JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 1, June 1988

Multipliers of Bergman Spaces

DO YOUNG KWAK AND GWANG-HUI KIM

ABSTRACT. In this paper, we study the multipliers of A_q^p into $L^{p'}$ when $0 < p' < p$. For this purpose, we study the condition on the measure μ satisfying $A_q^p \subset A^{p'}(d\mu)$. It turns out that the quotient $k_q = \mu/v_q$ over hyperbolic ball of radius less than 1 belongs to L_q^s, where $\frac{1}{s} + \frac{p'}{p} = 1$. For the proof, we replace the norm of k_q by the Riemann sum, and then use a result of interpolation theory.

1. Introduction.

Let $B = B^n$ be the unit ball in C^n and let $H(B)$ be the spaces of all holomorphic functions in the ball. For $q > 0$, we let

$$dv_q(z) = \frac{\Gamma(n+q)}{\pi^n \Gamma(q)} (1 - \|z\|^2)^{q-1} dv(z)$$

be the probability measure on B and define $L_q^p(B)$ to be the space of all measurable functions f on B for which $\|f\|_{p,q}^p = \int_B |f|^p \, dv_q$ is finite. Also we let $A_q^p(B) = L_q^p(B) \cap H(B)$ with the induced norm. Let μ be a positive finite Borel measure on B. When $n = 1$, a necessary and sufficient condition for μ to hold

$$(1.1) \qquad \left(\int_B |f|^{p'} \, d\mu \right)^{1/p'} \leq c \left(\int_B |f|^p \, dv_1 \right)^{1/p}$$

for all $f \in A_1^p(B)$ is well known provided $0 < p \leq p'$ [4]. For the case of polydisc in C^n we refer to [2]. To study the remaining case, $0 < p' < p$, we shall investigate measurable functions g which satisfy $gA_1^p \subset L_1^{p'}$.

Received by the editors on March 27, 1988.
1980 *Mathematics subject classifications*: Primary 32A.

LEMMA 1. *Given a measurable function g, a necessary and sufficient condition that $gA_1^p \subset L_1^{p'}$ is*

$$\left(\int |gf|^{p'}\, dv\right)^{1/p'} \leq c\left(\int |f|^p\, dv\right)^{1/p} \tag{1.2}$$

for all $f \in A_1^p$.

PROOF. Sufficiency is trivial. To show that it is also necessary, we let $T : A_1^p \to L_1^{p'}$ be the multiplication operator by g. Since $Tf \in L_1^{p'}$, $\int |gf|^{p'}\, dv$ is finite. Assume $\{f_n\}$ is a sequence in A_1^p whose limit is f, and if $Tf_n \to h$ in $L_1^{p'}$, then $\int |gf_n - h|^{p'}\, dv \to 0$. Now Fatou's lemma gives $\int |gf - h|^{p'}\, dv = 0$, i.e., $Tf = h$. By the colsed graph theorem, T is a continuous operator from A_1^p into $L_1^{p'}$, which implies (1.2).

2. Average of μ over $B(a,r)$.

For $a \in B$, we let

$$\varphi_a(z) = \left(a - \frac{\langle z,a\rangle}{\langle a,a\rangle}a - \sqrt{1-|a|^2}\left(z - \frac{\langle z,a\rangle}{\langle a,a\rangle}a\right)\right) \Big/ (1 - \langle z,a\rangle)$$

be the involution. We let $B(a,r) = \{z \in B : |\varphi_a(z)| < r\}$ and write $B(a)$ for $B(a,r)$ when there is no worry of confusion. It is known that $v_q(B(a,r)) = \left(\frac{1-|a|^2}{1-r^2|a|^2}\right)^{n+q}$. We let $S = B(a,r)$ and compute

$$\begin{aligned}
&\frac{1}{v_q(S)}\int_S |f|^{p'}\, dv_q \\
=&\frac{1}{v_q(S)}\int_{B(0)} |f\circ\varphi_a(z)|^{p'}\left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{n+q} dv_q(z) \\
\geq&\frac{v_q(B(0))}{v_q(B(a))}\left(1-|a|^2\right)^{n+q}|f(a)|^{p'} = \left(1-r^2|a|^2\right)^{n+q}|f(a)|^{p'} \\
\geq&(1-r^2)^{n+q}|f(a)|^{p'}.
\end{aligned}$$

Fubini's theorem gives

$$\int_B |f|^{p'}\, d\mu \leq \int_B |f(z)|^{p'}\int_{B(z,r)} (v_q(B(z,r)))^{-1}\, d\mu(\zeta)\, dv_q(z).$$

If $\zeta \in B(z,r)$ then

$$v_q(B(z,r))/v_q(B(\zeta,r)) \le \left(\frac{1-|z|^2}{(1-r^2)(1-|\zeta|^2)}\right)^{n+q}$$

Since $1-|\varphi_z(\zeta)|^2 = (1-|\zeta|^2)(1-|z|^2)/|1-\langle\zeta,z\rangle|^2 \ge 1-r^2$, we have

$$v_q(B(z,r))/v_q(B(\zeta,r)) \le 4^{n+q}/(1-r^2)^{2n+2q}.$$

If we put $k_q(z) = \mu(B(z,r))/v_q(B(z,r))$, then

$$\int |f|^{p'}\,d\mu \le c\int |f|^{p'} k_q\,dv_q.$$

COROLLARY. *If $k_q \in L^s_q$, $s = p/(p-p')$ then*

$$\left(\int |f|^{p'}\,d\mu\right)^{1/p'} \le c\left(\int |f|^p\,dv_q\right)^{1/p} \tag{2.1}$$

for all f in A^p_q.

The main theorem is the converse of (2.1).

THEOREM. *Let μ be a positive Borel measure on B and let $k_q(z) = \mu(B(z,r))/v_q(B(z,r))$, $0 < p' < p$. A necessary and sufficient condition that (2.1) holds for all f in A^p_q is that k_q belongs to L^s_q, where $1/s + p'/p = 1$.*

For the proof, we need some result from interpolation theory.

LEMMA 2. *$k_q \in L^s_q$ if and only if*

$$\sum (\mu(B_i)/v_q(B_i))^s\, v_q(B_i) < \infty, \tag{2.2}$$

where $\{B_i\} = \{B(z_i,\delta_i)\}$ is a disjoint cover of B.

PROOF. Assum $k_q \in L^s_q$. Then (2.2) is nothing but the Riemann sum of $\int |k_q|^s\,dv_q$.

DEFINITION. A sequence $\{a_i\}$ in B is said to be *separated* if there is a $\delta > 0$ such that $|\varphi_{a_i}(a_j)| > \delta$ whenever $i \ne j$. A separated sequence is called an *interpolation sequence* for A^p_q if the mapping $T : A^p_q \to \ell^p_q$ is onto, where ℓ^p_q is the space of sequences satisfying $\sum |a_i|^p(1-|a_i|^2)^{n+q} < \infty$. There is a result of R. Rochberg[**3**] which is a generalization of E. Amar on interpolation sequences[**1**].

THEOREM. *Suppose* $\inf|\varphi_{a_i}(a_j)| = K > 0$, *then* T *is a continuous map of* A^p_q *into* ℓ^p_q *and there is a* K_0 *such that if* $K > K_0$, *then* T *is onto.*

Now fix K so large that $\{a_i\}$ is an interpolation sequence. Let $r \in (0,1)$ be a small number. We construct a separated sequence $\{b_i\}$ such that the ellipses $\{B(b_i, r/2)\}$ cover B.

LEMMA 3. *There is a constant* A *such that if* r *is small, then for* $f \in A^p_q(B)$

$$\sum \int_{B(a_i,r)} |f(z) - f(a_i)|^{p'}\, d\mu(z) \tag{2.3}$$
$$\le A r^{p'} \|f\|_q^{p'} \left(\sum \mu(B(a_i,r)) v_q(B_i)^{1-s}\right)^{1/s}$$

where $B_i = B(a_i, K/2)$.

PROOF. $|f(z) - f(0)|^{p'} \le c r^{p'} \int_D |f|^{p'}\, dv_q$ if D is a small ball. The change of variable $z \to \varphi_{a_i}(a)$ gives

(2.4)
$$|f(z) - f(a_i)|^{p'} \le c r^{p'} \int_{B_i} |f|^{p'} \left(\frac{1 - |a_i|^2}{|1 - \langle z, a_i\rangle|^2}\right)^{n+q}$$
$$\le A r^{p'} \int_{B_i} |f|^{p'}\, dv_q / v_q(B_i).$$

Integration (2.4) w.r.t. μ over $B(a_i, r)$ and summing up, we see that the left hand side of (2.3) is less than

$$A r^{p'} \sum \int_{B_i} |f|^{p'}\, dv_q \mu(B(a_i,r)) v_q(B_i)^{-1}$$
$$\le A r^{p'} \sum \left(\int_{B_i} |f|^p\, dv_q\right)^{p'/p} \mu(B(a_i,r)) v_q(B_i)^{1/s-1}$$
$$\le A r^{p'} \left(\sum \int_{B_i} |f|^p\, dv_q\right)^{p'/p} \left(\sum \mu(B(a_i,r))^s v_q(B_i)^{1-s}\right)^{1/s}.$$

This gives (2.3). To complete the proof of the main theorem, we let μ be a measure satisfying (2.1). If we replace μ with $\chi_{\{|z|<r\}} \cdot \mu$ then

(2.1) is still valid. We only need to show $\|k_q\|_q^s$ is independent of r as $r \to 1$. Hence we assume without loss of generality that μ is compactly supported. From Lemma 3, if $f \in A_q^p$, $\|f\|_q^p \le K$ and $p' \le 1$, then with $S(k) = B(a_k, r)$,

(2.5)

$$\begin{aligned}
&\sum \int_{S(k)} |f|^{p'}\,d\mu \\
&\geq \sum \int_{S(k)} |f(a_k)|^{p'}\,d\mu - \sum \int_{S(k)} |f - f(a_k)|^{p'}\,d\mu \\
&\geq \sum |f(a_k)|^{P'} \mu(S(k)) - Ar^{p'} K^{p'} \left(\sum \mu(S(k))^s v_q(B_i)^{1-s}\right)^{1/s}.
\end{aligned}$$

Since $\{f(a_k)\}$ may assume the value of any sequence $\{c_k\}$ in ℓ_q^p, the sum $\sum |f(a_k)|^{p'} \mu(S(k))$ may assume the value

$$\begin{aligned}
&\left(\sum \mu(S(k))^s (1 - |a_k|^2)^{(n+q)(1-s)}\right)^{1/s} \\
&= \sup \sum |c_k|^{p'} \mu(B(a_k, r)),
\end{aligned}$$

where the supremum is taken over all $\{c_k\}$ with $\sum |c_k|^{p'} (1 - |a_k|^2)^{(n+q)} = 1$. Thus we have

$$\begin{aligned}
cK^{p'} &\geq \sum \int_{S(k)} |f|^{p'}\,d\mu \\
&\geq \left(1 - Ar^{p'} K^{p'}\right) \left(\sum \mu(S(k))^s v_q(B_k)^{1-s}\right)^{1/s}.
\end{aligned}$$

Now we choose $r^{p'} = 1/2AKp'$ and sum over the sequences $\{a_k\}$ to get

$$\left(\sum \mu(B(b_i, r))^s v_q(B_i)^{1-s}\right)^{1/s} \leq 2c^{p'} K^{p'}.$$

It remains to show that (2.5) implies $k_q \in L_q^s$. Set $\epsilon = r/2$ and define $k_q(z) = \mu(B(z, \epsilon))/v_q(B(z, \epsilon))$. If $z \in B(b_i, \epsilon)$ then $B(z, \epsilon) \subset B(b_i, r)$ and so $k_q(z) \leq \mu(B(b_i, r))/v_q(B(z, \epsilon)) \leq c\mu(B(b_i, r))/v_q(B_i)$. Thus

$$\begin{aligned}
\sum \int_{B(b_i, \epsilon)} k_q^s\,dv_q &\leq c \sum \mu(B(b_i, \delta))^s v_q(B_i)^{-s} v_q(B(b_i, \epsilon)) \\
&\leq c \sum \mu(B(b_i, r)) v_q(B_i)^{1-s} < C.
\end{aligned}$$

Since the ellipses $B(b_i, \epsilon)$ cover B we get $k_q \in L^p_q$, and the proof is complets.

References

1. E. Amer, *Suite d'interpolation pour le classes de Bergman de la boule et du polydisque de* $\mathbf{C}^n$, Canad. J. Math. **30** (1978), 711–737.
2. W. Hastings, *A carleson measure theorem for Bergman spaces*, Proc. Amer. Math. Soc. **52** (1975), 237–241.
3. R. Rochberg, *Interpolation by functions in Bergman spaces*, Michigan Math. J. **29** (1982), 229–236.
4. H.S. Shapiro and A.L. Shields, *On some interpolation problems for analytic functions*, Amer. J. Math. **83** (1961), 513–532.

Department of Mathematics
Korea Institute of Technology
Taejon, 302-338, Korea

Department of Mathematics
Chungnam National University
Taejon, 302-764, Korea