nilpotent groups. In order to compute signatures that we will develope in this paper,we have to use the G-signature theorem and representation theories. Their actual computations for specific boundary links will be done in other papers.

An *m-link* $L = K_1 \cup \cdots \cup K_m$ in S^{n+1} is an oriented, locally flat, codimension two submanifold whose each component K_i is homeomorphic to S^n. A *cobordism* of a link L_0 to a link L_1 is a submanifold C in $S^{n+2} \times [0,1]$ homeomorphic to $S^n \times [0,1]$ and intersecting $S^{n+2} \times i$ transversely at L_i, $i = 0,1$. Links L_0, L_1 are *cobordant* if there is a cobordism between them. A link L is *sliced* if it is cobordant to the trivial link. Since the trivial m-link bounds m disjoint disks, a sliced link bounds m disjoint disks in B^{n+3} bounded by the ambient sphere S^{n+2}. Let T be a tubular neighborhood of L in S^{n+2}. X denotes the

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exterior of L, that is, the closure of $S^{n+2} - T$. The link group $\pi_1(X)$ is denoted by π. A *meridian map* $\mu : \bigvee^m S^1 \to X$ is an embedding sending the i-th copy of S^1 to a loop consisted of a fiber of the tubular neighborhood of K_i and a path joined to the base point of X.

A *boundary m-link* L in S^{n+2} is an m-link which has m disjoint, oriented, locally flat submanifold $V = V_1 \cup \cdots V_m$ in S^{n+1} such that $\partial V_i = L_i$, the i-th component of L. An m-link L is a boundary link if and only if there is a surjection $\theta : \pi \to F$ such that $\theta\mu = id_F$ for some meridian map μ (see [**K**]).

In this paper we only consider higher dimensional links, i.e. $n \geq 2$.

By the Alexander duality, $H_1(X) \cong H^n(L) \cong \mathbf{Z}^m$ and $H_2(X) \cong H^{n-1}(L) = 0$. The Hopf sequence gives a surjection $H_2(X) \to H_2(\pi)$ and hence $H_2(\pi) = 0$.

For an group G, G_k denotes the k-th term of the lower central series of G, that is, $G_1 = G$, $G_i = [G, G_{i-1}]$. And $K(G,1)$ is the Eilenberg-MacLane space, i.e., a CW-comples with $\pi_1(K(G,1)) = G$ and $\pi_i(K(G,1)) = 0, i \geq 2$.

A meridian map $\mu : F \to \pi$ incuces an isomorphism $H_1(F) \to H_1(\pi)$ and $H_2(X) = 0$. By a Stalling's theorem[**S**], it induces isomorphisms $F/F_k \to \pi/\pi_k$ for all finite k. Thus μ induces homotopy equivalences $\bar{\mu}_k : K(F/F_k, 1) \to K(\pi/\pi_k, 1)$. Let $\phi_k : X \to K(F/F_k, 1)$ be the composition of the map $X \to K(\pi/\pi_k, 1)$ realizing the quotient map $\pi \to \pi/\pi_k$ with the homotopy inverse of $\bar{\mu}$.

The restriction $\phi_k|_{\partial X}$ is assumed to send the i-th component $K_i \times S^1$ to the i-th circle in $K(F/F_k, 1)$. Let $\bar{K}_k$ be the CW-complex obtained from $K(F/F_k, 1)$ by attaching m disks D^2 along m circles in $K(F/F_k, 1)$. By putting the tubular neighborhood $L \times D^2$ back to X, ϕ_k extends to a map $\bar{\phi}_k : S^{n+1} \to \bar{K}_k$ such that $\bar{\phi}_k^{-1}$(the centers of m disk D^2)$= L$.

Orr and Cochran showe that $\bar{\phi}_k$ is null-homotopic in [**O**, **C**]. In fact, the homotopy class $[\bar{\phi}_k] \in \pi_{n+2}(\bar{K}_k), k \geq 2$ was intended and proven to be an invariant under link cobordisms. But they all vanish probably except one defined on the infinite stage.

A pleasant consequence, however, follows from these vanishings.

LEMMA. *Let $L = K_1 \cup \cdots \cup K_m$ be a link in S^{n+2}, $n \geq 2$ and let B^{n+3} be the ball bounded by the ambient sphere S^{n+2}. Then there* *are m disjoint $(n+1)$-dimensional manifolds $V_1, \cdots, V_m$ in B^{n+3} such that for each $k \geq 2$,*

(1) $V_i \cap S^{n+2} = \partial V_i = K_i$;

(2) $\phi_k : X \to K(F/F_k, 1)$ *extends to* $\psi_k : Y \to K(F/F_k, 1)$ *where* ϕ_k *is defined above and* Y *is an exterior of* $\bigcup_i V_i$ *in* B^{n+3} *with* $\partial Y = X$;

(3) $\psi_k|_{\partial Y}$ *sends the* i*-th component* $V_i \times S^1$ *to the* i*-th circle of* $K(F/F_k, 1)$.

PROOF. Since $\bar{\phi}_k$ is null homotopic, it extends to $\bar{\psi}_k : B^{n+3} \to \bar{K}_k$. After perturbing $\bar{\psi}_k$ so that it is transverse to the centers of 2-disks in $\bar{K}_k$, let

$$V_i = \bar{\psi}_k^{-1}(\text{the center of the } i\text{-th 2-disk}).$$

Then (1) follows. The restriction $\psi_k = \bar{\psi}_k|_Y$ satisfies (2) and (3).

From now on, G denotes a finite nilpotent group. Let L be an m-link in S^{n+2} with the property that there is a homomorphism $f_* : \pi \to G$ and let $f : X \to K(G, 1)$ be the realization of f_*. For example, every boundary link has this property for any finite nilpotent group G since there is a homomorphism θ from π to the free group F.

Since $G/G_k \cong g$ for some $k \geq 2$ and $\pi/\pi_k \cong F/F_k$, the homomorphism $f_* : \pi \to G$ factors through F/F_k and $f : X \to K(G, 1)$ factors through $\phi_k : X \to K(F/F_k, 1)$. Thus we have the following as a corollary of Lemma.

THEOREM 1. *Let L a higher dimensional m-link with a map $f : X \to K(G, 1)$ for a finite nilpotent group G. Then there are m disjoint $(n+1)$-dimensional manifolds $V_1, \cdots, V_m$ in B^{n+3} such that*

(1) $V_i \cap S^{n+2} = \partial V_i = K_i$;

(2) f *extends to* $g : Y \to K(G, 1)$ *where* Y *is an exterior of* $\bigcup_i V_i$ *in* B^{n+3} *with* $\partial Y = X$;

(3) $g|_{\partial Y}$ *factor through the projection sending the* i*-th component* $V_i \times S^1$ *to the* i*-th circle of* $K(F/F_k, 1)$.

PROOF. $V_1, \cdots, V_m$ are obtained by the transversality of $\bar{\psi}_k$ where k is the nilpotent degree of G.

Let $p : \widetilde{Y} \to Y$ be the covering of Y associated to $g_* : \pi_1(Y) \to G$, i.e., G acts on $\widetilde{Y}$ as deck transformation. The restiction of p to the i-th component of $\partial \widetilde{Y}$ looks like a product of a cyclic covering $S^1 \to S^1$

with the identity $V_i \to V_i$. The degree of the cyclic cover is equal to the order of the element $f_*(i\text{-meridian})$ in G. Thus $\widetilde{Y}$ can be filled with standard branched coverings (branched G-bundle over D^2) $\times V_i \to D^2 \times V_i$ for $i = 1, \cdots, m$ and we obtain a G-branched cover $\widehat{Y}$ of B^{n+3} branched along V_i's.

In order to get signature invariants, we restrict our dimension $n = 2q-1, q \geq 2$ and so all links are odd dimensional ≥ 3 and $\widetilde{Y}$, $\widehat{Y}$ are $(2q+2)$-manifolds. Let $sign(G, \widetilde{Y})$ and $sign(G, \widehat{Y})$ be the G-signatures of G-manifolds $\widetilde{Y}$ and $\widehat{Y}$, respecitively ([**AS**]). Since $\widehat{Y} = \widetilde{Y} \cup (\bigcup_i V_i \times D^2)$ and G acts trivially on V_i in $\widehat{Y}$, the intersection pairing vanishes on $V_i \times D^2$ and hence $sign(G, \widetilde{Y}) = sign(G, \widehat{Y})$. For a link L in S^{2q+1} with a map $f : X \to K(G, 1)$, define

$$sign(L, f) = sign(G, \widehat{Y}).$$

THEOREM 2. *$sign(L, f)$ is well-defined.*

PROOF. When $sign(L, f)$ is defined, various extensions can be chosen and they result distinct branched covers of B^{2q+2}. Let $\widehat{Y}$ and $\widehat{Z}$ be G-branched covers of B^{2q+2} branched along V_i's and W_i's, respectively. Then $\widehat{Y} \cup (-\widehat{Z})$ is a G-branched cover of S^{2q+2} branched along $\cup_i V_i \cup (-\cup_i W_i)$. For an element $g \in G$, the value $sign(g, \widehat{Y} \cup (-\widehat{Z}))$ of $sign(G, \widehat{Y} \cup (-\widehat{Z}))$ is zero if the normal bundle of the fixed-point set of g is homeomorphic to $D^2 \times$ (fixed-point set of g) with the standard product action and the signature of the fixed point set vanishes. By the constructions of V_i and W_i, $V_i \cup (-W_i)$ is embedded and has the trivial normal bundle in S^{2q+2} and so in $\widehat{Y} \cup (-\widehat{Z})$. Thus the signature of $V_i \cup (-W_i)$ vanishes. Consequently $0 = sign(G, \widehat{Y} \cup (-\widehat{Z})) = sign(G, \widehat{Y}) - sign(G, \widehat{Z})$.

The following main result is obtained by restricting the order of G and utilizing the Smith Theory as in [**G**].

THEOREM 3. *Suppose that G is a finite nilpotent group and L is a $(2q-1)$-dimensional m-link, $q \geq 2$, with a map $f_* : \pi \to G$. If the order of G is a prime power p^r and L is sliced then $sign(L, f) = 0$.*

PROOF. The regular cover $\widetilde{Y}$ of Y defined by f_* can be decomposed into a sequence of coverings

$$\widetilde{Y} = Y_1 \to Y_2 \to \cdots \to Y_d = Y$$

where $Y_i \to Y_{i+1}$ is the regular cover with the group $\mathbf{Z}_p$. We prove $sign(L, f) = sign(G, \widetilde{Y}) = 0$ by showing $H_{q+1}(\widetilde{Y}; \mathbf{Q}) = 0$. Since L is sliced, we may take the V_i's to disks so that $H_{q+1}(Y; \mathbf{Z}) = 0$ by the Alexander duality and so $H_{q+1}(Y; \mathbf{Q}) = H_{q+1}(Y; \mathbf{Z}_p) = 0$. We will prove that if the $(q+1)$-st homologies of Y_{i+1} with coefficients in $\mathbf{Z}_p$ or $\mathbf{Q}$ vanish, then the $(q+1)$-st homologies of Y_i with coefficients in $\mathbf{Z}_p$ or $\mathbf{Q}$ vanish. Then the proof is completed by induction.

At this point, we introduce a Gilmer's result [G] modified to fit our circumstance. Let $C(Y_i)$ be the chain complex of Y_i and $C(Y_i, \mathbf{Z}_p)$ the chain complex with $\mathbf{Z}_p$ coefficients. Let $\delta = 1 - T_* : C(Y_i, \mathbf{Z}_p) \to C(Y_i, \mathbf{Z}_p)$ be a chian map where a deck transformation T is the generator of $\mathbf{Z}_p$. The kernel of δ^s, denoted by $C^{\delta^s}(\widetilde{Y})$, is a subcomplex of $C(\widetilde{Y}, \mathbf{Z}_p)$ for $1 \le s \le p$. Then we have among chains that $C^{\delta}(Y_i) = C(Y_{i+1}, \mathbf{Z}_p)$, $C^{\delta^p}(Y_i) = C(Y_i, \mathbf{Z}_p)$ and hence we have among their homologies that $H^{\delta}(Y_i) = H(Y_{i+1}; \mathbf{Z}_p)$, $H^{\delta^p}(Y_i) = H(Y_i; \mathbf{Z}_p)$. Moreover there is a short exact sequence

$$0 \to C^{\delta}(Y_i) \to C^{\delta^p}(Y_i) \to C^{\delta^{p-1}}(Y_i) \to 0$$

and the corresponding long exact sequence of homologies is given as

$$\cdots \to H_k^{\delta}(Y_i) \to H_k^{\delta^p}(Y_i) \to H_k^{\delta^{p-1}}(Y_i) \to H_{k-1}^{\delta}(Y_i) \to \cdots .$$

By dim, We will denote the rank of the free part of an integral homology which is of course equal to the dimension of the corresponding rational homology. By induction using the above long exact sequence, we have

(1)
$$\dim H_{q+1}^{\delta^{p-1}}(Y_i) \le (p-1) \dim H_{q+1}^{\delta}(Y_i) = (p-1) \dim H_{q+1}(Y_{i+1}; \mathbf{Z}_p).$$

From the above short exact sequence, we have

(2) $$\dim H_{q+1}(Y_i; \mathbf{Z}_p) \le \dim H_{q+1}(Y_{i+1}; \mathbf{Z}_p) + \dim H_{q+1}^{\delta^{p-1}}(Y_i)$$

[**G**, Proposition 1.3] gives

(3) $\dim H_{q+1}(Y_i) - \dim H_{q+1}(Y_{i+1}) \leq (p-1) \dim H_{q+1}(Y_{i+1}; \mathbf{Z}_p)$

By combining (1), (2), and (3), we see that

$$H_{q+1}(Y_{i+1}; \mathbf{Q}) = H_{q+1}(Y_{i+1}; \mathbf{Z}_p) = 0$$

implies $H_{q+1}(Y_i; \mathbf{Q}) = H_{q+1}(Y_i; \mathbf{Z}_p) = 0$.

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