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Generalized Zero in the Quotient Semiring

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ABSTRACT. Using the notion [1] of Q-ideal in a semiring we study some properties, especially g-zero, of quotient semiring.

1. Introduction

Allen [1] introduced the notion of Q-ideal and constructed the quotient structure of a semiring modulo a Q-ideal and Kim [4] studied some properties of quotient semiring and Jeter [3] studied a generalized zero in a semigroup. With this concept we study a generalized zero in the quotient semiring. A *semiring* is an algebra $(R, +, \cdot, 0)$ such that $(R, +)$ is a commutative semigroup, $(R, \cdot)$ is a semigroup, 0 is the zero, i.e., $x + 0 = x$ and $x \cdot 0 = 0 = 0 \cdot x$ for every $x \in R$, and $\cdot$ distributives over $+$ from the left and right. A subset I of a semiring R is called an *ideal* if $a, b \in I$ and $r \in R$ implies $a + b \in I$, $ra \in I$ and $ar \in I$.

DEFINITION 1.1 [1]. An ideal I in the semiring R is called a *Q-ideal* if there exists a subset Q of R satisfying the following conditions:

(1) $\{q + I\}_{q \in Q}$ is a partition of R; and
(2) if $q_1, q_2 \in Q$ such that $q_1 \neq q_2$, then $(q_1 + I) \cap (q_2 + I) = \emptyset$.

LEMMA 1.2 [1]. *Let I be a Q-ideal in the semiring R. If $x \in R$, then there exists a unique $q \in Q$ such that $x + I \subset q + I$.*

Let I be a Q-ideal in the semiring R. In view of the above results, one can define the binary operations $\oplus_Q$ and $\odot_Q$ on $\{q + I\}_{q \in Q}$ as follows:

(1) $(q_1 + I) \oplus_Q (q_2 + I) = q_3 + I$ where q_3 is the unique element in Q such that $q_1 + q_2 + I \subset q_3 + I$; and
(2) $(q_1 + I) \odot_Q (q_2 + I) = q_3 + I$ where q_3 is the unique element in Q such that $q_1 q_2 + I \subset q_3 + I$. The elements $q_1 + I$ and $q_2 + I$ in $\{q + I\}_{q \in Q}$ is called *equal* (denoted by $q_1 + I = q_2 + I$) if and only if $q_1 = q_2$.

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THEOREM 1.3 [1]. *If I is a Q-ideal in the semiring R, then*

$$(\{q+I\}_{q\in Q}, \oplus_Q, \odot_Q)$$

is a semiring and denoted by R/I.

THEOREM 1.4 [4]. *Let I be a Q-ideal in the semiring R. If $q_i + I \in R/I$ $(i = 1, \ldots, n)$, then*

$$(q_1 + I) \oplus_Q \cdots \oplus_Q (q_n + I) = q^* + I \text{ iff } q_1 + \cdots + q_n \in q^* + I$$
$$(q_1 + I) \odot_Q \cdots \odot_Q (q_n + I) = q^* + I \text{ iff } q_1 \cdot \cdots \cdot q_n \in q^* + I$$

2. Main results

THEOREM 2.1. *Let I be a Q-ideal of a semiring R with identity e. If $e \in q^* + I$ for some $q^* \in Q$, then $q^* + I$ is the identity of the quotient semiring R/I.*

PROOF. Since $e \in q^* + I$, there exists $i_1 \in I$ such that $e = q^* + i_1$. For any $q + I \in R/I$ we have $q = q \cdot e = q(q^* + i_1) = qq^* + qi_1$. This means that $q + I \subset qq^* + I$. Let $(q + I) \odot_Q (q^* + I) = q' + I$ where $q' \in Q$. It means that $qq^* + I \subset q' + I$. Hence $q + I \subset q' + I$. By the definition of Q-ideal $q+I = q'+I$. Therefore $(q+I)\odot_Q(q^*+I) = q+I$ for any $q + I \in R/I$. Similarly we have $(q^* + I) \odot_Q (q + I) = q + I$. The uniqueness is trivial.

DEFINITION 2.2 [2]. A *division semiring* R is a semiring with identity e such that for any non-zero a, there is an x in R with $ax = xa = e$.

THEOREM 2.3. *Let I be a Q-ideal of a division semiring R. Then the quotient semiring R/I is a division semiring.*

PROOF. Let $q^* \in Q$ with $e \in q^* + I$. For any $q + I \in R/I$ there exists $a \in R$ such that $qa = e = aq$, since R is a division semiring. Let $q_a \in Q$ with $a + I \subset q_a + I$. Then $a = q_a + i$ for some $i \in I$. Hence $e = qa = qq_a + qi \in qq_a + I \subset (q + I) \odot_Q (q_a + I)$. By the definition of Q-ideal we have $q^* + I = (q + I) \odot_Q (q_a + I)$. Similarly we can see $(q_a + I) \odot_Q (q + I) = q^* + I$. This completes the proof.

DEFINITION 2.4 [3]. An element z of a semiring R is called a *generalized zero* (or *g-zero*) if for all a, $b \in R$ it follows that $azb = bza$.

THEOREM 2.5. *Let R be a semiring having z as a g-zero. If I is a Q-ideal with $z \in q^* + I$ for some $q^* \in Q$, then the coset $q^* + I$ is a g-zero in the quotient semiring R/I.*

PROOF. For any $q_1 + I$ and $q_2 + I$ in R/I, let $(q_1 + I) \odot_Q (q^* + I) \odot_Q (q_2 + I) = q_3 + I$ and $(q_2 + I) \odot_Q (q^* + I) \odot_Q (q_1 + I) = q_4 + I$ where $q_3, q_4 \in Q$. By Theorem 1.4 we have $q_1 q^* q_2 \in q_3 + I$ and $q_2 q^* q_1 \in q_4 + I$. This means that $q_1 q^* q_2 = q_3 + i_1$ and $q_2 q^* q_1 = q_4 + i_2$ for some $i_1, i_2 \in I$. Since $z \in q^* + I$, there exists $i_3 \in I$ such that $z = q^* + i_3$. From this we can see that $q_1 z q_2 \in q_3 + I$ and $q_2 z q_1 \in q_4 + I$. Since z is a g-zero, $q_1 z q_2 = q_2 z q_1$ and hence $(q_3 + I) \cap (q_4 + I) \neq \emptyset$. By the definition of Q-ideal $q_3 + I = q_4 + I$. We completes the proof.

CORORALLY 2.6. *In the above Theorem 2.5, if z is an identity, then R/I is a commutative semiring with identity.*

PROPOSITION 2.7. *Let R be a semiring with identity e and I be a Q-ideal with $e \in q^* + I$. If R has a property that for any q_1 and q_2 in Q there exists b in $q^* + I$ with $q_1 q_2 = b q_2 q_1$, then R/I is a commutative.*

PROOF. It follows from Theorem 2.1 and Theorem 2.5.

PROBLEM: Is the coset $q^* + I$ is a unique g-zero in R/I if z is a unique g-zero in the semiring R in Theorem 2.5?

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