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Range of Operators and an Application to Existence of a Periodic Solution

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ABSTRACT. In this paper, we calculate the precise estimation of range of a Gateaux differentiable operator, and apply to the existence of a periodic solution of the second order nonlinear differential equation

$$z'' + Az' + G(z) = e(t) = e(t + 2\pi).$$

1. Introduction

Let X and Y be two Banach spaces and $T : X \to Y$ a nonlinear mapping. There are many approaches to studying solvability of the equation $Tx = y$ for $y \in Y$, a considerable number of which involve local or infinitesimal assumptions on the mapping T, by showing that T is surjective. In particular, in [**2**], it was shown that if T is a Gateaux differentiable mapping having closed graph such that for each x in X,

$$dT_x(\overline{B}(0:1)) \supseteq \overline{B}(0: c(\|x\|))$$

where $c : [0, \infty) \to (0, \infty)$ is a continuous function, then for any $K > 0$ (possibly $K = \infty$), $T(B(0:K))$ contains $B(T(0) : \int_0^K c(s)\,ds)$. This fact implies that T is an open mapping, therefore for any y in $B(T(0) : \int_0^K c(s)\,ds) \subseteq Y$, $Tx = y$ has a solution x in $B(0:K) \subseteq X$.

In the present paper we will give an application of the above theorem. Consider the second order nonlinear differentiable equation

$$z'' + Az' + G(z) = e(t) = e(t + 2\pi) \tag{1}$$

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where $e \in C(R, R^n)$, $G : R^n \to R^n$ is a function such that its partial derivatives are exist and its differential is symmetric but its partial derivatives need not be continuous and A is a constant symmetric matrix. In [**6**] it was shown that if there exist an integer $N > 0$ and positive constants r_1 and r_2 such that

$$N^2 < r_1 \leq r_2 < (N+1)^2 \tag{2}$$

and for any $u \in R^n$,

$$r_1 I \leq D^2\overline{G}(u) \leq r_2 I$$

where $G \in C^2(R^n, R)$, with $\overline{G} = \nabla G$, $D^2\overline{G}(u)$ denotes the Hessian of $\overline{G}$ at $u \in R^n$ and I is the $n \times n$ identity matrix, then the differential equation

$$z'' + Az' + \operatorname{grad} \overline{G}(z) = e(t) = e(t + 2\pi)$$

has a unique 2π-periodic solution. Also the uniqueness and existence of a 2π-periodic solution for the case $A = 0$ has been proved in [**1**] and [**5**]. In fact to prove the theorem in [**6**], the theorem 1.22 of [**7**] and theorem 5.4.4 of [**3**] are necessary, but our proof will be based on the theorem 3.1 of [**2**].

Now we give some definitions. Let X and Y be Banach spaces and T mapping from an open subset D of X to Y. We say that T is *Gateaux differentiable* if, for each $x \in D$, there is a linear function $dT_x : X \to Y$ satisfying

$$\lim_{t \to 0} \frac{T(x + ty) - T(x)}{t} = dT_x(y), \qquad y \in X.$$

Note that $x + ty \in D$ for small t since D is open. A Gateaux differentiable mapping, even from R^2 to R^1, need not be continuous. Also note that

$$dT_x(y) = \frac{d}{dt} T(x + ty)\big|_{t=0},$$

thus $dT_x(y)$ may be considered as a directional derivative.

We say that a mapping $T : D \to Y$ has closed graph if $\{x_n\} \subseteq D$ with $x_n \to x \in D$ and $Tx_n \to y$ as $n \to \infty$, it follows that $Tx = y$. We denote by $B(x : r)$ the set $\{y : \|y - x\| < r\}$, and $\overline{B}(x : r)$ its closure.

2. Appication

We will start this section with some preliminary results.

PROPOSITION 1. *Let E and F be two Banach spaces and $T : E \to F$ a Gateaux differentiable mapping with closed graph. If*

$$dT_x(\overline{B}(0:1)) \supseteq \overline{B}(0:c), \qquad x \in E,$$

where c is a positive constant, then T is onto.

PROOF: It is clear from the theorem 3.1 of [**2**].

Note that the linearity of dT_x is not necessary in the proof of Proposition 1. Using Proposition 1, we will prove the next proposition which will be needed in the proof of our theorem.

PROPOSITION 2. *Let H be a real Hilbert space and X and Y two closed subspace of H such that $H = X \oplus Y$. Let $T : H \to H$ be a Gateaux differentiable mapping with closed graph, and assume there exist two positive constants m_1 and m_2 such that*

$$\langle dT_u(x), x\rangle \leq -m_1\|x\|^2, \qquad \forall u \in H, \quad \forall x \in X, \tag{3}$$

$$\langle dT_u(y), y\rangle \geq m_2\|y\|^2, \qquad \forall u \in H, \quad \forall y \in Y, \tag{4}$$

$$\langle dT_u(x), y\rangle = \langle x, dT_u(y)\rangle, \quad \forall u \in H, \quad \forall x \in X, \quad \forall y \in Y. \tag{5}$$

Then under these condition we have T is onto.

PROOF. Let $u \in H$ and $v \in H$, $v = x + y$, $x \in X$, $y \in Y$. By linearity of dT_u, we have

$$\begin{aligned}&\langle dT_u(v), y - x\rangle \\ &\qquad = \langle dT_u(x), y\rangle - \langle dT_u(x), x\rangle + \langle dT_u(y), y\rangle - \langle dT_u(y), x\rangle .\end{aligned}$$

From (3), (4), and (5) we have that

$$\langle dT_u(v), y - x\rangle \geq m_1\|x\|^2 + m_2\|y\|^2. \tag{6}$$

On the other hand it is clear that

$$\|x \pm y\|^2 \leq 2\left[\|x\|^2 + \|y\|^2\right]. \tag{7}$$

Thus, first applying the Cauchy-Schwartz inequality to the left hand side of (6), taking the square of both sides of the resulting inequality, and then applying (7), we obtain

$$2c^2 \left[\|x\|^2 + \|y\|^2 \right] \leq \|dT_u(v)\|^2 \tag{8}$$

where $c = 1/2 \min\{m_1, m_2\}$. Applying again (7) to the left hand side of (8) we have

$$c\|v\| \leq \|dT_u(v)\|. \tag{9}$$

Following a straightforward argument it also follows from (9) that $dT_u(H)$ is a closed subspace of H. We will prove next that for each $u \in H$, $dT_u(H) = H$. To do this let us assume there exists a z in $[dT_u(H)]^\perp$, $z \neq 0$. Then $\langle z, dT_u(v)\rangle = 0$, $\forall v \in H$. We have that z can be decomposed as $z = h + k$, where $h \in X$ and $k \in Y$. Take $v = k - h$, then by linearity of dT_u, we have

$$\begin{aligned} 0 &= \langle z, dT_u(v)\rangle \\ &= \langle h, dT_u(k)\rangle - \langle h, dT_u(h)\rangle + \langle k, dT_u(k)\rangle - \langle k, dT_u(h)\rangle . \end{aligned}$$

From (3), (4) and (5) it follows that

$$0 = \langle z, dT_u(v)\rangle \geq m_1\|h\|^2 + m_2\|k\|^2,$$

which is a contradiction. Thus $[dT_u(H)]^\perp = 0$ and dT_u is onto H for each $u \in H$. Finally we claim that $dT_u(\overline{B}(0:1)) \supseteq \overline{B}(0:c)$, $u \in H$, $c = \frac{1}{2}\min\{m_1, m_2\} > 0$. Using (9) and the surjectivity of dT_u, $u \in H$, it is trivial. Therefore by proposition 1, T is onto and we complete the proof.

Note that for each $u \in H$, $dT_u : H \to H$ is one to one mapping. Now at this point, we are ready to prove our theorem. Let (,) and | | denote the euclidean inner product and norm in R^n, respectively. Using Proposition 2 we can prove the followings.

THEOREM. *Let $G : R^n \to R^n$ be a function such that its partial derivatives are exist and its differential is symmetric but its partial* *derivatives need not be continuous. If there exist two positive constants r_1 and r_2 satisfying the condition (2) such that for all $u \in R^n$, $v \in R^n$,*

$$r_1|v|^2 \le (dG_u(v), v) \le r_2|v|^2, \tag{10}$$

then there exists a 2π-periodic solution of (1).

Before proving, we give a function $G : R \to R$ satisfying the conditions in the Theorem. Define a function $G : R \to R$ by

$$G(x) = \begin{cases} 0, & \text{if } x = 0 \\ x(x \sin 1/x + M), & \text{if } |x| \le a \\ a(a \sin 1/a + M), & \text{if } |x| > a, \end{cases}$$

where M is any given real number and a is the zero of the equation $2a \sin 1/a - \cos a = 0$ satisfying $0 < a < 1$. Then the function $G : R \to R$ is differentiable but $G'(x)$ is not continuous at 0 and by choosing suitable $M > 0$, it can be proved that the function G satisfy the conditions (2) and (10).

PROOF. Let us say $v \in P_n$ if (a) $v : R \to R^n$ is 2π-periodic and absolutely continuous, (b) $\int_0^{2\pi} |v'(t)|^2 \, dt < \infty$. It is known that P_n is a real Hilbert space for the following inner product

$$\langle u, v\rangle = \int_0^{2\pi} [(u', v') + (u, v)] \, dt.$$

Let us define now two subspaces of P_n as follows, X consists of the $x \in P_n$ such that $x(t) = (a_0/2) + \sum_{k=1}^{N}(a_k \cos kt + b_k \sin kt)$ and Y consists of the $y \in P_n$ such that $y(t) = \sum_{k=N+1}^{\infty}(a_k \cos kt + b_k \sin kt)$, where a_k, $b_k \in R^n$ and $\sum_{k=1}^{\infty}(k^2 + 1)(|a_k|^2 + |b_k|^2) < \infty$. Then we have that X and Y are two closed subspaces of P_n such that $P_n = X \oplus Y$.

Next using the Riesz representation theorem let us define a mapping $T : P_n \to P_n$ by

$$\langle T(u), v\rangle = \int_0^{2\pi} [(u', v') - (Au', v) - (G(u), v)] \, dt \tag{11}$$

for all $v \in P_n$. We observe that T is defined implicity. From (11) and from the definition of G it can be proved that T is Gateaux differentiable having closed graph and that

$$(12) \qquad \langle dT_u(w), v\rangle = \int_0^{2\pi} [(w', v') - (Aw', v) - (dG_u(w), v)]\, dt$$

for all w, v, $u \in P_n$. We note that in general dT_u, $u \in P_n$, is not a self-adjoint operator. Again using the Riesz representation theorem let d be the unique element in P_n such that

$$\langle d, v\rangle = -\int_0^{2\pi} (e(t), v(t))\, dt,$$

for all $v \in P_n$. It can be proved that u is a 2π-periodic solution of (1) if and only if u satisfies the operator equation

$$T(u) = d.$$

We will next show that T satisfies the conditions of Proposition 2. This in turn will imply that (1) has a 2π-periodic solution.

Let $x \in X$, $y \in Y$ and let u be any element in P_n. Using symmetricity of dG, we have

$$\begin{aligned}\langle dT_u(x), y\rangle &- \langle x, dT_u(y)\rangle \\ &= -\int_0^{2\pi} (Ax'(t), y(t))\, dt + \int_0^{2\pi} (x(t), Ay'(t))\, dt = 0\end{aligned}$$

since Ax' is orthogonal to y and x is orthogonal to Ay' in $L^2[0, 2\pi]$. Thus (5) of Proposition 2 is satisfied. Next let us note that for $v \in P_n$ we have

$$(13) \qquad \int_0^{2\pi} (Av', v)\, dt = \frac{1}{2}(Av(t), v(t))\Big|_0^{2\pi} = 0.$$

From (12) and (13) and for any $u \in P_n$, any $x \in X$ and any $y \in Y$ we have that

$$(14) \qquad \langle dT_u(x), x\rangle = \int_0^{2\pi} [(x', x') - (dG_u(x), x)]\, dt$$

and

$$(15) \qquad \langle dT_u(y), y\rangle = \int_0^{2\pi} [(y', y') - (dG_u(y), y)]\, dt.$$

We also have that for $x \in X$ and $y \in Y$ the following inequalities are true

$$(16) \qquad \int_0^{2\pi} |x'(t)|^2\, dt \le N^2 \int_0^{2\pi} |x(t)|^2\, dt$$

$$(17) \qquad \int_0^{2\pi} |y'(t)|^2\, dt \le (N+1)^2 \int_0^{2\pi} |y(t)|^2\, dt.$$

From (2), (10), (14), (15), (16) and (17) it is possible to prove the existence of two positive constants m_1, m_2, such that (3) and (4) of Proposition 2 are satisfied. Hence we complete the proof.

Finally we remark the followings. Let H be a real Hilbert space and $f : H \to R$ of class C^2. We denote by $D^2 f$ the Hessian of f. It is known that if there exists a constant $m > 0$ such that for any $u \in H$ and any $w \in H$ we have

$$\left\langle D^2 f(u)w, w\right\rangle \ge m\|w\|^2$$

then there exists a unique $u_0 \in H$ such that $\operatorname{grad} f(u_0) = 0$ and $f(u_0) = \min\{f(u) : u \in H\}$. Furthermore, these results are extended as the following version. Let X and Y be two colsed subspace of H, such that $H = X \oplus Y$, X is finite dimensional and X and Y are not necessarily orthogonal. Let $T = \operatorname{grad} f$, then $T : H \to H$ and is a C^1 mapping. Its Frechet derivative at $u \in H$ is given by $T'(u) = D^2 f(u)$. Let m_1 and m_2 be two positive constants such that for any $u \in H$, any $x \in X$, and $y \in Y$ we have

$$\langle T'(u)x, x\rangle \le -m_1\|x\|^2$$
$$\langle T'(u)y, y\rangle \ge m_2\|y\|^2.$$

Under these conditions, in [4], it can be proved that there exists a unique $u_0 \in H$, such that $T(u_0) = 0$ and that u_0 satisfies

$$f(u_0) = \max_{x\in X}\min_{y\in Y} f(x+y).$$

Here we note that the existence of $u_0 \in H$ such that $T(u_0) = 0$ also derive from Proposition 2.

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