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H-Closed Spaces and W-Lindelöf Spaces

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ABSTRACT. We introduce the concept of a w-Lindelöf space which is a more general concept than that of a Lindelöf spaces. We obtain some characterization about H-closed sapces and w-Lindelöf spaces. Also, we investigate their invariance properties.

1. Introduction

In section 2, we obtain a characterization of H-closed spaces using the results in [1] and show that H-closedness is invariant under the σ-continuous surjections.

In section 3, we introduce the concept of a w-Lindelöf space which is a more general concept than that of a Lindelöf space. We give a counterexample and one characterization of the w-Lindelöf property. Finally, we show that the product of a H-closed space and a w-Lindelöf space is w-Lindelöf.

2. H-Closed spaces

DEFINITION 2.1 A space X is said to be *H-closed* if for each open cover $\{U_i\}$ of X there are finitely many i_k such that $X = \bigcup \mathrm{Cl}(U_{i_k})$.

DEFINITION 2.2 Let X be a space. A net (x_i) in X is said to *w-accumulate* to a point x of X, denoted by $x_i \overset{w}{\propto} x$, if for any neighborhood U of x and i there is an $i_1 \geq i$ such that $x_{i_1} \in \mathrm{Cl}(U)$. A net (x_i) in X is said to *w-converge* to a point x of X, denoted by $x_i \overset{w}{\longrightarrow} x$, if for each neighborhood U of x there is an i_1 such that $x_i \in \mathrm{Cl}(U)$ for all $i \geq i_1$.

It is easy to show that the following lemma holds.

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LEMMA 2.1. *Let X be a space. If an ultranet (x_i) in X w-accumulates to a point x of X, then (x_i) w-converges to x.*

DEFINITION 2.3 Let X be a space. For a subset A of X the *w-closure* $\mathrm{Cl}_w(A)$ of A is defined by the set

$$\mathrm{Cl}_w(A) = \{x \in X \mid A \cap \mathrm{Cl}(U) \neq \emptyset \text{ for all neighborhoods } U \text{ of } x\}.$$

It is clear that $A \subset \mathrm{Cl}(A) \subset \mathrm{Cl}_w(A)$.

LEMMA 2.2. *Let X be a space and A a subset of X. Then $x \in \mathrm{Cl}_w(A)$ if and only if there is a net (x_i) in A such that $x_i \xrightarrow{w} x$.*

PROOF. ($\Rightarrow$) Let (U_i) be the family of neighborhoods of x with the reverse inclusion order. For each i, since $A \cap \mathrm{Cl}(U_i) \neq \emptyset$, there is an $x_i \in A \cap \mathrm{Cl}(U_i)$. Then (x_i) is a net in A and $x_i \xrightarrow{w} x$.

($\Leftarrow$) Given any neighborhood U of x, there is an i_1 such that $x_i \in \mathrm{Cl}(U)$ for all $i \geq i_1$. Since $x_{i_1} \in A \cap \mathrm{Cl}(U)$, $A \cap \mathrm{Cl}(U) \neq \emptyset$. Thus $x \in \mathrm{Cl}_w(A)$.

DEFINITION 2.4 Let X be a space. A subset A of X is said to be *w-closed* if $\mathrm{Cl}_w(A) = A$.

LEMMA 2.3. *Let $\{X_k\}$ be a family of spaces. A net $\left((x_i^k)\right)$ in ΠX_k w-converges to a point (x^k) of ΠX_k if and only if the net $\left(x_i^k\right)$ in X_k w-converges to the point x^k of X_k for all k.*

PROOF. ($\Rightarrow$) Let U be a neighborhood of x^k. Since $p_k^{-1}(U)$ is a neighborhood of (x^k), there is an i_1 such that $(x_i^k) \in \mathrm{Cl}\left(p_k^{-1}(U)\right) = p_k^{-1}\left(\mathrm{Cl}(U)\right)$ for all $i \geq i_1$. Thus $x_i^k \in \mathrm{Cl}(U)$ for all $i \geq i_1$.

($\Leftarrow$) Let $\bigcap_{j=1}^{n} p_{k_j}^{-1}(U_j)$ be a basic neighborhood of (x^k). There is an i_1 such that $x_i^k \in \mathrm{Cl}(U_j)$ for all $i \geq i_1$. Thus we have

$$(x_i^k) \in \bigcap_{j=1}^{n} p_{k_j}^{-1}(\mathrm{Cl}(U_j)) = \mathrm{Cl}\left(\bigcap_{j=1}^{n} p_{k_j}^{-1}(U_j)\right)$$

for all $i \geq i_1$.

DEFINITION 2.5 A space X is said to be *completely Hausdorff* if for any two distinct points x and y of X there are neighborhoods U of x and V of y such that $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \emptyset$.

THEOREM 2.4. *Let X be a space. Then the following statements are equivalent.*

(1) *X is completely Hausdorff.*
(2) *Every net in X w-converges to at most one point of X.*
(3) *The diagonal $\Delta = \{(x,x) \mid x \in X\}$ is w-closed in $X \times X$.*

PROOF. (1) $\Rightarrow$ (2) Suppose that a net (x_i) in X w-converges to two distinct points x and y of X. Since X is completely Hausdorff there are neighborhoods U of x and V of y such that $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \emptyset$. Since $x_i \overset{w}{\longrightarrow} x$ and $x_i \overset{w}{\longrightarrow} y$, there is an i_1 such that $x_{i_1} \in \mathrm{Cl}(U)$ and $x_{i_1} \in \mathrm{Cl}(V)$. Thus $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) \neq \emptyset$, this is a contradiction.

(2) $\Rightarrow$ (3) Let $(x,y) \in \mathrm{Cl}_w(\Delta)$. There is a net (x_i) in X such that $(x_i, x_i) \overset{w}{\longrightarrow} (x,y)$. Since $x_i \overset{w}{\longrightarrow} x$ and $x_i \overset{w}{\longrightarrow} y$, $x = y$. Thus $(x,y) \in \Delta$.

(3) $\Rightarrow$ (1) Let x and y be two distinct points of X. Since $(x,y) \notin \Delta = \mathrm{Cl}_w(\Delta)$, there is a neighborhood W of (x,y) such that $\Delta \cap \mathrm{Cl}(W) = \emptyset$. Also, there are neighborhoods U of x and V of y such that $U \times V \subset W$. It is clear that $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \emptyset$.

DEFINITION 2.6 Let X be a space. A family $\{A_i\}$ of subsets of X is said to satisfy the *s-finite intersection condition* if for any finitely many i_k, $\bigcap \mathrm{Int}(A_{i_k}) \neq \emptyset$.

The following is a characterization of H-closed spaces [**1**].

THEOREM 2.5. *A space X is H-closed if and only if for each family $\{A_i\}$ of closed subsets of X satisfying the s-finite intersection condition, $\bigcap A_i \neq \emptyset$.*

LEMMA 2.6. *Let X be a H-closed space. Then for each net (x_i) in X there is an $x \in X$ such that $x_i \overset{w}{\propto} x$.*

PROOF. Suppose that $x_i \overset{w}{\not\propto} x$ for all $x \in X$. For each $x \in X$ there is a neighborhood U_x of x and an i_x such that $x_i \notin \mathrm{Cl}(U_x)$ for all $i \geq i_x$. Then $\{U_x\}$ is an open cover of X. Since X is H-closed, there are finitely many x_k such that $X = \bigcup \mathrm{Cl}(U_{x_k})$. There is an i_1 such that $i_1 \geq i_{x_k}$. Then $x_{i_1} \notin \bigcup \mathrm{Cl}(U_{x_k}) = X$, this is a contradiction. Thus there is an $x \in X$ such that $x_i \overset{w}{\propto} x$.

Our characterization of H-closed spaces is the following.

THEOREM 2.7. *A space X is H-closed if and only if every net in X has a w-convergent subnet.*

PROOF. ($\Rightarrow$) Let (x_i) be a net in X. Since every net has an ultrasubnet, (x_i) has an ultrasubnet (x_{i_k}). By lemma 2.6, there is an $x \in X$ such that $x_{i_k} \overset{w}{\propto} x$. Therefore we have $x_{i_k} \overset{w}{\longrightarrow} x$ by lemma 2.1.

($\Leftarrow$) Given any family $\mathcal{A}$ of closed subsets of X satisfying the s-finite intersection condition, set $\{A_i\}$ be the family of finite intersections of members of $\mathcal{A}$. Clearly, $\mathcal{A} \subset \{A_i\}$. Define $i_1 \leq i_2$ by $A_{i_2} \subset A_{i_1}$. For each i, since $\text{Int}(A_i) \neq \emptyset$, there is an $x_i \in \text{Int}(A_i)$. Then (x_i) is a net in X. Since (x_i) has a w-convergent subnet, we may assume that $x_i \overset{w}{\longrightarrow} x \in X$. Suppose that $\bigcap A_i = \emptyset$. Since $x \notin \bigcap A_i$, there is an i_1 such that $x \notin A_{i_1}$. Since $X - A_{i_1}$ is a neighborhood of x and $x_i \overset{w}{\longrightarrow} x$, there is an $i_2 \geq i_1$ such that $x_{i_2} \in \text{Cl}(X - A_{i_1})$. But

$$x_{i_2} \in \text{Int}(A_{i_2}) \subset \text{Int}(A_{i_1}) = X - \text{Cl}(X - A_{i_1})$$

and so we have a contradiction. Thus $\bigcap A_i \neq \emptyset$. Since $\bigcap A_i \subset \bigcap \mathcal{A}$, $\bigcap \mathcal{A} \neq \emptyset$. By theroem 2.5, X is H-closed.

THEOREM 2.8. *Let X be a H-closed space. If A is a w-closed subset of X, then A is H-closed.*

PROOF. Let (x_i) be a net in A. Then (x_i) is a net in X. Since X is H-closed, (x_i) has a w-convergent subnet. Let $x_i \overset{w}{\longrightarrow} x \in X$. Since $x \in \text{Cl}_w(A) = A$, A is H-closed.

It is easy to show that the following theorem holds.

THEOREM 2.9. *Let X be completely Hausdorff space. Then every H-closed subset of X is w-closed.*

DEFINITION 2.7 Let X and Y be spaces. A function $f : X \to Y$ is said to be ***σ-continuous at*** a point x of X if for each neighborhood $U$of $f(x)$ there is a neighborhood V of x such that $f(\text{Cl}(V)) \subset \text{Cl}(U)$. f is said to be ***σ-continuous*** if f is σ-continuous at all $x \in X$.

Clearly, continuous functions are σ-continuous. σ-continuity is characterized by the w-convergence property in the following.

THEOREM 2.10. *Let X and Y be spaces. A function $f : X \to Y$ is σ-continuous at $x \in X$ if and only if for any net (x_i) in X which w-converges to x the net $(f(x_i))$ in Y w-converges to $f(x)$.*

PROOF. ($\Rightarrow$) Given any neighborhood U of $f(x)$, there is a neighborhood V of x such that $f(\mathrm{Cl}(V)) \subset \mathrm{Cl}(U)$. Also, there is an i_1 such that $x_i \in \mathrm{Cl}(V)$ for all $i \geq i_1$. Since $f(x_i) \in f(\mathrm{Cl}(V)) \subset \mathrm{Cl}(U)$ for all $i \geq i_1$, we have $f(x_i) \xrightarrow{w} f(x)$.

($\Leftarrow$) Suppose that f is not σ-continuous at x. Then there is a neighborhood U of $f(x)$ such that $f(\mathrm{Cl}(V)) \not\subset \mathrm{Cl}(U)$ for all neighborhoods V of x. Let (V_i) be the family of neighborhoods of x with the reverse inclution order. For each i, since $f(\mathrm{Cl}(V_i)) \not\subset \mathrm{Cl}(U)$, there is an $x_i \in \mathrm{Cl}(V_i)$ such that $f(x_i) \notin \mathrm{Cl}(U)$. Then the net (x_i) in X w-converges to x but the net $(f(x_i))$ in Y does not w-converge to $f(x)$. Thus we have a contradiction. Hence f is σ-continuous at x.

DEFINITION 2.8 Let X and Y be spaces. A function $f : X \to Y$ is said to have *w-closed graph* if its graph $G(f) = \{(x, f(x)) \mid x \in X\}$ is w-closed subset of $X \times Y$.

THEOREM 2.11. *Let X and Y be spaces. A function $f : X \to Y$ has a w-closed graph if and only if for any net (x_i) in X, $x_i \xrightarrow{w} x \in X$ and $f(x_i) \xrightarrow{w} y \in Y$ implies $y = f(x)$.*

PROOF. ($\Rightarrow$) Since $((x_i, f(x_i)))$ is a net in $G(f)$ and $(x_i, f(x_i)) \xrightarrow{w} (x, y)$, $(x, y) \in \mathrm{Cl}_w(G(f)) = G(f)$. Thus $y = f(x)$.

($\Leftarrow$) Let $(x, y) \in \mathrm{Cl}_w(G(f))$. There is a net (x_i) in X such that $(x_i, f(x_i)) \xrightarrow{w} (x, y)$. Since $x_i \xrightarrow{w} x$ and $f(x_i) \xrightarrow{w} y$, $y = f(x)$. Thus $(x, y) \in G(f)$. Hence $G(f)$ is w-closed.

THEOREM 2.12. *Let Y be a completely Hausdorff space. Then every σ-continuous function $f : X \to Y$ has a w-closed graph.*

PROOF. Let $(x, y) \in \mathrm{Cl}_w(G(f))$. There is a net (x_i) in X such that $(x_i, f(x_i)) \xrightarrow{w} (x, y)$. Then $x_i \xrightarrow{w} x$ and $f(x_i) \xrightarrow{w} y$. Since f is σ-continuous at x, $f(x_i) \xrightarrow{w} f(x)$. Since Y is completely Hausdorff, $y = f(x)$. This implies $(x, y) \in G(f)$. Hence $G(f)$ is w-closed.

The converse of the above theorem does not hold.

THEOREM 2.13. *Let Y be a H-closed space. If a function $f : X \to Y$ has a w-closed graph, then f is σ-continuous.*

PROOF. Let (x_i) be a net in X and $x_i \xrightarrow{w} x \in X$. Since Y is H-closed, the net $(f(x_i))$ in Y has a w-convergent subnet by theorem 2.7. Let $f(x_i) \xrightarrow{w} y \in Y$. Since $(x_i, f(x_i)) \xrightarrow{w} (x, y)$, $(x, y) \in \mathrm{Cl}_w(G(f)) = G(f)$. Thus $y = f(x)$ and so $f(x_i) \xrightarrow{w} f(x)$. This means that f is σ-continuous at x.

The σ-continuous image of a H-closed space is also H-closed.

THEOREM 2.14. *Let X be a H-closed space and Y a space. If $f : X \to Y$ is a σ-continuous surjection, then Y is H-closed.*

PROOF. Let (y_i) be a net in Y. For each i, there is an $x_i \in X$ such that $y_i = f(x_i)$. Since X is H-closed, there is a subnet (x_{i_k}) of (x_i) and an $x \in X$ such that $x_{i_k} \xrightarrow{w} x$. Since f is σ-continuous at x, $f(x_{i_k}) \xrightarrow{w} f(x)$. Thus Y is H-closed.

THEOREM 2.15. *Let $\{X_k\}$ be a family of sapces. Then ΠX_k is H-closed if and only if X_k is H-closed for all k.*

PROOF. ($\Rightarrow$) By theorem 2.14, X_k is$H$-closed for all k.
($\Leftarrow$) Let (x_i) be a net in ΠX_k. For each k, since $(p_k(x_i))$ is a net in X_k and X_k is H-closed, $(p_k(x_i))$ has a w-convergent subnet. We may assume that $p_k(x_i) \xrightarrow{w} x_k \in X_k$. Let $x = (x_k) \in \Pi X_k$. Then $x_i \xrightarrow{w} x$. Thus ΠX_k is H-closed.

3. W-Lindelöf spaces

DEFINITION 3.1 A space X is said to be *w-Lindelöf* if for each open cover $\{U_i\}$ of X there are countably many i_k such that $X = \bigcup \mathrm{Cl}(U_{i_k})$.

Clearly the Lindelöf property implies the w-Lindelöf property. But the following example shows that the converse need not hold.

EXAMPLE. Let $X = \{(x, y) \in R^2 \mid y \geq 0\}$. For any $(x, y) \in X$ and $r > 0$, let

$$N_r(x,y) = \begin{cases} B_r(x,y) \text{ if } y > 0 \text{ and } r \leq y \\ B_r(x,r) \cup \{(x,0)\} \cup B_r(0,r) \text{ if } y = 0. \end{cases}$$

We take $\{N_r(x,y)\}$ as a basis for the topology. We shall show that X is not Lindelöf. Let

$$\mathcal{U} = \{N_1(x,y) \mid x \in R, y \geq 1\} \cup \{N_1(x,0) \mid x \in R\}.$$

Then $\mathcal{U}$ is an open cover of X. Note that $(z,0) \notin N_1(x,y)$ if $y \geq 1$ and $(z,0) \notin N_1(x,0)$ if $x \neq z$. Thus $\mathcal{U}$ has no countable subcover. This implies that X is not Lindelöf.

Let us show that X is w-Lindelöf. Given any open cover $\mathcal{U}$ of X, let

$$\beta_1 = \{B \in \beta \mid \text{there is an } U \in \mathcal{U} \text{ such that } B \subset U\}$$

where $\beta = \{N_r(x,y) \mid x, y \text{ and } r \text{ are rational numbers}, 0 < r \leq y\}$. Clearly, β_1 is countable and so we let $\beta_1 = \{B_n\}$. For each n, there is an $U_n \in \mathcal{U}$ such that $B_n \subset U_n$. Also, there exists a $V \in \mathcal{U}$ such that $(0,0) \in V$. We shall show that $X = \bigcup \mathrm{Cl}(U_n) \cup \mathrm{Cl}(V)$. Let $(x,y) \in X$.

Case 1 : $y > 0$. There is an $U \in \mathcal{U}$ such that $(x,y) \in U$. Also, there is a positive rational number r such that $N_{2r}(x,y) \subset U$. We have $(a,b) \in N_r(x,y)$ for some rational numbers a and b. Then

$$(x,y) \in N_r(a,b) \subset N_{2r}(x,y) \subset U.$$

Thus $N_r(a,b) \in \beta_1$ and so $N_r(a,b) = B_{n_1}$ for some n_1. Therefore

$$(x,y) \in B_{n_1} \subset U_{n_1} \subset \mathrm{Cl}(U_{n_1}).$$

Case 2 : $y = 0$. For any neighborhood W of $(x,0)$ there is an $r_1 > 0$ such that $N_{r_1}(x,0) \subset W$. Since V is a neighborhood of $(0,0)$, there is an $r_2 > 0$ such that $N_{r_2}(0,0) \subset V$. It is clear that $W \cap V \neq \emptyset$. Thus $(x,0) \in \mathrm{Cl}(V)$. Hence X is w-Lindelöf.

In regular spaces, w-Lindelöfness implies Lindelöf.

THEOREM 3.1. *Let X be a regular space. If X is w-Lindelöf, then X is Lindelöf.*

PROOF. Let $\{U_i\}$ be an open cover of X. For each $x \in X$, there is an i_x such that $x \in U_{i_x}$. Since X is regular, there is a neighborhood V_x of x such that $\mathrm{Cl}(V_x) \subset U_{i_x}$. Then $\{V_x\}$ is an open cover of X. Since X is w-Lindelöf, there are countably many x_k such that $X = \bigcup \mathrm{Cl}(V_{x_k}) \subset \bigcup U_{i_k}$. This proves that X is Lindelöf.

Now we obtain a characterization of w-Lindelöf spaces.

THEOREM 3.2. *A space X is w-Lindelöf if and only if for each family $\{A_i\}$ of closed subsets of X satisfying $\bigcap \mathrm{Int}(A_{i_k}) \neq \emptyset$ for all countably many i_k, $\bigcap A_i \neq \emptyset$.*

PROOF. ($\Rightarrow$) Let $\bigcap A_i = \emptyset$. Since $X = X - \emptyset = X - \bigcap A_i = \bigcup(X - A_i)$, $\{X - A_i\}$ is an open cover of X. Since X is w-Lindelöf, there are countably many i_k such that $X = \bigcup \mathrm{Cl}(X - A_{ik})$. Then

$$\emptyset = X-X = X-\bigcup \mathrm{Cl}(X-A_{i_k}) = \bigcap(X-\mathrm{Cl}(X-A_{i_k})) = \bigcap \mathrm{Int}(A_{i_k}).$$

This is a contradiction. Thus $\bigcap A_i \neq \emptyset$.

($\Rightarrow$) Suppose that X is not w-Lindelöf. There is an open cover $\{U_i\}$ of X such that $\bigcup \mathrm{Cl}(U_{i_k}) \neq X$ for all countably many i_k. Since $\{X - U_i\}$ is a family of closed subsets of X and

$$\bigcap \mathrm{Int}(X - U_{i_k}) = \bigcap(X - \mathrm{Cl}(U_{i_k})) = X - \bigcup \mathrm{Cl}(U_{i_k}) \neq \emptyset$$

for all countably many i_k, $\bigcap(X - U_i) \neq \emptyset$. Then $\bigcup U_i = X - \bigcap(X - U_i) \neq X$. This is a contradiction. Thus X is w-Lindelöf.

DEFINITION 3.2 Let X be a space. A filter $\mathcal{A}$ in X is said to *w-accumulate* to a point x of X, denoted by $\mathcal{A} \overset{w}{\propto} x$, if $A \cap \mathrm{Cl}(U) \neq \emptyset$ for all neighborhoods U of x and $A \in \mathcal{A}$.

THEOREM 3.3. *Let X be a space. The following statements are equivalent.*

(1) *X is w-Lindelöf.*
(2) *For any open filterbase $\{U_i\}$ in X, if $\bigcap U_{i_k} \neq \emptyset$ for all countably many i_k, then there is a point x of X such that $\{U_i\}$ w-accumulates to x.*

PROOF. (1) $\Rightarrow$ (2) Suppose that $\{U_i\} \overset{w}{\not\propto} x$ for all $x \in X$. For each $x \in X$, there is a neighborhood V_x of x and i_x such that $\mathrm{Cl}(V_x) \cap U_{i_k} = \emptyset$. Clearly, $U_{i_k} \subset X - \mathrm{Cl}(V_{x_k})$ and $\{V_x\}$ is an open cover of X. Since X is w-Lindelöf, there are countably many x_k such that $X = \bigcup \mathrm{Cl}(V_{x_k})$. From the observation of

$$\bigcap U_{i_{x_k}} \subset \bigcap(X - \mathrm{Cl}(V_{x_k})) = X - \bigcup \mathrm{Cl}(V_{x_k}) = X - X = \emptyset,$$

we have $\bigcap U_{i_{x_k}} = \emptyset$. This is a contradiction. This proves (2).

(2) $\Rightarrow$ (1) Suppose that X is not w-Lindelöf. Then there is an open cover $\{U_i\}$ of X such that $\bigcup \mathrm{Cl}(U_{i_k}) \neq X$ for all countably many i_k. Let $\mathcal{U}$ be the family of open subsets of X containing $\bigcap(X - \mathrm{Cl}(U_{i_k}))$ for some countably many i_k. Since $\bigcap(X - \mathrm{Cl}(U_{i_k})) = X - \bigcup \mathrm{Cl}(U_{i_k}) \neq \emptyset$ for all countably many i_k, $\mathcal{U}$ is an open filterbase in X satisfying the countable intersection condition. By (1), there is an $x \in X$ such that $\mathcal{U} \overset{w}{\propto} x$. Then $x \in U_{i_1}$ for some i_1. Since $X - \mathrm{Cl}(U_{i_1}) \in \mathcal{U}$, $(X - \mathrm{Cl}(U_{i_k})) \cap \mathrm{Cl}(U_{i_1}) \neq \emptyset$. This is a contradiction. Thus X is w-Lindelöf.

For the invariance property of w-Lindelöf spaces, we have

THEOREM 3.4. *Let X be a w-Lindelöf space and Y a space. If $f : X \to Y$ is a continuous surjection, then Y is w-Lindelöf.*

PROOF. Let $\{U_i\}$ be an open cover of Y. Then $\{f^{-1}(U_i)\}$ is an open cover of X. Since X is w-Lindelöf, there are countably many i_k such that $X = \bigcup \mathrm{Cl}(f^{-1}(U_{i_k}))$. Thus we have

$$\begin{aligned} Y = f(X) &= f\left(\bigcup \mathrm{Cl}\left(f^{-1}(U_{i_k})\right)\right) \\ &= \bigcup f\left(\mathrm{Cl}\left(f^{-1}(U_{i_k})\right)\right) \subset \bigcup \mathrm{Cl}\left(ff^{-1}\left(U_{i_k}\right)\right) = \bigcup \mathrm{Cl}(U_{i_k}). \end{aligned}$$

Consequently, Y is w-Lindelöf.

The product of two w-Lindelöf spaces need not be w-Lindelöf. However, we improve this by imposing H-closedness.

THEOREM 3.5. *Let X be a w-Lindelöf space and Y a H-closed space. Then the product space $X \times Y$ is w-Lindelöf.*

PROOF. Let $\{W_i\}$ be an open cover of $X \times Y$. For each $(x, y) \in X \times Y$, there is an $i(x, y)$ such that $(x, y) \in W_{i(x,y)}$. Also, there are neighborhoods $U_{(x,y)}$ of x and $V_{(x,y)}$ of y such that $U_{(x,y)} \times V_{(x,y)} \subset W_{i(x,y)}$. Clearly, $\{V_{(x,y)} \mid y \in Y\}$ is an open cover of Y. Since Y is H-closed, there are finitely many $y_1, \dots, y_{n(x)}$ such that $Y = \bigcup_{j=1}^{n(x)} \mathrm{Cl}(V_{(x,y_j)})$. Let $U_x = \bigcap_{j=1}^{n(x)} U_{(x,y_j)}$. Then U_x is a neighborhood of x and

$$U_x \times V_{(x,y_j)} \subset U_{(x,y_j)} \times V_{(x,y_j)} \subset W_{i(x,y_j)}$$

for all $j = 1, \dots, n(x)$. Since X is w-Lindelöf, for an open cover $\{U_x \mid x \in X\}$, there are contably many $x_1, x_2, \dots$ such that $X = \bigcup_{k=1}^{\infty} \mathrm{Cl}(U_{x_k})$. Then we have

$$X \times Y = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{n(x_k)} \mathrm{Cl}(U_{x_k}) \times \mathrm{Cl}\left(V_{(x_k, y_j)}\right) \subset \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{n(x_k)} \mathrm{Cl}\left(W_{i_{(x_k, y_j)}}\right).$$

Thus $X \times Y$ is w-Lindelöf.

Finally, w-Lindelöfness and σ-compactness are equivalent in locally compact spaces.

THEOREM 3.6. *Let X be a locally compact space. Then X is w-Lindelöf if and only if X is σ-compact.*

PROOF. ($\Rightarrow$) For each $x \in X$ there is a neighborhood U_x of x such that $\mathrm{Cl}(U_x)$ is compact. $\{U_x\}$ is an open cover of X. Since X is w-Lindelöf, there are countably many x_k such that $X = \bigcup \mathrm{Cl}(U_{x_k})$. Thus X is σ-compact.

($\Rightarrow$) It is clear.

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