Numerical Evaluation of the Rayleigh Integral Using the FFT Method for Transient Sound Radiation

FFT 방법을 이용한 음압복사에 대한 Rayleigh Integral의 수치해석적 연구

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ABSTRACT

In this paper, the sound radiation from a clamped circular plate in an infinite baffle is calculated by using the FFT technique. The radiated sound fields are obtained by two-dimensional fast Fourier transform method in the spatial domain instead of a direct numerical evaluation of Rayleigh integral for economy of the computation time. The computation time is consumed at least by 1/200 times of the direct numerical evaluation on the Rayleigh integral in acoustic fields. The FFT method can be applicable to any shaped geometry as well as circular plate. The FFT solution could be very powerful in predicting the near and far fields of complex structures.

요 약

이 논문에서는 무한 격리벽내의 가장자리가 고정된 원형평판으로부터의 음압복사를 FFT 기법을 이용하여 계산하였다. 음장은 컴퓨터 계산시간을 절약하기 위해 Rayleigh integral을 직접 수치해석적으로 구하는 대신에 공간영역에서 2차원 FFT방법을 이용하였다. 그 결과 1/200의 시간을 절약할 수 있었다. FFT방법은 원형평판형상 뿐만아니라 어떤 형상에도 적용 가능하며 복잡한 형상의 근거리 및 원거리 음장을 예측하는데 상당히 유효하다.

I. INTRODUCTION

Rayleigh's integrals are a special case of the Helmholtz integral which stands the foundation of the theory of the sound radiation from vibrating sources. The forward propagation of acoustic field is to obtain the pressure field from the velocity distributions and the backward propagation is to obtain the velocity distributions from the pressure field.

For the case of planar vibrators, the 2-dimensional Fourier transform method particularly attracts as the rapid means of evaluating

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the transforms, i.e., the use of FFT method.

Copley[1], Schenck[2], and Chertock[3] developed solutions for the radiation from rotating bodies such as *finite* cylinders and spheroids. Recently, Kristiansen[4] dealt with the solution for the nearfield vector intensity from a vibrating membrane. Stepanishen et al.[5] had appeared on the numerical calculation of the nearfield of planar sources using FFT method. And E.G. William et al.[6,7] studied the numerical evaluation of Rayleigh integral for planar radiators using FFT technique.

In this paper, we represent an extremely *fast numerical solution of Rayleigh's integral* for a baffled clamped circular plate with the velocity responses obtaining from Ref. 10 and 11. The acoustic field for an arbitrary plane is calculated by using the two dimensional fast Fourier transform. It is shown to decrease the computation time at least about 1/200 than the direct numerical integration for the 64x64 lattices used in this paper. In order to calculate the sound field with the FFT method, the meshes are divided to 64x64 and the complex acoustic pressure field is obtained. We compare the direct numerical results of Rayleigh integral with those of FFT technique. The results using the FFT method is considerably acceptable to calculate the acoustic field. It takes about 250 sec on a Cyber 170-835.

II. COMPUTATIONAL TECHNIQUE WITH FFT METHOD

The well-known Rayleigh integral on the radiated sound pressure from the vibrating surfaces shown in Fig. 1 is given as

$$p(r,\theta,t)=\frac{\rho_0}{2\pi}\,\frac{\cos\Psi}{\cos\Psi+\beta}\int A(r',\theta,t-R/c)/R\;ds \tag{1}$$

where

$$R=[(x-x')^2+(y-y')^2+z_0^2]^{\frac{1}{2}}$$

$$r=[x^2+y^2]^{\frac{1}{2}},\quad r'=[x'^2+y'^2]^{\frac{1}{2}}$$

and $A(r',\theta,t\text{-}R/c)$ is the acceleration having a time delay, c is the sound speed in ambient medium, β is the acoustic admittance, $o_o\,c/z(\omega)$, $z(\omega)$ is the impedance of the vibrating surface, and s is the vibrating surface area.

We can divide the acceleration term of Eq. (1) into terms in the spatial and time domain as following Eq.(2).

$$A(x',y',t-R/c)/R=A(x',y')R(R)T(t) \tag{2}$$

Hence, in the Eq.(1), the integration is rewitten in condensed form by the defining a kernel.

$$k(x,y,y_0)=R(R)\cdot\frac{\cos\Psi}{\cos\Psi+\beta}\,\frac{\rho_0}{2\pi} \tag{3}$$

where

$$R=[x^2+y^2+z^2]^{\frac{1}{2}}$$

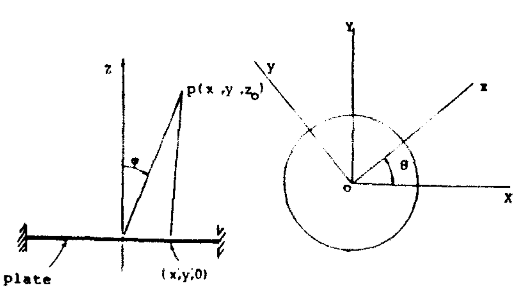


Fig. 1. Sound radiation model from the plate.

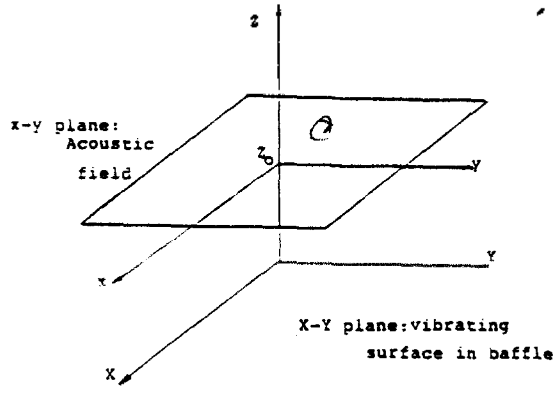


Fig. 2. Geometry of acoustic field at $z=z_o$.

If the integration of Eq.(1) is calculated in spatial domain at $z=z_o$ shown in Fig.2, we rewrite Eq.(1) with a certain time independence as following equation.

$$p(x, y, z_0) = \int A(x', y')\, R(R)\, ds \cdot \frac{\rho_0 \cos\Psi}{2\pi(\cos\Psi + \beta)} \quad (4)$$

Using the kernel, we have, since Eq.(4) is two dimensional convolution at $z=z_o$,

$$p(x, y, z_0) = A(x, y) ** k(x, y, z_0) \quad (5)$$

where symbol $**$ denotes a two-dimensional convolution(see Appendix). Applying the convolution theorem to Eq.(5), Eq.(5) is given as

$$p(x, y, z_0) = F^{-1}[\hat{A}(k_x, k_y)\, \hat{K}(k_x, k_y, z_0)] \quad (6)$$

where F^{-1} denotes the inverse Fourier transform and $\hat{A}$ and $\hat{K}$ are the Fourier transform of A and k, respectively. Expressing symbolically the discrete Fourier transform(DFT)

$$p_D(x, y, z_0) = D^{-1}[\hat{A}(k_x, k_y)\, \hat{K}(k_x, k_y, z_0)] \quad (7)$$

where the subscript D denotes the result of calculation via the DFT and D^{-1} represents the inverse DFT. The international mathematical and statistical library(IMSL) is used as a FFT program for this study.

1. Relation of the DFT to the Continuous Fourier Transform

The two-dimensional DFT of A(x,y) may be derived from the continuous Fourier transform as following equation.

$$\hat{A}_D(k_x, k_y) = D[A(x, y)] = \int_{-\infty}^{\infty} A(x, y)\, \text{III}(x/a, y/a)\, \Pi(x/L, y/L) \times \exp(-ik_x x - ik_y y)\, dx\, dy \quad (8)$$

where a is the sample spacing and L is the aperture window. The substitution of special functions in Eq.(8) (from Appendix) gives the discrete Fourier transform for A(x,y).

$$\hat{A}_D(m\Delta k, n\Delta k) = a^2 \sum_{l,q=-N/2}^{N/2} A(la, qa) \times \exp(-i2\pi lm/N - i2\pi qn/N) \quad (9)$$

where $N=L/a$, $k_x = m\Delta k = m(2\pi/L)$ and $k_y = n\Delta k = n(2\pi/L)$. Here Δk represents the smallest spatial frequency which can be represented by a single wavelength across the aperture L.

The discrete Fourier transform on the radiated sound pressure field from the vibrating plate is given as following equation.

$$p_D(x, y, z_0) = (1/2\pi)^2 \int_{-\infty}^{\infty} dk_x\, dk_y\, \hat{A}_D(k_x, k_y)\, \hat{K}_D(k_x, k_y, z_0) \times \text{III}(k_x/\Delta k, k_y/\Delta k)\, \Pi(k_x/2k_m, k_y/2k_m) \times \exp(ik_x x + ik_y y) \quad (10)$$

where $\Delta k = 2\pi/L$ and $k_m = \pi/a$ Note that k_m is the maximum spatial frequency(one wavelength across two samples).

2. The Errors Introduced by the DFT

While the theory of the DFT is precise and self-consistent and exactly describes the manipulation performed on actual data samples when a Fourier transform is to be computed, the question remains to what degree the DFT approximates the Fourier transform of the function underlying the data samples. Clearly, the DFT can be only an approximation for the continuous numerical evaluation since it provides only for a finite set of discrete frequencies. But these discrete values are not correct, if the initial samples are not sufficiently closely spaced to represent high-frequency components presented in the underlying function, then both the DFT values and a smooth curve passing through them will be falsified by aliasing.

With the technique described in the previous section we can derive a relation between the estimated pressure $P(x,y,z_o)$ and the actual pressure $p(x,y,z_o)$.

At first, the convolution theorem is applied to Eq.(10)

$$p_D(x,y,z_o) = F^{-1}[\hat{A}_D(k_x,k_y)] ** F^{-1}[\hat{K}_D(k_x,k_y,z_o)]$$
$$** F^{-1}[\text{III}(k_x/(2\pi/L), k_y/(2\pi/L))]$$
$$** F^{-1}[\Pi(k_x/(2\pi/a), k_y/(2\pi/a))] \quad (11)$$

From Eq.(8)

$$F^{-1}[\hat{A}_D(k_x,k_y)] = A(x,y)\,\text{III}(x/a, y/a)\,\Pi(x/L, y/L) \quad (12)$$

So that Eq.(11) becomes(refer to Appendix)

$$p_D(x,y,z_o) = \sum_{l,q=-N/2}^{N/2} a^2 A(x,y)\,\delta(x-la)\,\delta(y-qa)$$
$$** k(x,y,z_o) ** \text{III}(x/L, y/L)/L^2$$
$$** \operatorname{sinc}(x/a, y/a)/a^2 \quad (13)$$

where the sum represents a double sum over l and q, respectively, and N=L/a. The first convolution in Eq.(13) yields

$$p_N(x,y,z_o) = \sum_{l,q=-N/2}^{N/2} a^2 A(la,qa)\,k(x-la, y-qa, z_o) \quad (14)$$

which looks like a discretized version of the Eq.(4) obtained by replacing the double integral with a double sum and setting dxdy equal to a^2. Of course, as $a \to 0$, $p_N(x,y,z_o) \to p(x,y,z_o)$.

From Eq.(13) and (14),

$$p_D(x,y,z_o) = \sum_{l,q} p_N(x-lL, y-qL, z_o)$$
$$** \operatorname{sinc}(x/a, y/a)/a^2 \quad (15)$$

If a is small, we can replace p_N with p to give

$$p_D(x,y,z_o) = \sum_{l,q=-\infty}^{\infty} P(x-lL, y-qL, z_o)$$
$$** \operatorname{sinc}(x/a, y/a)/a^2 \quad (16)$$

The above equation shows that the relation between p and the actual pressure radiated by the source. The summations operate only on sound pressure p and represent the influence of an two-dimensional set of planar sources all vibrating in phase with the same velocity distribution as the actual source, each located at a node in an infinite lattice with internodal distance L. These are called replicated sources generating the error in the calculation of acoustic fields, that is the aliasing effect. Their influence is smoothed slightly by the action of the convolutions with the sinc functions. The replicated sources themselves are the inevitable results of sampling in k space. The influence of the replicated sources with the acoustic field p(x,y) from the real plate source represents the aliasing effects in k space. The influence of replicated source causing the errors can be reduced by the decrease of the ratio of the vibrating surface and aperture, and the calculation of the farfield region of sound pressure radiated from the vibrating structures.

III. APPLICATION OF FFT METHOD ON THE SOUND RADIATION FROM A CLAMPED CIRCULAR PLATE

The sound pressure radiated from a vibrating surface is evaluated by using the Rayleigh integral that approximated Helmholtz integral at acoustic farfield. The sound pressure radiating from a plate as shown in Fig.1 is written as the form having the time delay on each element of plate.The equation on the sound pressure is given as[8]

$$p(r,\theta,t) = [illegible] \frac{\rho_o [illegible] \cos\Psi}{2\pi [illegible] \cos\Psi [illegible] \theta} [illegible] \quad (17)$$

$w(r',\theta,t-R/c)$ is the acceleration-time response of the clamped circular plate to the central impact. In this paper, the plate vibrates in a rigid infinite baffle where reflection or diffraction of sound does not occur at boundaries. As using the normal mode method, Laplace transform and convolution integral, the displacement response of a clamped circular plate, initially underformed and at rest, to impact force Eq.(12) in Ref. 10($\lambda^*=0$) can be written as

$$w(r', \theta, t) = \sum_{n=1} \frac{F_o W_n(0) W_n(r')}{\omega_n^*} \int_0^t \sin\omega_n^*(t-\tau) \cdot \sin\omega_o \tau \, d\tau. \tag{18}$$

The acceleration response of plate is obtained by the differentiation of Eq.(18), twice.

In order to obtain the sound field by FFT method, we can separate the acceleration of plate into spatial and time domain.

$$A(x, y) = \frac{F_o W_n(0) W_n(r)}{\omega_n^*} \frac{\omega_o \omega_n^*}{\omega_o^2 - \omega_n^{*2}} \frac{\rho_o}{2\pi} \tag{19}$$

and R(R) is given as

$$\begin{aligned} R\,R(R)\,T(t) &= -\omega_o \sin\omega_o t \cos\omega_o R/c - \omega_o \cos\omega_o t \\ &\quad \cos\omega_o t \sin\omega_o R/c - \omega_n^* \sin\omega_n^* t \cos\omega_n^* R/c \\ &\quad - \omega_n^* \cos\omega_n^* t \sin\omega_n^* R/c, \quad t \leqq d \\ &= -\omega_n^* \sin\omega_n^* t \cos\omega_n^* R/c + \omega_n^* \cos\omega_n^* t \sin\omega_n^* R/c \\ &\quad \omega_n^* \sin\omega_n^*(t-d) \cos\omega_n^* R/c \\ &\quad + \omega_n^* \cos\omega_n^*(t-d) \sin\omega_n^* R/c, \quad t > d \end{aligned} \tag{20}$$

where $R = [x^2 + y^2 + z_o^2]^{\frac{1}{2}}$

From Eqs.(5),(6),(19) and (20), the acoustic pressure field is calculated by using FFT method at a fixed time. Examples of sound pressure field to the central impact in accordance with the acoustic fields at $z=z_o$ and the vibration mode of the plate are shown in Figs.4,5 and 6 for the case of impact of the steel ball of 1.504 cm-diam. on a 2 mm-thick, steel plate of 0.28 m diameter, respectively. In Fig.7 the results of the FFT method are compared with the direct numerical evaluation on z-axis in time domain.

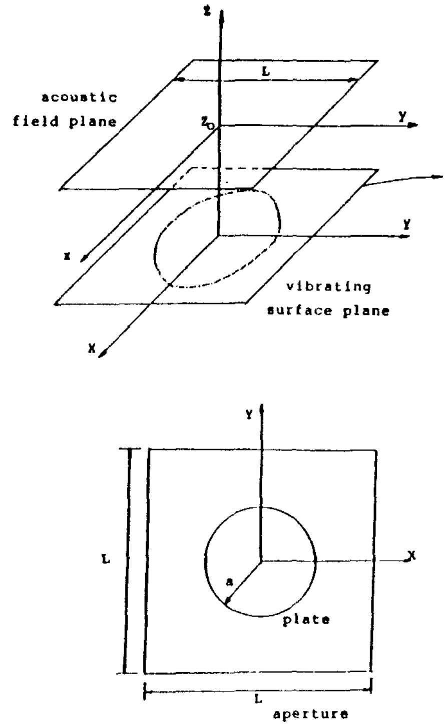


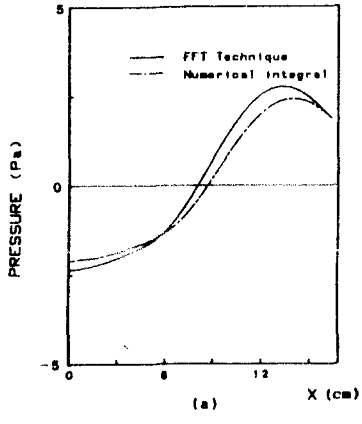
Fig. 3. Geometry on the location of the plate within the aperture (64x64 array).

In Fig.3, the space aperture is a square of length L on a side and the size of the aperture is chosen so that it occupies larger than 4 times of the area of the vibrating plate. The aperture is divided into a lattice of 64x64 points. The data files of values(real or complex) for A(x,y) and $k(x,y,z_o)$ at each the lattice point are used as the starting point of calculations. Two-dimensional Fourier transform of A(x,y) and $k(x,y,z_o)$ are computed by using the FFT program included in IMSL subroutine package. Following Eq.(6) the results obtained by FFT technique are multiplied each other and in order to obtain the sound fields the results are computed by the two-dimensional inverse Fourier transform with FFT algorithm. Thus a 64x64 array containing values of $p_r(x,y,z_o)$ is obtained. These 4096 complex data points are computed in about 8 sec for one mode on Cyber 170-835. The direct numerical evaluation of Rayleigh integral is computed in about 820 sec for one mode.

The length L of the aperture is fiexed at 89.6 cm and the lattice spacing is 1.4 cm. The actual edge of the plate occurs at x/L=0.15625. The agreement is excellent for x/L<0.25 and begins to differ slightly as the aperture boundary is approached. In this case the nearest replicated source contributes to this errors.

For (1,0) mode of the clamped circular plate the radiated sound pressure is shown in Fig.4 and for (3,0) mode is given in Figs.5 and 6. Figures 4 and 5 show the results for z_o=10 cm. As the acoustic plane moves away from the surface of the plate to z_o/λ, the contribution of the near replicated source becomes relatively stronger and the errors increase, where λ is the wavelength of interested mode and z_o/λ is 0.077 and 0.673, respectively. Since the radiation efficiency rapidly changes below the critical frequency(the flexural wave speed has the same speed as one in the ambient medium), that is, while the radiation efficiency is one above the critical frequency, for the frequency below the critical frequency that rapidly decreases. In this paper, the critical frequency for the bare steel 2 mm-thickness is about 5800 Hz.

As shown in Fig.6, for the acoustic field at z_o=25cm, the results of the sound pressure field by FFT technique agree well with the direct numerical integration of Rayleigh integral. As z_o, the distance from the vibrating surface to the acoustic plane, increases, the influence of replicated source decreases.



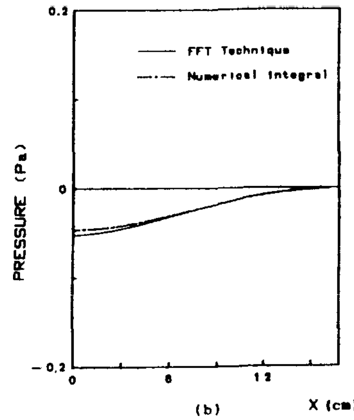
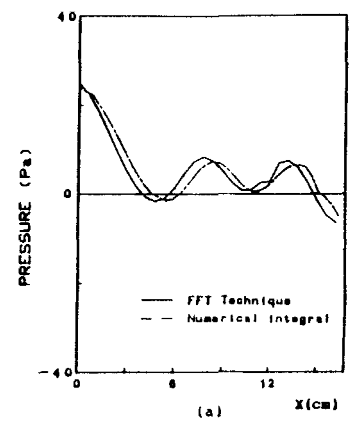


Fig. 4. Real(a), and imaginary(b) components of the pressure at the surface z_o=10 cm. (1.0) mode computed by a direct numerical integral of the Rayleigh integral and by the FFT technique. L=89.6 cm, t=0.81314 msec, 2mm-thick. steel plate.



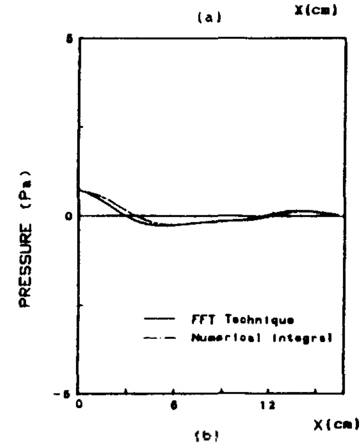


Fig. 5. Real(a), and imaginary(b) components of the pressure at the surface z_o=10 cm. (3.0) mode computed by a direct numerical integral of the Rayleigh integral and by the FFT technique. L=89.6 cm, t=0.81314 msec, 2 mm-thick. steel plate.

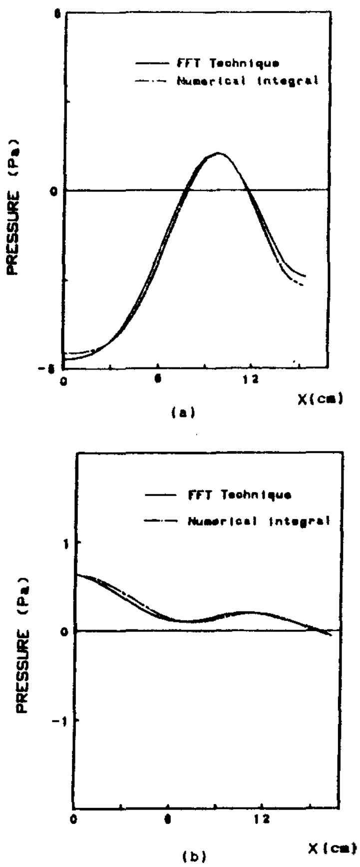


Fig. 6. Real(a), and imaginary(b) components of the pressure at the surface z_0=25 cm. (3,0) mode computed by a direct numerical integral of the Rayleigh integral and by the FFT technique. L=89.6 cm, t=0.81314 msec, 2 mm-thick. steel plate.

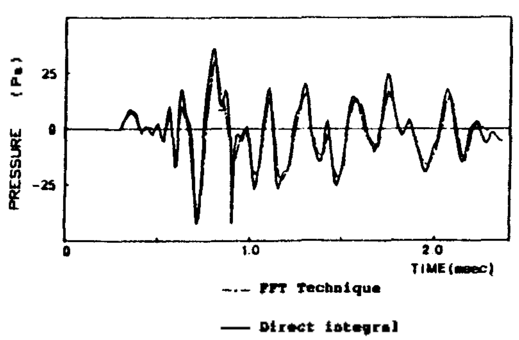


Fig. 7. Comparison with the sound pressure radiated from the plate(thick, 2mm) calculated by the direct numerical integral and the FFT technique in according to the variation of time. Bare steel plate radius 14 cm, measurement point (0,0,10).

The sound pressure-time history is shown in Fig.7. The results of FFT technique agree with those of the direct numerical integration. The results of DFT can be only an approximation since it provides only for a finite set of discrete frequencies the waveform is more smooth than the direct integral. The magnitudes of pressure have some difference each other but the acoustic sound pressure field is well predicted.

IV. CONCLUSION

We have tried to provide an insight into the solution of Rayleigh integral using the two-dimensional fast Fourier transform for the transient sound radiation. The results of FFT method agree well with the direct numerical evaluation and also the computation time is saved by using the fast Fourier transform algorithm for the evaluation of the sound pressure field over 1/200 times of the direct numerical calculation for the 64x64 lattice. But as the results, the detailed development of the physical significance of aliasing was presented. As the distance of the interested acoustic field from the vibrating surface increases, the influence of the replicated sources decreases. The effect of the replicated source is severe, in the case that the driving frequency of the external forces is higher than the critical frequency of the plate. Since the radiation efficiency of the plate is about one at the higher frequency than the critical one. While, for below critical frequency, the radiation efficiency rapidly decreases. Thus the errors by the replicated sources changes in accordance with the driving frequency of the exciting forces.

In this paper, the application of the FFT technique have concentrated on rectangular boundaries for baffled circular vibrating source. Also the FFT method can be applicable to any shaped geometry as well as this one. The FFT solution could be very powerful in predicting the near and far fields of complex structures. The computation by direct numerical evaluation of Rayleigh integral may take about 24 hour and by using the FFT technique it requires about 8 min. for the acoustic fields in according to time variation. Since the computation time in FFT method is reduced than the direct one, because the direct one requires the caculation of the vibration in order to obtain the sound pressure at each mesh point in acoustic field but FFT method is not. This is indeed a significant savings. With the growing use of array processors in FFT analysis further time saving can be gained with their use.

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APPENDIX

SPECIAL FUNCTIONS AND THEIR FOURIER TRANSFORMS

$*$: One-dimensional convolution

$**$: Two-dimensional convolution

$$f(x, y) ** g(x, y) = \iint f(x', y')\, g(x-x', y-y')\, dx'dy'$$

$$\Pi(x/L) = 1 \quad |x| < L/2$$
$$= 1/2 \quad |x| = L/2$$
$$= 0 \quad |x| > L/2$$

$$\text{III}(x/a) = a \sum_{n=-\infty}^{\infty} \delta(x - na)$$

$$\text{III}(-x) = \text{III}(x)$$

$$\text{III}(x+n) = \text{III}(x) \qquad n \text{ integer}$$

$$\text{III}(x - \tfrac{1}{2}) = \text{III}(x + \tfrac{1}{2})$$

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \text{III}(x)\, dx = 1$$
$$\text{III}(x) = 0 \quad x \neq n$$

$\text{III}(x)$ is periodic with period.

$$\text{III}(x) f(x) = \sum_{n=-\infty}^{\infty} f(n)\, \delta(x-n)$$

$$\text{III}(x+m, y+n) = \text{III}(x, y) \quad m, n \text{ integer}$$

δ = Dirac delta function

$$\int_{-\infty}^{\infty} \delta(x-a)\, f(x)\, dx = f(a)$$

$$\int_{-\infty}^{\infty} \delta(x)\, f(x-a) = f(-a)$$

$$\delta(x) * f(x) = f(x) * \delta(x) = f(x)$$

$$\delta(-x) = \delta(x)$$

$$\text{sinc}(x/a) = \sin(\pi x/a) / (\pi x/a)$$

$$\Pi(x, y) = \Pi(x)\, \Pi(y)$$

$$\text{III}(x, y) = \text{III}(x)\, \text{III}(y)$$

$$\text{sinc}(x, y) = \text{sinc}(x)\, \text{sinc}(y)$$

$F(f)$ or f = Fourier transform (2-dimension)

$$= \int dxdy\, (f)\, e^{-ik_x x - ik_y y}$$

$F^{-1}(f)$ = inverse Fourier transform (2-dimension)

$$= \left(\frac{1}{2\pi}\right)^2 \int dxdy\, (f)\, e^{ik_x x + ik_y y}$$

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