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A Note on Hermitian Elements of a Banach Algebra

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ABSTRACT. In this paper, the abelian property of Hermitian elements holds not generally in Banach algebra, but in the case that some conditions satisfy, they are abelian. By using property of [1], [2], the Hermitian elements a and b in Banach algebras have been shown that $ab = ba$.

I. Introduction

Throughout this paper, A denotes a complex unital Banach algebra. An element h of A is *Hermitian* if its numerical range is real. Let H denote the set of all Hermitian elements of A. This paper deals with the following questions; If a, b, $ab \in H$, does it then follow that $ab = ba$?

II. Preliminaries

The following results can be found in [1] and [2].

LEMMA 2.1.

i) *H is a real linear subspace of A.*
ii) $H \cap iH = \{0\}$

LEMMA 2.2. *If h, $k \in H$, then $i(hk - kh) \in H$.*

LEMMA 2.3. *(Sinclair's theorem) If $h \in H$, then $r(h) = \|h\|$, where r denotes spectral radius.*

We use the following statement, which was proved independently by Kleinecke [4] and Shirokov [6].

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LEMMA 2.4. *Let A be a Banach algebra. Let $x, y \in A$. Let x commute with $xy - yx$. Then $xy - yx$ is quasi-nilpotent, i.e., $r(xy - yx) = 0$.*

In general, although h belongs to H, it does not follow that $h^2 \in H$(Crabb in [1] constructed counterexample). Berkson [1] has proved various partial positive results; one of them is that, if a, b, ab, a^2 and b^2 are all Hermitians, then $ab = ba$.

III. Main Results

The purpose of this paper is to prove the following positive results.

THEOREM 3.1. *Let $a, b, x \in H$.*

i) *If $x^2 = 0$ and $xa = 0$, then $ax = xa$.*
ii) *If $x = i(ab - ba)$, $x^2 = 0$ and $ax = 0$, then $x = 0$, i.e., $ab = ba$.*

PROOF. i) Let $a, x \in H$, $x^2 = 0$ and $xa = 0$. Then by Lemma 2.2, $i(ax - xa) \in H$. And also $x^2a - 2xax + ax^2 = 0$. And equivalently, $x(x(ia) - (ia)x) = (x(ia) - (ia)x)x$. Thus by Lemmas 2.3 and 2.4

$$0 = r(x(ia) - (ia)x) = r(i(xa - ax)) = \|i(xa - ax)\| = \|ax - xa\|.$$

Therefore $ax = xa$.

ii) Let a, b, $x = i(ab - ba) \in H$. Suppose $x^2 = 0$ and $xa = 0$. Then by Theorem 3.1 i), $ax = xa$. And equivalently, $a(a(ib) - (ib)a) = (a(ib) - (ib)a)a$. And by Lemma 2.2, $i(ab - ba) \in H$. Thus by Lemmas 2.3 and 2.4,

$$0 = r(a(ib) - (ib)a) = r(i(ab - ba)) = \|i(ab - ba)\| = \|ab - ba\|.$$

Therefore $x = 0$, i.e., $ab = ba$.

LEMMA 3.2. *Let $a, b, ab \in H$.*

i) *$a^2b = ba^2$ if and only if $a^2b - ba^2 \in H$.*
ii) *$a^2b = aba$ if and only if $a^2b - aba \in H$.*
iii) *$ab^2 = bab$ if and only if $ab^2 - bab \in H$.*
iv) *$a^2b - ba^2 \in H$ if and only if $a^2b = aba$.*

PROOF. Let a, b, $ab \in H$. i) Apply Lemma 2.2 with $h = a$, $k = ab$, so $i(a(ab) - (ab)a) = i(a^2b - aba) \in H$. Clearly, it suffices to show that if $a^2b - ba^2 \in H$, then $a^2b = ba^2$. Suppose $a^2b - ba^2 \in H$. And by Lemma 2.2, $i(ab - ba) \in H$. Hence, apply Lemma 2.2 with $h = a$, $k = i(ab-ba)$, so $i(a(i(ab-ba))-i(ab-ba)a) = -(a^2b-2aba+ba^2) \in H$, i.e., $a^2b - 2aba + ba^2 \in H$. Hence, $2(a^2b - aba) = (a^2b - ba^2) + (a^2b-2aba+ba^2) \in H$. i.e., $a^2b-aba \in H$. Thus $a^2b-aba \in H \cap iH$. Hence, by Lemma 2.1 ii), $a^2b = aba$. And so,

$$aba - ba^2 = -(a^2b - 2aba + ba^2) + (a^2b - aba) \in H.$$

And now, let $x = aba - ba^2$. Then $x \in H$,

$$\begin{aligned} x^2 &= (aba - ba^2)(aba - ba^2) \\ &= aba^2ba - ababa^2 - ba^3ba + ba^2ba^2 \\ &= aba^2ba - aba^2ba - ba^3ba + ba^3ba \\ &= 0 \end{aligned}$$

and $ax = a(aba-ba^2) = a^2ba - aba^2 = a^2ba - a^2ba = 0$. Since $x^2 = 0$ and $ax = 0$, $ax = xa$. And equivalently,

$$a(a(ba) - (ba)a) = (a(ba) - (ba)a)a.$$

Thus by Lemmas 2.3 and 2.4,

$$0 = r(a(ba) - (ba)a) = \|aba - ba^2\|.$$

Therefore $aba = ba^2$. Consequently, $a^2b = ba^2$.

ii) Clearly, it suffices to show that if $a^2b-aba \in H$, then $a^2b = aba$. And so, suppose $a^2b - aba \in H$. Apply Lemma 2.2 with $h = a$, $k = ab$. So $i(a^2b-aba) \in H$. Thus $a^2b-aba \in H \cap iH$. Therefore by Lemma 2.1 ii), $a^2b = aba$.

iii) Clearly, it suffices to show that if $ab^2 - bab \in H$, then $ab^2 = bab$. Suppose $ab^2 - bab \in H$. Apply Lemma 2.2 with $h = b$, $k = ab$, so $i(bab - ab^2) \in H$. Thus $ab^2 - bab \in H \cap iH$. Therefore by Lemma 2.1 ii), $ab^2 = bab$.

iv) If $a^2b - ba^2 \in H$, then we know that $a^2b = aba$ in the process of the proof of Lemma 3.2 i). And so, it is enough to show that if

$a^2b = aba$, then $a^2b - ba^2 \in H$. Suppose $a^2b = aba$. Apply Lemma 2.2 with $h = a$, $k = b$, so $i(ab - ba) \in H$. Apply Lemma 2.2 with $h = b$, $k = i(ab - ba)$, so $i(a(i(ab - ba)) - (i(ab - ba))a) = -(a^2b - 2aba + ba^2) \in H$, i.e., $a^2b - 2aba + ba^2 \in H$. Hence $-aba + ba^2 = -(a^2b - aba) + (a^2b - 2aba + ba^2) \in H$, i.e., $aba - ba^2 \in H$. Therefore $a^2b - ba^2 = (a^2b - aba) + (aba - ba^2) \in H$.

THEOREM 3.3. *Let a, b, $ab \in H$.*

i) *If $a^2b = aba$, then $ab = ba$.*
ii) *If $ba^2 = aba$, then $ab = ba$.*

PROOF: Let a, b, $ab \in H$. i) Suppose $a^2b = aba$. Apply Lemma 2.2 with $h = a$, $k = b$, so $i(ab - ba) \in H$. Apply Lemma 2.2 with $h = a$, $k = i(ab - ba)$. So $i(a(i(ab - ba)) - i(ab - ba)a) = -(a^2b - 2aba + ba^2) \in H$, i.e., $a^2b - 2aba + ba^2 \in H$. And by assumption, $a^2b - aba = 0$. Thus $ba^2 - aba \in H$. And now, let $x = ba^2 - aba$. Then $x \in H$,

$$\begin{aligned} x^2 &= (ba^2 - aba)(ba^2 - aba) \\ &= ba^2ba^2 - ba^3ba - ababa^2 + aba^2ba \\ &= ba^2ba^2 - ba^2ba^2 - ababa^2 + ababa^2 \\ &= 0 \end{aligned}$$

and $ax = a(ba^2 - aba) = aba^2 - a^2ba = aba^2 - aba^2 = 0$. Since $x^2 = 0$ and $ax = 0$, $ax = xa$. And equivalently $a(a(ba) - (ba)a) = (a(ba) - (ba)a)a$. Therefore, by Lemmas 2,3 and 2.4,

$$0 = r(a(ba) - (ba)a) = \|aba - ba^2\|.$$

Thus $aba = ba^2$. Hence $a^2b - 2aba + ba^2 = 0$, and equivalently,

$$a(a(ib) - (ib)a) = (a(ib) - (ib)a)a.$$

Thus by Lemmas 2.3 and 2.4,

$$0 = r(a(ib) - (ib)a) = r(i(ab - ba)) = \|i(ab - ba)\| = \|ab - ba\|.$$

Therefore $ab = ba$.

ii) Suppose $ba^2 = bab$. Apply Lemma 2.2 with $h = a$, $k = b$, so $i(ab - ba) \in H$. Apply Lemma 2.2 with $h = a$, $k = i(ab - ba)$, so $(i(a(i(ab - ba)) - (i(ab - ba))a) = -(a^2b - 2aba + ba^2) \in H$. i.e., $a^2b - 2aba + ba^2 \in H$. But by assumption $ba^2 = aba$. Thus $a^2b - aba \in H$. Apply Lemma 2.2 with $h = a$, $k = ab$, so $i(a(ab) - (ab)a) = i(a^2b - aba) \in H$. Hence $a^2b - aba \in H \cap iH$. Therefore by Lemma 2.1 ii), $a^2b = aba$. Hence $a^2b - 2aba + ba^2 = 0$. And equivalently, $a(a(ib) - (ib)a) = (a(ib) - (ib)a)a$. Thus by Lemmas 2.3 and 2.4,

$$0 = r(a(ib) - (ib)a) = r(i(ab - ba)) = \|i(ab - ba)\| = \|ab - ba\|.$$

Therefore $ab = ba$.

COROLLARY 3.4. *Let a, b, $ab \in H$.*

i) *If a^2b, $aba \in H$, then $ab = ba$.*
ii) *If ba^2, $aba \in H$, then $ab = ba$.*

PROOF. We omit its proof, for its statements are special cases of Theorem 3.3.

THEOREM 3.5. *Let a, b, $ab \in H$. If $a^2b \in H$ and $a^3 \in H$, then $ab = ba$.*

PROOF. Let $a, b, ab, a^2b, a^3 \in H$. Then by Lemma 2.2, $i(ab - ba) \in H$, $i(a^2b - aba) \in H$. And also, $i(a(i(ab - ba)) - (i(ab - ba))a) = -(a^2b - 2aba + ba^2) \in H$, i.e., $a^2b - 2aba + ba^2 \in H$. But since $a^2b \in H$, $-2aba + ba^2 \in H$. Hence,

$$\begin{aligned}&i(-2a^2ba + 3aba^2 - ba^3)\\ =&i(-2a^2ba + aba^2 + 2aba^2 - ba^3)\\ =&i(a(-2aba + ba^2) - (-2aba + ba^2)a) \in H.\end{aligned}$$

And also, apply Lemma 2.2 with $h = a$, $k = a^2b - 2aba + ba^2$. So

$$\begin{aligned}&i(a(a^2b - 2aba + ba^2) - (a^2b - 2aba + ba^2)a)\\ =&i(a^3b - 2a^2ba + aba^2 - a^2ba + 2aba^2 - ba^3)\\ =&i(a^3b - 3a^2ba + 3aba^2 - ba^3) \in H.\end{aligned}$$

But since $i(a^2b - aba) \in H$, by Lemma 2.2,

$$\begin{aligned} &i(a(i(a^2b - aba)) - i(a^2b - aba)a) \\ &= -(a^3b - a^2ba - a^2ba + aba^2) \in H. \end{aligned}$$

i.e., $a^3b - 2a^2ba + aba^2 \in H$. Apply Lemma 2.2. with $h = b$, $k = a^3$, so $i(a^3b - ba^3) \in H$. From these results,

$$\begin{aligned} &i(a^3b - 3a^2ba + 3aba^2 - ba^3) - i(a^3b - ba^3) \\ &= i(-3a^2ba + 3aba^2) \in H, \end{aligned}$$

i.e., $i(aba^2 - a^2ba) \in H$. On the other hand,

$$\begin{aligned} &i(a^3b - 3a^2ba + 3aba^2 - ba^3) \\ &\quad - i(-2a^2ba + 3aba^2 - ba^3) + i(aba^2 - a^2ba) \\ &= i(a^3b - 3a^2ba + 2a^2ba + 3aba^2 - ba^3 + ba^3 + aba^2 - a^2ba) \\ &= i(a^3b - 2a^2ba + aba^2) \in H. \end{aligned}$$

Thus $a^3b - 2a^2ba + aba^2 \in H \cap iH$. Therefore by Lemma 2.1 ii), $a^3b - 2a^2ba + aba^2 = 0$. And equivalently

$$a[a\{a(ib)\} - \{a(ib)\}a] - [a\{a(ib)\} - \{a(ib)\}a]a.$$

But since $i(a^2b - aba) \in H$, by Lemmas 2.3 and 2.4,

$$\begin{aligned} 0 &= r(a(a(ib)) - a(ib)a) = r(i(a^2b - aba)) \\ &= \|i(a^2b - aba)\| = \|a^2b - aba\|. \end{aligned}$$

Thus $a^2b = aba$. Therefore by Theorem 3.3 i), $ab = ba$.

THEOREM 3.6. *Let a, b, $ab \in H$. If a^3, a^4, a^2ba^3 and $a^5b \in H$, then $ab = ba$.*

PROOF. Let a, b, ab, a^3, a^4, a^2ba^3, $a^5b \in H$. Then by Lemma 2.2, $i(ab - ba) \in H$. Now, apply Lemma 2.2 with $h = a$, $k = ab$, so

$$i(a^2b - aba) \in H.$$

Apply Lemma 2.2 with $h = a$, $k = i(ab - ba)$, so

$$-i(a(i(ab - ba)) - i(ab - ba)a) = a^2b - 2aba + ba^2 \in H.$$

Apply Lemma 2.2 with $h = a$, $k = a^2b - 2aba + ba^2$, so

$$\begin{aligned} &i(a(a^2b - 2aba + ba^2) - (a^2b - 2aba + ba^2)a) \\ =&i(a^3b - 3a^2ba + 3aba^2 - ba^3) \in H. \end{aligned}$$

Apply Lemma 2.2 with $h = a^3$, $k = b$, so

$$i(a^3b - ba^3) \in H.$$

Hence $3i(a^2ba - aba^2) = i(a^3b - ba^3) - i(a^3b - 3a^2ba + 3aba^2 + ba^3) \in H$, i.e., $i(a^2ba - aba^2) \in H$. Apply Lemma 2.2 with $h = a^3$, $k = ab$, so

$$i(a^4b - aba^3) \in H.$$

Apply Lemma 2.2 with $h = a^4$, $k = b$, so

$$i(a^4b - ba^4) \in H.$$

Apply Lemma 2.2 with $h = a^3$, $k = i(ab - ba)$, so

$$-i(a^3(i(ab - ba)) - i(ab - ba)a^3) = a^4ba - a^3ba - aba^3 + ba^4 \in H.$$

Apply Lemma 2.2 with $h = a$, $k = i(a^4b - aba^3)$, so

$$-i(a(i(a^4b - aba^3)) - i(a^4b - aba^3)a) = a^5b - a^4ba - a^2ba^3 + aba^4 \in H.$$

Apply Lemma 2.2 with $h = a$, $k = i(a^2ba - aba^2)$, so

$$-i(a(i(a^2ba - aba^2)) - i(a^2ba - aba^2)a) = a^3ba - 2a^2ba^2 + aba^3 \in H.$$

Apply Lemma 2.2 with $h = a$, $k = a^3ba - 2a^2ba^2 + aba^3$, so

$$\begin{aligned} &i(a(a^3ba - 2a^2ba^2 + aba^3) - (a^3ba - 2a^2ba^2 + aba^3)a) \\ =&i(a^4ba - 3a^3ba^2 + 3a^2ba^3 - aba^4) \in H. \end{aligned}$$

Apply Lemma 2.2 with $h = a$, $k = i(a^4b - ba^4)$, so

$$-i(a(i(a^4b - ba^4)) - i(a^4b - ab^4)a) = a^5b - a^4ba - aba^4 + b^5 \in H.$$

Apply Lemma 2.2 with $h = a^3$, $k = a^2b - 2aba - ba^2$, so

$$\begin{aligned} &i(a^3(a^2b - 2aba + ba^2) - (a^2b - 2aba + ba^2)a^3) \\ =&i(a^5b - 2a^4ba + a^3ba^2 - a^2ba^3 + 2aba^4 - ba^5) \in H. \end{aligned}$$

Hence, $i(a^5b - 5a^4ba + 10a^3ba^2 - 10a^2ba^3 + 5aba^4 - ba^5) \in H$. And by the following results it holds.

$$\begin{aligned} &2(a^5b - 2a^4ba - a^2ba^3 + aba^4) \\ =&(a^5b - 3a^4ba - 2a^2ba^3 + 3aba^4 - ba^5) \\ &+ (a^5b - a^4ba - a^2ba^3 + ba^5) \in H, \end{aligned}$$

i.e., $a^5b - 2a^4ba - a^2ba^3 + aba^4 \in H$. Hence $-a^2ba^3 + 2aba^4 - ba^5 = (a^5b - a^4ba - a^2ba^3 + aba^4) - (a^5b - a^4ba - aba^4 - ba^5) \in H$. And also,

$$a^4ba = (a^5b - a^4ba - a^2ba^3 + aba^4) - (a^5b - 2a^4ba - a^2ba^3 + aba^4) \in H.$$

Hence $a^5b - a^2ba^3 + aba^4 = (a^5b - 2a^4ba - a^2ba^3 + aba^4) + 2a^4ba \in H$. And $(3(a^4ba - 2a^3ba^2 + a^2ba^3) = (a^5b - a^4ba - a^2ba^3 + aba^4) - (a^5b - 4a^4ba + 6a^3ba^2 - 4a^2ba^3 + aba^4) \in H$. i.e., $a^4ba - 2a^3ba^2 + a^2ba^3 \in H$. But since $a^4ba \in H$, $2a^3ba^2 - a^2ba^3 \in H$. Besides, $a^5b \in H$, $a^2ba^3 \in H$. Thus from the above results,

$$\begin{aligned} &a^5b - 5a^4ba + 10a^3ba^2 - 10a^2ba^3 + 5aba^4 - ba^5 \\ =&(-2a^4b - a^4ba - a^2ba^3 + aba^4) + (-a^2ba^3 + 2aba^4 - ba^5) \\ &+ 5(2a^3ba^2 - a^2ba^3) + 3(a^5b - a^2ba^3 + aba^4) \in H. \end{aligned}$$

Therefore $a^5b - 5a^4ba + 10a^3ba^2 - 10aba^3 + 5aba^4 - 6a^5 \in H \cap iH$. Thus by Lemma 2.1 ii), $a^5b - 5a^4ba + 10a^3ba - 10a^2ba^3 + 5aba^4 - ba^5 = 0$. Equivalently, $a(a(y) - (y)a) = (a(y) - (y)a)a$, where $y = a^3b - 3a^2ba + 3aba^2 - ba^3$. Then $iy \in H$ and $i(a(iy) - (iy)a) \in H$, i.e., $ay - ya \in H$. Therefore by Lemmas 2.3 and 2.4, $0 = r(ay - ya) = \|ay - ya\|$. Thus $ay - ya$. And again, equivalently, $a(a(iz) - (iz)a) = (a(iz) - (iz)a)a$, where $z = a^2b - 2aba + ba^2$, also $z \in H$. Thus by Lemmas 2.3 and 2.4, $0 = r(a(iz) - (iz)a) = r(i(az - za)) = \|i(az - za)\| = \|az - za\|$. Therefore $az = za$, i.e.,

$$a(a(ab - ba) - (ab - ba)a) = (a(ab - ba) - (ab - ba)a)a.$$

Hence by Lemmas 2.3 and 2.4,

$$0 = r(a(ab - ba) - (ab - ba)a) = \|a(ab - ba) - (ab - ba)a\|.$$

Therefore $a(ab - ba) = (ab - ba)a$. And again, equivalently,

$$a(a(ib) - (ib)a) = (a(ib) - (ib)a)a.$$

Hence, by Lemmas 2.3 and 2.4,

$$0 = r(a(ib) - (ib)a) = \|a(ib) - (ib)a\| = \|i(ab - ba)\| = \|ab - ba\|.$$

Therefore $ab = ba$.

References

1. E. Berkson, *Hermitian projections and orthogonality in Banach spaces*, Proc. London Math. Soc. **24** (1972), 102–118.
2. F. F. Bonsall and J. Duncan, "Complete normed algebras," Springer Verlag, Berlin, 1973.
3. H. R. Dowson, T. A. Gillespie and P. G. Spain, *A commutativity theorem for Hermitian operators*, Math. Ann. **220** (1976), 215–217.
4. D. C. Kleinecke, *On operator commutators*, Proc. Amer. Math. Soc. **8** (1957), 535–536.
5. K. B. Laurson, *Central factorization in C^*-algebras and continuity of homorphisms*, J. London Math. Soc. **28** (1983), 123–130.
6. F. V. Shirokov, *Proof of a conjecture of Kaplansky*, Uspchi. Math. Nauk **11** (1956), 167–168.
7. J. Vukman, *A result concerning addtive functions in Hermitian Banach * algebras and an application*, Proc. Amer. Math. Soc. **11** (1984), 367–372.

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