

Recurrent Discrete Flows on Totally Disconnected Spaces

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ABSTRACT. In this paper dynamical properties of recurrent discrete flows are stated. The pointwise periodic discrete flows on totally disconnected spaces are of characteristic 0.

1. It is wellknown that any nonempty compact invariant set contains at least one compact minimal set. In this note, first, we consider a discrete flow such that each compact invariant set contains only one compact minimal set. Next, we consider the stability of the periodic trajectory in certain discrete flows.

It is wellknown fact that there exists deep relations between the periodicity and the zero characteristic concepts of flows. An wellknown result is that of R. Knight [4], which shows that if a real flow is pointwise periodic without rest points, then it is of characteristic 0. But it is uncertain that whether the above result remains valid if the flow is assumed to have rest points. Recently, K. Ahmad and S. Elaydi have shown that if the assumption of the above result is strenghten to uniform periodicity (there exists a $s \in R^+$ with $xs = x$ for all $x \in X$), then the result remains valid even if the flow has rest points[1]. The purpose of the present paper is to point out that for discrete flows on totally disconnected phase spaces the Knight's result is generalized to the recurrent flows which may have rest points. Also it is proved that on countable compact spaces, the periodicity, recurrent concept, and zero characteristic concepts of a discrete flow are all equivalent.

2. Throughout this paper we shall let (X, f) be a given discrete flow induced by a homeomorphsim f from X onto itself, where the phase space X is a locally compact Hausdorff, and totally disconnected space. Most notations and definitions used herein are standard and found in references in [1] and [2]. However, a few are enumerated

Received by the editors on April 29, 1988.

1980 *Mathematics subject classifications*: Primary 34D.

for convenience. The trajectory, trajectory closure, limit, prolongational limit, and prolongational relations are denoted, respectively, by C , K , L , J , and D with unilateral versions identified by the appropriate superscript. A flow is said to be of characteristic 0 provided $D(x) = K(x)$ for all $x \in X$. A flow is said to be nearly periodic if $D^+(x) = L^+(x)$ and $D^-(x) = L^-(x)$ for all $x \in X$. It was shown that in Lagrange stable flow the nearly periodicity and the zero characteristic concepts are equivalent. A flow is called to be pointwise periodic or compact provided each point in X is periodic or rest point. A flow is said to be Poisson stable if $x \in L^+(x) \cap L^-(x)$ for all $x \in X$. A flow is said to be recurrent if each trajectory closure in X is a compact minimal set.

3. LEMMA 1. *Let (X, f) be a Lagrange stable and Poisson stable flow and $x \in X$. Then for any y in $J(x)$ the two trajectory closures $K(x)$ and $K(y)$ must intersect.*

PROOF. We shall verify this result by contradiction. First, suppose that $y \in J^+(x)$ and the two trajectory closures $K(x)$ and $K(y)$ are disjoint. A compact open neighborhood V of $K(x)$ with $K(y) \subset X - V$ exists. Since y belongs to $J^+(x) - V$, there are nets (x_i) in V and (n_i) in Z^+ such that $f^{n_i}(x_i)$ is in $X - V$, $x_i \rightarrow x$, $n_i \rightarrow +\infty$, and $f^{n_i}(x_i) \rightarrow y$. Let $r_i = \max\{n \in Z^+ : f^n(x_i) \in V \text{ and } 0 \leq n \leq n_i\}$. Thus $f^{r_i+1}(x_i) \in X - V$. Assume that $f^{r_i}(x_i) \rightarrow z \in V$. Then we have $f^{r_i+1}(x_i) \rightarrow f(z) \in X - (V \cup K(y))$. Choose a compact open neighborhood W of z such that $W \subset V$ and $f(W) \subset X - (V \cup K(y))$. Denote $\min\{n > 0 : f^n(x) \in W\}$ by n_z for each point z in $f(W)$. Clearly $n_z < +\infty$ since the flow is Poisson stable. Also define $N = \sup\{n_z : z \in f(W)\}$. We shall show that N is bounded. For any $z \in f(W)$, by the continuity of f , we have $n_p \leq n_z + 1$ for each point p in some neighborhood V_z of z . The covering $\{V_z : z \in f(W)\}$ of $f(W)$ contains a finite subcovering $\{V_{z_i} : i = 1, \dots, k\}$ since $f(W)$ is compact. For each point a in $f(W)$, $n_a \leq \max\{n_{z_i} + 1, i = 1, \dots, k\}$. Hence we have $f^{n_i+1}(x_i) = f^{n_i-r_i-1}(f^{r_i+1}(x_i)) \in \bigcup\{f^n(f(W)) : 0 \leq n \leq N\}$. Thus $y \in \bigcup\{f^{n+1}(W) : 0 \leq n \leq N\}$ and this implies $K(y)$ and W intersect. But this is a contradiction. Next, in the case that $y \in J^-(x)$ the proof is similar. Hence we concluded that $K(x)$ and $K(y)$ intersect for any y in $J(x)$. This completes the proof of Lemma 1.

THEOREM 2. *Let (X, f) be Poisson stable and Lagrange stable. If X be a compact trajectory closure of a point in X . Then any two trajectory closures in X intersect. Further, X contains a unique compact minimal subset.*

PROOF. By assumption we may assume that $X = L^+(x)$ for some $x \in X$. Let $y, z \in X$. Then it is easily shown that $y \in D^+(z)$ and $z \in D^+(y)$. Therefore by the above Lemma $K(y)$ and $K(z)$ intersect. Furthermore, let M and N be two compact minimal sets. Then they are the trajectory closures of some points which intersect. By the minimality of M and N they must coincide. This completes the proof of Theorem 2.

COROLLARY 3. *Let (X, f) be Lagrange stable and Poisson stable. Then any trajectory closure contains only one compact minimal set.*

THEOREM 4. *Let (X, f) be recurrent. Then it is of characteristic 0.*

PROOF. Let $x \in X$ and suppose that $y \in D(x) - K(x)$. Then by Lemma 1 the two trajectory closures $K(x)$ and $K(y)$ must intersect. Since they are minimal they must equal. This contradiction derives that the flow (X, f) is of characteristic 0.

COROLLARY 5. *If the flow (X, f) be pointwise periodic, then it is of characteristic 0.*

PROOF. Obviously pointwise periodic flow is recurrent.

4. In general two conditions, the pointwise periodicity and the zero characteristic concepts of a flow are not equivalent. Here we show that on certain totally disconnected space these two conditions are equivalent. We are now in position to characterize the discrete flows on the countable compact spaces. In [3], G. Faulkner, J. Franke, and L. Janos have proved the elegant result, which shows that if the phase space X is a countable compact metric space then the flow (X, f) is pointwise periodic if and only if it is Liapunov stable. We can extend this result to the recurrent flows and flows satisfying the zero characteristic concepts. This generalization is that the original hypothesis are replaced by weaker conditions. From this we obtain the two conditions, the pointwise periodicity and the zero characteristic concepts, are equivalent on the countable compact space.

LEMMA 6. *If (X, f) is of characteristic 0, then all trajectory closures in X are minimal.*

PROOF. Let $x \in X$, whenever $L(x) = \emptyset$ $K(x)$ is clearly minimal. So we assume that $L(x) \neq \emptyset$. For any $y \in L(x)$ we have $x \in D(y) = K(y)$. Thus $K(x) \subset K(y)$ and consequently we get $K(x) = K(y)$ for any $y \in K(x)$. Therefore we concluded that $K(x)$ is minimal.

LEMMA 7. *Let the phase space X be a countable compact space. Then the following are equivalent.*

- (1) (X, f) is pointwise periodic.
- (2) (X, f) is of characteristic 0.

PROOF. (1) implies (2): See Corollary 5.

(2) implies (1): By Lemma 6 all trajectory closures in X are compact minimal. Let $x \in X$. We shall show that x is a periodic point. By virtue of the Baire category theorem, $K(x)$ has at least one isolated point. Let $y \in K(x)$ be an isolated point. Then the set $\{y\}$ is open. Hence $K(x) - \bigcup_{n=1}^{\infty} f^{-n}(y)$ is closed positively invariant subset of $K(x)$. Therefore by the minimality and compactness of $K(x)$, we have $K(x) = K(y) = \bigcup_{n=1}^{\infty} f^{-n}(y)$. Thus $y = f^k(y)$ for some $k > 0$. Thus y is a periodic point and so is x . This completes the proof of Lemma 7.

This lemma finishes the proof of the following theorem.

THEOREM 8. *Let X be a compact countable space. Then the following conditions are pairwise equivalent.*

- (1) (X, f) is pointwise periodic.
- (2) (X, f) is recurrent.
- (3) (X, f) is of characteristic 0.
- (4) (X, f) is nearly periodic.

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