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A Noncommutative but Internal Addition on the Banach Algebra A_t

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1. Introduction.

In [**1**], Johnson and Lapidus introduced a family $\{A_t | t > 0\}$ of Banach algebras of functionals on Wiener space and showed that for every F in A_t, the analytic operator-valued function space integral $K_\lambda^t(F)$ exists for all nonzero complex numbers λ with nonnegative real part. In [**2**, **3**] Johnson and Lapidus introduced a noncommutative multiplication $*$ having the property that if $F \in A_{t_1}$ and $G \in A_{t_2}$ then $F * G \in A_{t_1+t_2}$ and

$$K_\lambda^{t_1+t_2}(F * G) = K_\lambda^{t_1}(F) K_\lambda^{t_2}(G).$$

Note that for F, G in A_t, $F * G$ is not in A_t but rather is in A_{2t} and so the multiplication $*$ is not internal to the Banach algebra A_t. In [**4**], the second author and professor Skoug introduced an internal noncommutative multiplication $\otimes$ on A_t having the property that for F, G in A_t, $F \otimes G$ is in A_t and

$$K_\lambda^t(F \otimes G) = K_{2\lambda}^t(F) K_{2\lambda}^t(G)$$

for all nonzero λ with nonnegative real part.

In this paper we introduce an internal noncommutative addition $\oplus$ on A_t having the property that for F, G in A_t, $F \oplus G$ is in A_t and

$$K_\lambda^t(F \oplus G) = K_{2\lambda}(F) \exp(-2\lambda H_0/t) + \exp(-2\lambda H_0/t) K_{2\lambda}(G)$$

for all nonzero λ with nonnegative real part where H_0 is a free Hamiltonian.

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2. Preliminaries.

We will adopt much of the notation and terminology used in [**1**, **3**]. However, we will include a brief description of the Banach algebra A_t and the operator-valued function space interal $K_\lambda^t(F)$.

Let C, C_+, and $C_+^\sim$ denote the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part respectively. Let $L^2(R^N)$ denote the space of Borel measurable, C-valued functions ψ on R^N such that $|\psi|^2$ is integrable with respect to Lebesgue measure on R^N.

For given $t > 0$, let $C[0,t]$ denote the R^N-valued continuous on $[0,t]$ and let $C_0[0,t]$ denote Wiener space; that is the set of all functions in $C[0,t]$ that vanish at 0. Let $m_{[0,t]}$ denote Wiener measure on $C_0[0,t]$.

Let $F : C[0,t] \to C$ be Borel measurable. For given $\lambda > 0$, $\psi \in L^2(R^N)$ and $\xi \in R^N$, consider the expression

$$(1)\quad \left(K_\lambda^t(F)\psi\right)(\xi) = \int_{C_0[0,t]} F(\lambda^{-1/2}x+\xi)\psi(\lambda^{-1/2}x(t)+\xi)\, dm_{[0,t]}(x).$$

The operator-valued function space integral $K_\lambda^t(F)$ exists for $\lambda > 0$ if (1) defines $K_\lambda^t(F)$ as an element of $(L_2(R^N))$, the space of bounded linear operators on $L_2(R^N)$. If, in addition, $K_\lambda^t(F)$, as a function of λ, has an extension to an analytic function on C_+ and to a strongly continuous function on $C_+^\sim$, we say that $K_\lambda^t(F)$ exists for $\lambda \in C_+^\sim$. When λ is purely imaginary $K_\lambda^t(F)$ is called the analytic operator-valued Feynman integral of F.

Let $M[0,t)$ denote the space of C-valued Borel measures on $[0,t)$. Given $\eta \in M[0,t)$, let $L_{\infty 1;\eta}[0,t)$ denote the class of all Borel measurable function $\theta : [0,t) \times R^N \to C$ such that

$$\|\theta\|_{\infty 1;\eta} = \int_{[0,t)} \|\theta(s,\cdot)\|_\infty |\eta|(s) < \infty.$$

A_t consists of all functions (actually equivalence classes of functions) on $C[0,t]$ of the form

$$F(x) = \sum_{n=0}^{\infty} \prod_{k=1}^{m_n} \int_{[0,t)} \theta_{n,k}(s,x(s))\, d\eta_{n,k}(s)$$

where

$$(2) \qquad \sum_{n=0}^{\infty} \prod_{k=1}^{m_n} \|\theta_{n,k}\|_{\infty 1;\eta_{n,k}} < \infty.$$

For $F \in A_t$, let $\|F\|_t$ be the infimum of the left-hand side of (2) over all such representations of F. In [**1**, Theorem 6.1], Johnson and Lapidus show that $(A_t, \|\cdot\|_t)$ is a commutative Banach algebra under pointwise multiplication and addition. In addition they show that given $F \in A_t$, $K_\lambda^t(F)$ exists for all $\lambda \in C_+^\sim$ and saitsfies $\|K_\lambda^t(F)\| \leq \|F\|_t$. For λ in $C_+^\sim$, ψ in $L^2(R^N)$, ξ in R^N and positive real number t,

$$\exp(-\lambda H_0/t) = \left(\frac{\lambda}{2t}\right)^{\frac{N}{2}} \int_{R^N} \psi(u) \exp\left(\frac{-\lambda|u-\xi|^2}{2t}\right) dm_L(u),$$

where if N is odd we always choose $\lambda^{\frac{1}{2}}$ with nonnegative real part and if $\operatorname{Re}\lambda = 0$ the integral in the above should be interpreted in the mean just as in the theory of the L_2 Fourier transform and H_0 is a free Hamiltonim. Then as a function λ, $\exp(-\lambda H_0/t)$ is analytic in C_+, weakly continuous in $C_+^\sim$ and $\|\exp(-\lambda H_0/t)\| \leq 1$.

Let a and b be positive real numbers. Let $E_{a,b} : C[0,a] \to C[0,b]$ be given by the formula

$$(3) \qquad E_{a,b}(x)(s) = \sqrt{\frac{b}{a}}\, x\left(\frac{as}{b}\right).$$

for $0 \leq s \leq b$. Then $E_{a,b}$ is bijective and continuous under the topology of uniform convergence. From [4], we obtain the following lemma.

LEMMA 1. $m_{[0,a]} = m_{[0,b]}E_{a,b}$.

Let F be a function on A_a. Then we can write F in the form

$$(4) \qquad F(x) = \sum_{n=0}^{\infty} \prod_{k=1}^{m_n} \int_{[0,a)} \theta_{n,k}(s, x(s))\, d\eta_{n,k}(s)$$

where each $\eta_{n,k}$ is in $M[0,a)$ and each $\theta_{n,k}$ is in $L_{\infty 1;\eta_{n,k}}$. Now for each n and k we define a measure $\bar{\eta}_{n,k}$ in $M[0,\frac{a}{2})$ by the formula

$$\bar{\eta}_{n,k}(B) = \eta_{n,k}(2B)$$

for each Borel subset B of $[0,\frac{a}{2})$. We also define $\bar{\theta}_{n,k}$ in $L_{\infty 1;\bar{\eta}_{n,k}}$ by $\bar{\theta}_{n,k}(s,v)=\theta_{n,k}(2s,v)$ for all $(s,v)\in[0,\frac{a}{2})\times R^N$. We note that $\|\eta_{n,k}\|=\|\bar{\eta}_{n,k}\|$ and $\|\bar{\theta}_{n,k}\|_{\infty 1;\bar{\eta}_{n,k}}=\|\theta_{n,k}\|_{\infty 1;\eta_{n,k}}$. Now we define $\overline{F}:C[0,\frac{a}{2}]\to C$ by

$$\overline{F}(y)=\sum_{n=0}^{\infty}\prod_{k=1}^{m_n}\int_{[0,\frac{a}{2})}\bar{\theta}_{n,k}(t,y(t))\,d\bar{\eta}_{n,k}(t). \tag{5}$$

It is quite easy to verify that $\overline{F}$ is in $H_{a/2}$ with $\|\overline{F}\|_{a/2}=\|\overline{F}\|_a$.

From [4], we obtain the following lemma.

LEMMA 2. *Let F in H_a be given by (4) and let $\overline{F}$ be given by (5). Then*

$$K_\lambda^a(F)=K_{\lambda/2}^{a/2}(\overline{F})\quad \text{for all}\quad \lambda \text{ in } C_+^\sim.$$

For $x\in C[0,a]$ let $R_1(x)$ be the restriction of x to $[0,\frac{a}{2}]$; that is to say $R_1:C[0,a]\to C[0,\frac{a}{2}]$ is given by $R_1(x)(s)=x(s)$ for $0\le s\le a/2$. Also for $x\in C[0,a]$ let $R_2(x)$ be the restriction of x to $[a/2,a]$; that is to say $R_2:C[0,a]\to C[a/2,a]$ is given by $R_2(x)(s)=x(s)$ for $a/2\le s\le a$. Also let $T:C[a/2,a]\to C[0,a/2]$ be the translation map

$$T(x)(s)=x(s+a/2),\qquad 0\le s\le a/2.$$

DEFINITION. For F and G in H_a, we define $F\otimes G:C[0,a]\to C$ by the formula

$$(F\otimes G)(x)=\overline{F}(R_1(x)\overline{G}(T(R_2(x))). \tag{6}$$

REMARK: In view of the definition of $*$ given by equation (3,2) of [3] we see that $F\otimes G=\overline{F}*\overline{G}$.

From [4], we obtain the following lemma.

LEMMA 3. *For F, G in A_a, $F\otimes G$ is in A_a and $K_\lambda^a(F\otimes G)=K_{2\lambda}^a(F)K_{2\lambda}^a(G)$ for all λ in $C_+^\sim$.*

3. The addition operator $\oplus$.

In this section we introduce the internal noncommutative addition $\oplus$ on A_a.

DEFINITION. For F and G in A_a, we define $F \oplus G : C[0,a] \to C$ by the formula

$$(F \oplus G)(x) = \overline{F}(R_1(x)) + \overline{G}(T(R_2(x))). \tag{7}$$

DEFINITION. Let I be a functional on $C[0,a]$ with $I(x) = 1$ for all x in $C[0,a]$.

Easily, we have $\overline{I}(x) = 1$ for $x \in C[0,a/2]$ and $I \in H_a$. In generally, $F \otimes I$ and F are not same but $I \otimes I = I$. Since $F \otimes I = \overline{F} \circ R_1(x)$ and $I \otimes G = \overline{G} \circ T \circ R_2$, $F \oplus G = F \otimes I + I \otimes G$. Hence we have the following theorem.

THEOREM 1. *For F and G in A_a, $F \oplus G$ in A_a and $K_\lambda^a(F \oplus G) = K_{2\lambda}^a(F)\exp(-2\lambda H_0/a) + \exp(-2\lambda H_0/a)K_{2\lambda}^a(G)$ for all λ in $\tilde{C}_+$.*

PROOF: Since

$$\begin{aligned} K_\lambda^a(F)\psi &= \int_{C_0[0,a]} I(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(a) + \xi)\, dm_{[0,a]}(x) \\ &= \int_{C_0[0,a]} \psi(\lambda^{-1/2}x(a) + \xi)\, dm_{[0,a]}(x) = [\exp(-\lambda H_0/a)\psi](\xi) \end{aligned}$$

for $\psi \in L_2(R^N)$, $\lambda > 0$ and $\xi \in R^N$,

$$\begin{aligned} &K^a(F \oplus G) = K^a(F \otimes I + I \otimes G) \\ =&K_{2\lambda}^a(F)K_{2\lambda}^a(I) + K_{2\lambda}^a(I)K_{2\lambda}^a(G) \\ =&K_{2\lambda}^a(F)\exp(-2\lambda H_0/a) + \exp(-2\lambda H_0/a)K_{2\lambda}^a(G) \quad \text{for all } \lambda \text{ in } \tilde{C}_+. \end{aligned}$$

For F in A_a, we easily check that $(\overline{F})^p = \overline{F^p}$ for any natural number p.

THEOREM 2. *For F, G in A_a,*

$$(F \oplus G)^n = \sum_{p+q=n} \frac{n!}{p!\,q!} F^p \otimes G^q.$$

Here, we adopt the conventions $F^0 = I = G^0$.

PROOF. Let $x \in C_0[0,a]$. Then

$$\begin{aligned}(F \oplus G)^n(x) &= \left(\overline{F} \circ R_1(x) + \overline{G} \circ T \circ R_2(x)\right)^n \\ &= \sum_{p+q=n} \frac{n!}{p!\,q!}(\overline{F} \circ R_1(x))^p(\overline{G} \circ T \circ R_2(x))^q \\ &= \sum_{p+q=n} \frac{n!}{p!\,q!}\overline{F^p} \circ R_1(x)\overline{G^q} \circ T \circ R_2(x) \\ &= \sum_{p+q=n} \frac{n!}{p!\,q!}F^p \otimes G^q.\end{aligned}$$

THEOREM 3. *For F, G in A_a, $e^{F\oplus G} \in A_a$ and*

$$\exp(F \oplus G) = \exp(F) \otimes \exp(G).$$

Moreover,

$$K^a_{2\lambda}(e^{F\oplus G}) = K^a_{2\lambda}(\exp F)K^a_{2\lambda}(\exp G) \quad \textit{for all } \lambda \textit{ in } C^{\sim}_{+}.$$

PROOF. Let $x \in C_0[0,a]$. Then

$$\begin{aligned}\exp(F \oplus G)(x) &= \sum_{n=0}^{\infty} \frac{1}{n!}(F \oplus G)^n(x) = \sum_{n=0}^{\infty} \sum_{p+q=n} \frac{1}{p!\,q!}(F^p \otimes G^q)(x) \\ &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{1}{p!\,q!}F^p \otimes G^q(x) = \sum_{n=0}^{\infty} \frac{1}{p!}F^p \otimes \sum_{n=0}^{\infty} \frac{1}{q!}G^q \\ &= \exp(F) \otimes \exp(G).\end{aligned}$$

The rest part of this proof is trivial.

REMARK. It is well-known that if $\oplus$ is a noncommutative operator, then the formula

$$\exp(A \oplus B) = \exp(A)\exp(B)$$

may fail. Nevertheless, in this case, this rule holds.

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