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Characterization of Weak Asplund Space in Terms of Positive Sublinear Functional

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ABSTRACT. For each continuous convex function ϕ defined on an open convex subset A_ϕ of a Banach space X, if we define a positively homogeneous sublinear functional σ_x on X by $\sigma_x(y) = \sup\{f(y) : f \in \partial\phi(x)\}$, where $\partial\phi(x)$ is a subdifferential of ϕ at x, then we get the following characterization theorem of Gateaux differentiability (weak Asplund) sapce.

THEOREM. *For every ϕ above, $D_\phi = \{x \in A : \sup_{\|u\|=1} \sigma_x(u) + \sigma_x(-u) = 0\}$ contains dense (dense G_δ) subset of A_ϕ if and only if X is a Gateaux differentiability (weak Asplund) space.*

A Banach space X is said to be a Gateaux differentiability space (GDS) (or weak Asplund space ;WAS) if every continuous convex function defined on an open convex subset A of X is Gateaux differentiable on a dense (dense G_δ) subset of A. This weak Asplund space has been characterized by means of separability of the space X [**5**], rotundness of the dual space X^* of X, or more generally by means of the space X of weakly compactly generated [**1**]. It has been tried for times to characterize it by smoothness of the space X [4]. (X is said to be smooth if the norm on X is Gateaux differentiable at every point except 0). But it failed. Recently J. M. Borwein and D. Preiss proved a Banach space X is at least a GDS if it is smooth [**2**]. This paper characterizes weak Asplund space by means of some particular positively homogeneous sublinear functionals.

DEFINITIONS. A real function ϕ on an open convex subset A of X is said to be *Gateaux differentiable at $x \in A$ in the direction $y \in X$* if

$$\lim_{t\to 0} \frac{\phi(x+ty) - \phi(x)}{t} \qquad \text{exists.}$$

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If ϕ is continuous and convex, then right-hand [left-hand] Gateaux derivative at x in the direction y denoted by $\phi'_+(x)(y)[\phi'_-(x)(y)]$ exists and $\phi'_+(x)$ is a continuous sublinear functional on X.

A real function ψ on a linear space X is said to be positively homogeneous if

$$\psi(tx) = t\psi(x) \quad \text{for all} \quad t > 0 \quad \text{and} \quad x \in X.$$

If ψ is positively homogeneous and convex, it is called *sublinear functional*. A real function ϕ on an open convex subset A of X said to be *Gateaux differentiable at* $x \in A$ if

$$\phi'_-(x)(y) = \phi'_+(x)(y) \quad \text{for all} \quad y \in X.$$

The real functional f on X which satisfies

$$\phi'_-(x)(y) \le f(y) \le \phi'_+(x)(y) \quad \text{for all} \quad y \in X$$

is called a *subgradient of* ϕ *at* x and we denote the set of all such subgradients by $\partial\phi(x)$, *the subdifferential of* ϕ *at* x.

LEMMA 1. *Let X be a Banach space. For a continuous convex function ϕ on an open convex subset A of X, define σ_x on X by*

$$\sigma_x(y) = \sup\{f(y) : f \in \partial\phi(x)\}, \qquad y \in X.$$

Then

(1) *for every $x \in X$ and $u \in X$,*

$$\lim_{t\to 0+} \frac{\phi(x+tu) - \phi(x)}{t} = \sigma_x(u),$$

and

$$\lim_{t\to 0+} \frac{\phi(x-tu) - \phi(x)}{-t} = -\sigma_x(-u).$$

(2) *If $\sigma_x(u) + \sigma_x(-u) = 0$, then for any $y \in X$,*

$$\sigma_x(y+u) = \sigma_x(y) + \sigma_x(u), \quad \text{and} \quad \sigma_x(y-u) = \sigma_x(y) - \sigma_x(u).$$

PROOF. 1. By the definition of subdifferential of ϕ at x,

$$f_x(tu) \le \phi(x+tu) - \phi(x) \le f_{x+tu}(tu)$$

for all $f_x \in \partial\phi(x)$ and $f_{x+tu} \in \partial\phi(x+tu)$. Hence if $t > 0$,

$$f_x(u) \le \frac{\phi(x+tu) - \phi(x)}{t} \le f_{x+tu}(u).$$

By the weak* upper semi-continuity of subgradient mapping $x \mapsto \partial\phi(x)$, f_{x+tu} weak* converges to $f_x \in \partial\phi(x)$ as $t \to 0$ [**3**, P.132]. Hence

$$\sigma_x(u) \le \lim_{t\to 0+} \frac{\phi(x+tu) - \phi(x)}{t} \le \sigma_x(u), \quad \text{that is,}$$

$$\lim_{t\to 0+} \frac{\phi(x+tu) - \phi(x)}{t} = \sigma_x(u).$$

With same arguement, we get the second part.

2. By the property of convexity of ϕ, we have

$$-\sigma_x(-u) = \phi'_-(x)(u) \le \phi'_+(x)(u) = \sigma_x(u).$$

But since $\sigma_x(u) + \sigma_x(-u) = 0$, we have $\phi'_-(x)(u) = \phi'_+(x)(u)$, that is, if $f \in \partial\phi(x)$, then $f(u) = \sigma_x(u) = -\sigma_x(-u)$. Consider $\sigma_x(y)$ for any $y \in X$. Since $\partial\phi(x)$ is a weak* compact convex subset of X^*, there exists a $f \in \partial\phi(x)$ such that $f(y) = \sigma_x(y)$. Now for such f,

$$f(y+u) = f(y) + f(u) \le \sigma_x(y+u) \le \sigma_x(y) + \sigma_x(u).$$

Since for any $f \in \partial\phi(x)$, we have $f(u) = \sigma_x(u)$, and hence

$$\sigma_x(y) + \sigma_x(u) \le \sigma_x(y+u) \le \sigma_x(y) + \sigma_x(u).$$

Therefore

$$\sigma_x(y+u) = \sigma_x(y) + \sigma_x(u).$$

Likewise we get the second part.

THEOREM 2. *Let X be a Banach space. For every continuous convex function ϕ defined on some open convex subset A_ϕ of X, if we define σ_x as in Lemma 1, we have the following characterizations.*

(1) *For every ϕ, $D_\phi = \{x \in A_\phi : \sup_{\|u\|=1} \sigma_x(u) + \sigma_x(-u) = 0\}$ contains dense (dense G_δ) subset of A_ϕ if and only if X is a GDS(WAS).*

(2) *In a GDS, for every ϕ on some open convex subset A_ϕ of X, $\{x \in A : \sigma_x$ is Gateaux differentiable at $x\}$ is dense G_δ subset of A_ϕ if and only if X is a WAS.*

PROOF. 1. Suppose X is a GDS(WAS). ϕ is Gateaux differentiable on a dense (dense G_δ) subset D of A_ϕ. Hence for each $x \in D$, $\partial\phi(x)$ is a singleton set [**3**, p.122], that is, σ_x is a linear functional on X. Therefore $A_\phi \subset D$.

Cinversely, for a given ϕ on A_ϕ, suppose D_ϕ contains dense (dense G_δ) subset D of A_ϕ. For each $x \in D$,

$$\begin{aligned}&\lim_{t\to 0^+} \frac{\phi(x+tu)+\phi(x-tu)-2\phi(x)}{t}\\ =&\sigma_x(u)+\sigma_x(-u) \le \sup_{\|u\|=1} \sigma_x(u)+\sigma_x(-u) = 0.\end{aligned}$$

Hence ϕ is Gateaux differentiable on D.

2. A real function $x \mapsto \sigma_x(x)$ is upper semi-continuous. Suppose $\|x_n - x\| \to 0$, then

$$\limsup_{n\to\infty} \sigma_{x_n}(x_n) = \lim_{k\to\infty} \sigma_{x_{n_k}}(x_{n_k}) = \lim_{k\to\infty} f_{n_k}(x_{n_k})$$

for some $f_{n_k} \in \partial\phi(x_{n_k})$ for the weak* compact convexness of $\partial\phi(x_{n_k})$. Again by using weak* upper semi-continuity of subgadient mapping, $x \mapsto \partial\phi(x)$, we get the weak* convergence of $\{f_{n_k}\}$ to some point $f_x \in \partial\phi(x)$, and because of local boundedness of $\partial\phi(x)$,

$$\lim_{k\to\infty} f_{n_k}(x_{n_k}) = f_x(x) \quad \text{for some} \quad f_x \in \partial\phi(x),$$

which implies

$$\limsup_{n\to\infty} \sigma_{x_n}(x_n) \le \sigma_x(x).$$

$x \mapsto \sigma_x(-x)$ is slso upper semi-continuous and so is $x \mapsto \sigma_x(x) + \sigma_x(-x)$. Hence

$$D_n = \{x \in A_\phi : \sigma_x(x) + \sigma_x(-x) < \frac{1}{n}\}$$

is an open set for each n. But since X is GDS, D_n is dense in A_ϕ for each n. Let $D_1 = \bigcap_{n=1}^{\infty} D_n$. By assumption $D_2 = \{x \in A : \sigma_x$ is Gateaux differentiable at $x\}$ is dense G_δ set, hence so is $D = D_1 \cap D_2$. For each $x \in D$ and any $u \in X$, $\|u\| = 1$,

$$\begin{aligned}
&\limsup_{t \to 0^+} \frac{\phi(x+tu) + \phi(x-tu) - 2\phi(x)}{t} \\
=&\sigma_x(u) + \sigma_x(-u) = \frac{\sigma_x(tu) + \sigma_x(-tu)}{t} \qquad (t > 0) \\
=&\frac{\sigma_x(x+tu) + \sigma_x(x-tu) - 2\sigma_x(x)}{t},
\end{aligned}$$

by Lemma 1, and right-hand side is Gateaux differentiable. Hence ϕ is Gateaux differentiable on D.

The converse is obvious.

If X is smooth, then X is a GDS [**2**], hence following corollary is obvious.

COROLLARY 3. *If X is smooth and for every ϕ on some open convex subset A_ϕ of X, $\{x \in A_\phi : \sigma_x$ is Gateaux differentiable at $X\}$ is dense G_δ subset of A_ϕ, then X is a WAS.*

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