JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 1, June 1988

A Note on a Theorem by Parida and Sen

SUNG-MO IM AND WON KYU KIM

ABSTRACT. In a recent paper, Parida and Sen obtained a variational-like inequality. In this note, we obtain another variational-like inequality using Fan's minimax inequality [1] and Michael's selection theorem [2]. Also we generalize the Parida-Sen theorem in Banach spaces.

In a recent paper [3], Parida and Sen obtained the following variational-like inequality:

THEOREM [3]. *Let S be a compact convex set in R^n, and C a closed convex set in R^p. Let $V : S \to P(C)$ be upper semicontinuous and $\phi : S \times C \times S \to R$ continuous. Suppose that*

(i) *$\phi(x, y, x) \geq 0$ for each $x \in S$.*
(ii) *for each fixed $(x, y) \in S \times C$, $\phi(x, y, u)$ is quasi-convex in $u \in S$.*

Then there exists $\bar{x} \in S$, $\bar{y} \in V(\bar{x})$ such that

$$\phi(\bar{x}, \bar{y}, x) \geq 0 \quad \text{for all } x \in S.$$

In this note, we obtain another variational-like inequality using Fan's minimax inequality [1] and Michael's selection theorem [2]. Also we generalize the Parida-Sen theorem in Banach spaces. Most of the definitions and terminologies of this paper can be found in Parida and Sen [3].

First we state the well-known minimax inequality due to Fan.

LEMMA (K. FAN [1]). *Let S be a compact convex set in a Hausdorff topological vector space. Let g be a real-valued continuous function defined on $S \times S$. If for each fixed $x \in S$, $g(x, y)$ is a quasi-convex function of y on S, then there exists a point $\bar{x} \in S$ such that*

$$g(\bar{x}, y) \geq g(\bar{x}, \bar{x}) \quad \text{for all } y \in S.$$

Now we prove the following.

Received by the editors on February 10, 1988.
1980 *Mathematics subject classifications*: Primary 49A29.

THEOREM 1. *Let S be a compact convex set in R^n, and C a closed convex set in R^p. Let $V : S \to P(C)$ be lower semicontinuous and $\phi : S \times C \times S \to R$ continuous. Suppose that for each fixed $(x, y) \in S \times C$, $\phi(x, y, u)$ is quasi-convex in $u \in S$. Then there exist $\bar{x} \in S$, $\bar{y} \in V(\bar{x})$ such that*

$$\phi(\bar{x}, \bar{y}, y) \geq \phi(\bar{x}, \bar{y}, \bar{x}) \quad \textit{for all } y \in S.$$

PROOF. By Michael's selection theorem [**2**], there exists a continuous selection $h(x)$ of $V(x)$. Now we define $g : S \times S \to R$ by

$$g(x, y) := \phi(x, h(x), y) \quad \text{for each } (x, y) \in S \times S.$$

Then g satisfies the whole assumptions of the lemma, so that there exists a point $\bar{x} \in S$ such that $g(\bar{x}, y) \geq g(\bar{x}, \bar{x})$ for all $y \in S$. Therefore there exists $\bar{x} \in S$, $\bar{y} = h(\bar{x}) \in V(\bar{x})$ such that

$$\phi(\bar{x}, \bar{y}, y) \geq \phi(\bar{x}, \bar{y}, \bar{x}) \quad \text{for all } y \in S.$$

It completes the proof.

REMARK. Parida and Sen obtained the same conclusion when V is upper semicontinuous [**3**], but in theorem 1, V is only assumed to be lower semicontinuous. Therefore, replacing the upper semicontinuity of V by the lower semicontinuity in [**3**], we can obtain several applications of Theorem 1.

Using the method of Parida and Sen, we prove a generalized variational-like inequality, which is a generalization of Theorem 1 of [**3**].

THEOREM 2. *Let C be a compact convex subset of a Banach space E and K be a closed convex subset of a Banach space F. Let $S : C \to 2^K$ be upper semicontinuous with each $S(x)$ compact convex and $T : C \to 2^C$ be upper semicontinuous with each $T(x)$ compact convex. Let $\phi : C \times K \times C \to R$ be continuous such that for each fixed $(x, y) \in C \times K$, $\phi(x, y, u)$ is quasi-concave in $u \in C$.*

Then there exist $\bar{x} \in C$, $\bar{y} \in S(\bar{x})$ such that

$$\phi(\bar{x}, \bar{y}, \bar{x}) \geq \phi(\bar{x}, \bar{y}, x) \quad \textit{for all } x \in T(\bar{x}).$$

PROOF. For each $(x,y) \in C \times K$, let

$$f(x,y) := \left\{ s \in T(x) \mid \phi(x,y,s) = \max_{u \in T(x)} \phi(x,y,u) \right\}$$

Then each $f(x,y)$ is nonempty compact since $T(x)$ is compact and ϕ is continuous. Furthermore, each $f(x,y)$ is convex since ϕ is quasi-concave in $u \in C$. Now we show that $f(x,y)$ is upper semicontinuous on $C \times K$. For each $x_0 \in C$, $y_0 \in K$, $u \in T(x_0)$, since ϕ is continuous at (x_0, y_0, u), we can associate open neighborhoods $N_u(x_0)$, $N(y_0)$, $N(u)$ with x_0, y_0, u, respectively, such that

$$\phi(x,y,u') \leq \phi(x_0,y_0,u) + \varepsilon$$

for every

$$x \in N_u(x_0), \quad y \in N(y_0), \quad u' \in N(u).$$

Since $T(x_0)$ is compact, it can be covered by n open neighborhoods $N(u_i)$ of $u_i \in T(x_0)$ such that $u_0 \in T(x_0) \subset \bigcup_{i=1}^{n} N(u_i)$. Let $N := \bigcup_{i=1}^{n} N(u_i)$. Since T is upper semicontinuous, there exists an open neighborhood $N(x_0)$ of x_0 such that $T(N(x_0)) \subset N$. Let $\overline{N} := N(x_0) \cap \left(\bigcup_{i=1}^{n} N_{u_i}(x_0) \right)$ be an open neighborhood of x_0. Then we have an open neighborhood $\overline{N} \times N(y_0)$ of (x_0, y_0). For each $(x,y) \in \overline{N} \times N(y_0)$ and $u \in T(x)$, $u \in N(u_i)$ for some $i \in \{1, \ldots, n\}$ and $x \in N_{u_i}(x_0)$ for all $1 \leq i \leq n$, so that

$$\phi(x,y,u) \leq \phi(x_0,y_0,u_i) + \varepsilon \leq f(x_0,y_0) + \varepsilon \quad \text{for all } 1 \leq i \leq n.$$

Therefore f is upper semicontinuous at $(x_0, y_0) \in C \times K$.

Let $G_1 := \bigcup_{x \in C} S(x)$ be the image of S, then G_1 is a compact subset of K, so that $G := \overline{\text{co}}(G_1)$ is also compact convex in K. Define

$F: C \times G \to C \times G$ be $F(x,y) = (f(x,y), S(x))$ for each $(x,y) \in C \times G$. Then F is upper semicontinuous on $C \times G$ and each $F(x,y)$ compact convex. By Fan's fixed point theorem, there exists a point $(\bar{x},\bar{y}) \in C \times G$ such that $(\bar{x},\bar{y}) \in F(\bar{x},\bar{y}) = (f(\bar{x},\bar{y}), S(\bar{x}))$, so that $\bar{y} \in S(\bar{x})$ and $\phi(\bar{x},\bar{y},\bar{x}) = \max_{x \in T(\bar{x})} \phi(\bar{x},\bar{y},u)$, i.e., $\phi(\bar{x},\bar{y},\bar{x}) \geq \phi(\bar{x},\bar{y},u)$ for all $x \in T(\bar{x})$. It completes the proof.

REMATKS. (1) If K is compact convex, then the given spaces E and F can be changed by locally convex Hausdorff topological vector spaces without affection the conclusion.

(2) By following the method of Parida and Sen, we can obtain some applications as in [**3**].

REFERENCES

1. K. Fan, "A minimax inequality and applications," Inequalities III(Ed. O. Shisha), Academic Press, 1972, pp. 103–113.
2. E. Michael, *A selection theorem*, Proc. Amer. Math. Soc. **17** (1966), 1404--1406.
3. J. Parida and A. Sen, *A variational-like inequality for multifuctions with applications*, J. Math. Anal. Appl. **124** (1987), 73–81.

Department of Mathematics
Chungbuk National University
Cheongju, 360-763, Korea