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Stability Theory for Ordinary Differential Equations

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ABSTRACT. For a given autonomous system $x' = f(x)$, we obtain some properties on the location of positive limit sets and investigate some stability concepts which are based upon the existence of Liapunov functions.

1. Introduction and Preliminaries

The classical theorem of Liapunov on stability of the origin $x = 0$ for a given autonomous system $x' = f(x)$ makes use of an auxiliary function $V(x)$ which has to be positive definite. Also, the time derivative $V'(x)$ of this function, as computed along the solution, has to be negative definite.

It is our aim to investigate some stability concepts of solutions for a given autonomous system which are based upon the existence of suitable Liapunov functions.

Let $U \subset \mathbf{R}^n$ be an open subset containing $0 \in \mathbf{R}^n$. Consider the autonomous system $x' = f(x)$, where x' is the time derivative of the function $x(t)$, defined by the continuous function $f : U \to \mathbf{R}^n$. When f is C^1, to every point t_0 in $\mathbf{R}$ and x_0 in U, there corresponds the unique solution $x(t) = x(t, x_0)$ of $x' = f(x)$ such that $x(t_0) = x_0$. For such a solution, let us denote $I(x_0) = (\alpha, \omega)$ the interval where it is defined (α, ω may be infinity).

Let $f : (a, b) \to \mathbf{R}$ be a continuous function. Then f is decreasing on (a, b) if and only if the upper right-sided *Dini derivative*

$$D^+f(t) = \limsup_{h \to 0+} \frac{f(t+h) - f(t)}{h} \leq 0$$

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for every $t \in (a,b)$ [6]. Moreover, the decreasing function $f : [a,b] \to \mathbf{R}$ is differentiable almost everywhere on $[a,b]$, the derivative f' of f is integrable and one has the following formula:

$$f(b) - f(a) \leq \int_a^b f'(s)\,ds$$

[7]. Since the derivative of f, when it exists, equals to the upper right-sided Dini derivative, the above formula can be written as

$$f(b) - f(a) \leq \int_a^b D^+ f(s)\,ds.$$

Let us consider an auxilary function of the type $V : U \to \mathbf{R}$. As is well known [2], the Dini derivative of V estimated along the solutions of $x' = f(x)$, namely

$$D^+V(x(t)) = \limsup_{h \to 0+} \frac{V(x(t+h)) - V(x(t))}{h}$$

may be computed without knowing these solutions, for it is equal to

$$\limsup_{h \to 0+} \frac{V(x + hf(x)) - V(x)}{h}.$$

Furthermore, if $D^+V(x(t)) \leq 0$ for every $t \in [a,b] \subset I(x_0)$, then $V(x(t))$ is decreasing and we have

$$V(x(b)) - V(x(a)) \leq \int_a^b D^+V(x(s))\,ds$$

by the above observations. In case $V(x(t))$ is C^1, we can replace $D^+V(x(t))$ be the time derivative

$$V'(x(t)) = \operatorname{grad} V(x(t)) \cdot f(x).$$

A *Liapunov function* for $x' = f(x)$ is the continuous function $V : U \to \mathbf{R}$ which satisfies

(i) $V(x)$ is positive definite on U, i.e., $V(x) > 0$ for $x \in U - \{0\}$ and $V(0) = 0$,
(ii) $D^+V(x(t)) \leq 0$.

Also, the condition (i) means that there exists a continuous strictly increasing function $k : \mathbf{R}^+ \to \mathbf{R}^+$ with $k(0) = 0$ such that $V(x) \geq k(|x|)$ for all $x \in U$, where $|\ |$ denotes the Euclidean norm on $\mathbf{R}^n$. In this case, we call k the function of class $\mathcal{K}$ and write $k \in \mathcal{K}$.

Finally, we need the concept of limit sets. The *positive semitrajectory* starting at $x_0 \in U$ is the set

$$\gamma^+(x_0) = \{x(t) : t \in [\,t_0, \omega)\}$$

and the *positive limit set* of $x_0 \in U$ is the set

$$\begin{aligned} &\wedge^+ (x_0) \\ &= \{y \in \mathbf{R}^n : x(t_n) \to y \text{ for some net } (t_n) \subset I(x_0) \text{ with } t_n \to \omega\}. \end{aligned}$$

A subset $M \subset \mathbf{R}^n$ is *positively invariant* if for every $x_0 \in M$ for all $t \geq 0$ with $t \in I(x_0)$. We can easily show (as in [**1**]) that $\overline{\gamma^+(x_0)}$, the *orbit colsure* of $\gamma^+(x_0)$, is equal to $\gamma^+(x_0) \cup \wedge^+(x_0)$ and $\wedge^+(x_0) = \cap\{\overline{\gamma^+(y)} : y \in \gamma^+(x_0)\}$. Also, $\wedge^+(x_0)$ is an nonempty compact positively invariant set if $\gamma^+(x_0)$ is bounded. In this case, $I(x_0) = (\alpha, \omega) = (-\infty, \infty)$.

2. Main Results

THEOREM 2.1. *Let V be a C^1 Liapunov function for the system $x' = f(x)$, defined on $U = B_\varepsilon(0) = \{x \in R^n : |x| < \varepsilon, \varepsilon > 0\}$. Suppose that for every $x \in U$, $V(x) \to 0$ as $x \to 0$. Then for all $t > t_0$, $V(x(t)) \to 0$ as $t \to \infty$.*

PROOF. It suffices to show that there is no $M > 0$ such that $V(x(t)) > M$ for all $t > t_0$ because $V(x(t))$ is decreasing. Assume that such an M exists. Then there is a $\delta > 0$ with $0 < \delta < \varepsilon$ such that $|x| < \delta$ implies $V(x) < M$ by the hypothesis. From the definition of $V(x)$, $x \in U$ implies that $V(x) \geq k_1(|x|)$ and $V'(x) \leq -k_2(|x|)$ for some functions k_1, $k_2 \in \mathcal{K}$. Then we have $V(x) \geq k_1(\delta)$ and $V'(x) \leq -k_2(\delta) < 0$ whenever $\delta < |x| < \epsilon$. Therefore

$$\begin{aligned} k_1(\delta) \leq V(x(t)) &\leq V(x_0) + \int_{t_0}^{t} V'(x(s))\,ds \\ &\leq V(x_0) - k_2(\delta)(t - t_0). \end{aligned}$$

This leads a contradiction as $t \to \infty$ because we assumed that $V(x(t)) > M$ for all $t > t_0$.

COROLLARY 2.2. *Under the hypotheses of Theorem 2.1, we have* $x(t) \to 0$ *as* $t \to \infty$.

PROOF. Assuming the contrary, there exist an $N > 0$ and a sequence $t_n \to \infty$ such that $|x(t_n)| > N$. By the positive definiteness of V, we have $V(x(t_n)) > M$, where $M = \min\{V(x) : N \le |x| \le \varepsilon\} > 0$, which contradicts to Theorem 2.1.

LaSalle's invariance principle [4] states that, if the positive limit sets have an invariance property, then Liapunov functions can be used to obtain information on the location of positive limit sets. We have a similar form of [**3**, Theorem 2].

THEOREM 2.3. *Let* V *be a* C^1 *Liapunov function on an open subset* U *of* R^n. *Let* $K \subset U$ *be a compact subset and* $E = \{x \in K : V'(x) = 0\} \subset K$. *If* M *is the union of all positive trajectories that remains in* E *and* x_0 *is any point of* U *with* $\gamma^+(x_0) \subset K$, *then* $x(t) \to M$ *as* $t \to \infty$.

PROOF. Note that $\Lambda^+(x_0)$ is positively invariant and V is constant on $\Lambda^+(x_0)$ [**1**]. Thus we have $V'(x) = 0$ for all $x \in \Lambda^+(x_0)$. Then $\Lambda^+(x_0) \subset M \subset K$ and so $\Lambda^+(x_0)$ is nonempty compact. Hence we have $x(t) \to \Lambda^+(x_0)$ as $t \to \infty$ since $\Lambda^+(x_0) \subset M$.

For example, we consider the differential equation

$$ax'' + bx' + cx = 0$$

where a, b, and c are positive constants, in the plane. The equivalent form is

$$\begin{aligned} x' &= y \\ y' &= -\frac{c}{a}x - \frac{b}{a}y. \end{aligned}$$

If we take

$$V(x, y) = \frac{a}{2}y^2 + \frac{c}{2}x^2,$$

then V is a C^1 Liapunov function for the above equation since

$$V'(x, y) = \frac{\partial V}{\partial x}x' + \frac{\partial V}{\partial y}y' = -by^2 \le 0.$$

Moreover the set E is the x-axis and $M = \{0\}$. Clearly every trajectory is bounded. Therefore, by Theorem 2.3, we have $x(t) \to 0$ as $t \to \infty$.

A property about the location of the limit set is appeared in [**2**, Theorem 1] by using their three lemmas. But if we use only one lemma [**2**, Lemma 3] implying the set E contains no invariant subset, then we obtain Theorem 1 in [**2**].

THEOREM 2.4. *Let V, E and M be defined as in Theorem 2.3. Suppose that for any $x \in E$ there exist a neighborhood N_x of x in U and a C^1 function $W_x : N_x \to \mathbf{R}$ such that $W_x(y) = 0$ for all $y \in N_x \cap E$ and $W_x'(y) \neq 0$ for all $y \in N_x$. Then one has $\wedge^+ = \wedge^+(x(t)) \subset M$ for every solution $x(t)$ whose positive trajectory is bounded.*

PROOF. It suffices to show that $\wedge^+ \cap U = \emptyset$. Suppose that the contrary holds. Since V is constant on the invariance set $\wedge^+ \cap U$, we have $\wedge^+ \cap U \subset E$ and so $\wedge^+ \cap U$ is an invariant set in E. But the hypotheses on W_x implies that E has no invariant subset [**2**, Lemma 3]. This is a contradiction.

Thus Theorem 2.3 is a special case of Theorem 2.4.

Now we consider some stability concepts for the given autonomous system $x' = f(x)$, $x(t_0) = x_0$.

The trivial solution $x(t) = 0$ of the system $x' = f(x)$ is *stable* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $|x_0| < \delta$ implies $|x(t)| < \varepsilon$ for all $t \geq t_0$. Also, $x(t) = 0$ is *uniformly stable* if the δ in the above definition of stability is independent of t_0.

The well-known stability theorem is [**8**, Theorem 6.2] and its converse is [**6**, Theorem 4.2]. Also, the uniform stability theorem is appeared in [**8**, Theorem 6.3]. In [**6**] its converse is given without proof.

THEOREM 2.5. *Suppose that $x(t) = 0$ is uniformly stable. Then there is a continuous function $V : B_\rho(0) \to \mathbf{R}$ satisfying*

(i) $V(0) = 0$,
(ii) $a(|x|) \leq V(x) \leq b(|x|)$ *for some* $a, b \in \mathcal{K}$,
(iii) $V(x(t))$ *is decreasing for every solution* $x(t)$ *with* $|x(t)| < \rho$.

PROOF. We take

$$V(x) = \sup_{\tau \geq 0} |x(t + \tau, x)|.$$

$V(x)$ is defined for $t \geq 0$ and $|x| < \rho = \sup \delta(\varepsilon)$. Clearly, $V(0) = 0$. Since we may take the inverse function $\varepsilon(\delta)$ of $\delta(\varepsilon)$, we have $|x(t + \tau, x)| < \varepsilon(|x|)$. Thus

$$|x| = |x(t,x)| \leq V(x) \leq \varepsilon(|x|).$$

Taking $t_1 > t_2$ and $s = t_1 - t_2$, we have

$$\begin{aligned} V(x(t_1, x_0)) &= \sup_{\tau \geq 0} |x(t_1 + \tau, x_0)| = \sup_{\tau \geq 0} |x(t_2 + s + \tau, x_0)| \\ &= \sup_{\tau \geq s} |x(t_2 + \tau, x_0)| \\ &\leq \sup_{\tau \geq 0} |x(t_2 + \tau, x_0)| = V(x(t_2, x_0)). \end{aligned}$$

The trivial solution $x(t) = 0$ of $x' = f(x)$ is *asymptotically stable* if it is stable and if there is a $\delta_0 = \delta_0(t_0) > 0$ such that $|x_0| < \delta_0$ implies that $x(t, x_0) \to 0$ as $t \to \infty$. $x(t) = 0$ is *uniformly asymptotically stable* if it is uniformly stable and if for any $\varepsilon > 0$ there exists a δ_0 and a $T(\epsilon) > 0$ such that $|x_0| < \delta_0$ implies $|x(t, x_0)| < \varepsilon$ for all $t \geq t_0 + T(\varepsilon)$.

The asymptotic stability theorem and a counterexample that asymptotic stability does not imply uniform asymptotic stability are appeared in [**8**, Theorem 7.10 and Example 7.4]. For uniform asymptotic stability we can obtain a weak form of [**8**, Theorem 7.9] based upon the existence of C^1 Liapunov functions.

THEOREM 2.6. *Suppose that there exists a continuous function* $V : B_\rho(0) \to \mathbf{R}$ *which satisfies the following conditions:*

(i) $a(|x|) \leq V(x) \leq b(|x|)$ *for some* a, $b \in \mathcal{K}$,
(ii) $D^+V(x(t)) \leq -c(|x(t)|)$, $c \in \mathcal{K}$.

Then $x(t) = 0$ *is uniformly asymptotically stable.*

PROOF. Let $\varepsilon > 0$ and $\delta(\varepsilon) < b^{-1}(a(\varepsilon))$. If $|x_0| < \delta(\varepsilon)$, then

$$\begin{aligned} a(|x(t, x_0)| &\leq V(x(t, x_0)) \leq V(x_0) \\ &\leq b(|x_0)| < b(\delta) < a(\varepsilon). \end{aligned}$$

Thus we have $|x(t, x_0)| < \varepsilon$ for all $t \geq t_0$.

Now we put $\delta_0 = \delta(\rho)$ and $T(\varepsilon) = b(\delta_0)/c(\delta(\varepsilon))$. We show that $|x(t,x_0)| < \varepsilon$ whenever $|x_0| < \delta_0$ and $t \geq t_0 + T(\varepsilon)$. Assume that $|x(t,x_0)| \geq \delta(\varepsilon)$ for $t_0 \leq t \leq t_0 + T(\varepsilon)$. Then, for $t_0 \leq t \leq t_0 + T$,

$$\limsup_{h\to 0+} \frac{V(x(t+h,x_0)) - V(x(t,x_0))}{h} \leq -c(\delta(\varepsilon)).$$

By integrating, we have $V(x(t,x_0)) - V(x_0) \leq -c(\delta(\varepsilon))(t-t_0)$. Thus

$$V(x(t)) \leq V(x_0) - c(\delta(\varepsilon))(t-t_0) \leq b(\delta_0) - c(\delta(\varepsilon))(t-t_0).$$

For $t = t_0 + T$, we get $V(x(t)) \leq b(\delta_0) - c(\delta(\varepsilon))T = 0$, which is a contradiction. Therefore for every $t \geq t_0 + T(\varepsilon)$ we have $|x(t,x_0)| < \varepsilon$ when $|x_0| < \delta_0$.

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