On Cn-Semistratifiable over $\alpha$

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1. Preliminaries

In this section we now introduce some definitions and results which are used throughout this paper.

DEFINITION 1.1. (Vaughan) An ordinal number $\alpha$ is called an initial ordinal provided that for every ordinal $\beta < \alpha$, there exists an injection from $\beta$ to $\alpha$, but there does not exist an injection from $\alpha$ to $\beta$. We assume that cardinal numbers and initial ordinal numbers are the same.

Let $\omega$ stand for the first infinite ordinal.

DEFINITION 1.2. (Vaughan) Let $(X, J)$ be a $T_1$-space and let $\alpha$ be an initial ordinal. $\alpha \geq \omega$. The space $(X, J)$ is said to be stratifiable over $\alpha$ or linearly stratifiable provided that there exists a map $S \colon \alpha \times J \rightarrow J$ (called an $\alpha$-stratification) which satisfies the following (where we denote $S(\beta, U)$ by $U_\beta$)

$LS1$: $U_\beta \subseteq U$ for all $\beta < \alpha$ and all $U \in J$.

$LS2$: $U \upharpoonright U_\beta : \beta < \alpha \upharpoonright U$ for all $U \in J$.

$LS3$: If $U \subseteq W$, then $U_\beta \subseteq W_\beta$ for all $\beta < \alpha$.

$LS4$: If $r < \beta < \alpha$, then $U_r \subseteq U_\beta$ for all $U \in J$.

DEFINITION 1.3. (Vaughan) A $T_1$-space $X$ is called $\alpha$-stratifiable provided that $\alpha$ is the smallest initial ordinal for which $X$ is stratifiable over $\alpha$. A space which is stratifiable over $\omega$ is called stratifiable, and the map $S$ is called a stratification.

DEFINITION 1.4. (Vaughan) A collection $P$ of pairs $P=\{(p_1, p_2)\}$ of subsets of a topological space $(X, J)$ is said to be a linearly cushioned collection of pairs with respect to a linear order $\leq$ provided that $\leq$ is a linear order on $P$ such that $(U \upharpoonright P_1 : P=\{(p_1, p_2) \in P_1 \} \subseteq U \upharpoonright P_2 : p=\{(p_1, p_2) \in P_1 \}$ for every subset $P_1$ of $P$ which is majorized (i.e., has an upper bound) with respect to $\leq$.

DEFINITION 1.5. (Ceder) A collection $P$ of pairs is called a pairbase for $(X, J)$ provided that (1) for each $P=\{(p_1, p_2) \in P, P_1$ is open and (2) for every $x$ in $x$ and every open set $W$ containing $x$, there exists $P=\{(p_1, p_2) \in P$ such that $x \in P_1 \subseteq P_2 \subseteq W$.

THEOREM 1.6. (Vaughan) If $(X, J)$ is a $T_1$-topological space and $\alpha$ an infinite initial ordinal, then the following are equivalent.
(1) \((X, \mathcal{J})\) is stratifiable over \(\alpha\).

(2) \((X, \mathcal{J})\) has a linearly cushioned pair-base \(P\) and \(\alpha\) is cofinal with \(P\).

(3) There exists a family \(\{g_\beta: \beta < \alpha\}\) of functions with domain \(X\) and range \(\mathcal{J}\) such that the following hold.

(a) \(x \in g_\beta(x)\) for all \(\beta < \alpha\)

(b) For every \(F \subseteq X\), if \(y \in [(U \downarrow g_\beta(x): x \in F)]\) for all \(\beta < \alpha\), then \(y \in F\).

(c) If \(\beta < \gamma < \alpha\), then \(g_\beta(x) \supseteq g_\gamma(x)\) for all \(x\).

A linearly semistratifiable space is introduced by K.B. Lee [4]. The new class of spaces is an extension of semistratifiable spaces, definitions and main results of which are collected in the following.

DEFINITION 1.7. (K.B. Lee) Let \((X, \mathcal{J})\) be a topological space and \(\alpha\) be an initial ordinal not less than \(\omega\). The space \(X\) is said to be sewistratifiable over \(\alpha\) or linearly semistratifiable provided that there exists a map \(S: \alpha \times \mathcal{J} \rightarrow \{\text{closed subsets of } X\}\) (called an \(\alpha\)-semistratification) which satisfies the following:

LSS1: For every \(U \in \mathcal{J}\), \(U = U \upharpoonright S(\beta, U): \beta < \alpha\)

LSS2: If \(U, V \in \mathcal{J}\) and \(U \subseteq V\), then \(S(\beta, U) \subseteq S(\beta, V)\) for all \(\beta < \alpha\)

LSS3: If \(\gamma < \beta < \alpha\), then \(S(\gamma, U) \subseteq S(\beta, U)\) for all \(U \in \mathcal{J}\).

DEFINITION 1.8. (K.B. Lee) A collection \(P\) of pairs \((p_1, p_2)\) of subsets of a space \((X, \mathcal{J})\) is called a pair-net provided that for every \(x\) in \(X\) and every open \(U\) containing \(x\), there exists a \(P = (p_1, p_2)\) such that \(x \in P_1 \subseteq P_2 \subseteq U\).

THEOREM 1.9. (K.B. Lee) If \((X, \mathcal{J})\) is a space and \(\alpha\) an infinite initial ordinal, then the following are equivalent:

(1) \(X\) is semistratifiable over \(\alpha\).

(2) \(X\) has a linearly cushioned pair-net \(P\) with which \(\alpha\) is cofinal.

(3) There is a function \(g\) from \(\alpha \times X\) into \(\mathcal{J}\) such that

(a) for each \(x \in X\), \(x \in g(\beta, x)\) if \(\beta < \alpha\)

(b) if \(x \in g(\beta, x_0)\) for each \(\beta < \alpha\), then the net \(\{x_\beta: \beta < \alpha\}\) accumulates at \(x\)

(c) if \(\gamma < \beta < \alpha\), then \(g(\gamma, x) \supseteq g(\beta, x)\) for every \(x \in X\).

(4) There is a function \(g\) from \(\alpha \times X\) into \(\mathcal{J}\) such that (a) for each \(x \in X\), \(\cap g(\beta, x) = \{x\}\) if \(x \in g(\beta, x_0)\) for each \(\beta < \alpha\), then the net \(\{x_\beta: \beta < \alpha\}\) converges to \(x\); and (c) if \(\gamma < \beta < \alpha\) then \(g(\gamma, x) \supseteq g(\beta, x)\) for every \(x \in X\).

DEFINITION 1.10. A pair-net is called an \(\alpha\)-pairnet if given any convergent net \(\chi_\alpha \rightarrow x\) and an open subset \(U\) containing \(x\), there is a \(P = (P_1, P_2) \in P\) such that \(\chi \in P_1 \subseteq P_2 \subseteq U\) and \(\chi_\alpha\) is eventually in \(P_1\).

2. Definition of \(cn\)-semistratifiable over \(\alpha\) and some characterizations

DEFINITION 2.1. Let \((X, \mathcal{J})\) be a topological space and \(\alpha\) be an initial ordinal not less than \(\omega\). The space \(X\) is said to be \(cn\)-semistratifiable over \(\alpha\) or linearly \(cn\)-semistratifiable provided that there exists a map \(S: \alpha \times \mathcal{J} \rightarrow \{\text{closed subsets of } X\}\) (called an \(cn\)-semistratification) which satisfies the following.

a) For every \(U \in \mathcal{J}\), \(U = U \upharpoonright S(\beta, U) = \beta < \alpha\)
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b) If $\emptyset, V \in \mathcal{J}$ and $U \subseteq V$, then $S(\beta, U) \subseteq S(\beta, V)$ for all $\beta < \alpha$

c) If $\gamma : \beta < \alpha$, then $S(\gamma, U) \subseteq S(\beta, U)$ for all $U \in \mathcal{J}$

d) For each convergent net $\chi_\beta \rightarrow \chi$ and $U \in \mathcal{J}$, containing $\chi$, there is a $\beta < \alpha$ such that $\chi \in S(\beta, U)$ and $|\chi_\beta \setminus \beta < \alpha|$ is eventually in $S(\beta, U)$.

DEFINITION 2.2. A topological space $X$ is called $\alpha$-cn-semistratifiable provided that $\alpha$ is the smallest initial ordinal for which $X$ is cn-semistratifiable over $\alpha$. A space which is cn-semistratifiable over $\omega$ is called cs-semistratifiable.

THEOREM 2.3. If $(X, \mathcal{J})$ is a space and $\alpha$ an infinite initial ordinal, then the following are equivalent:

1) $X$ is cn-semistratifiable over $\alpha$.

2) $X$ has a linearly cushioned cn-pairnet $P$ with which $\alpha$ is cofinal.

3) There is a function $g$ from $\alpha \times X$ into $\mathcal{J}$ satisfying Theorem 1.9(1) and an additional condition:

4) Given a convergent net $|x_\beta \setminus \beta < \alpha| \rightarrow x$ and an open subset $U$ containing $x$, there is a $\beta < \alpha$ such that $x \in U, g(\beta, y)$ and $|\beta < \alpha| |x_\beta \setminus \beta < \alpha| \rightarrow x \in U, g(\beta, y)$ such that $J = |x \setminus y \leq \beta < \alpha|$ is cofinal.

Proof. For (1) $\leftrightarrow$ (2), see the proof of Theorem 1.13

For (2) $\leftrightarrow$ (3), let $P$ be a linearly cushioned cn-pairnet for $X$, and $\alpha$ cofinal with $P$.

There is a subclass $P' = |P_\beta : \beta < \alpha|$ such that for every $P \in P$ there is a $\beta < \alpha$ such that $P \subseteq P_\beta$.

For each $x$ in $X$ and each $\beta < \alpha$, define $g(\beta, x) = X \cap (U | P_1 : x \in P_1$ and $P = (P_1, P_2) \subseteq P_\beta|).

Lee K.B proved $g$ is a linearly stratifiable function. To show $g$ satisfies (d).

Consider the following.

$$U = \bigcup_{y \in \mathcal{V}} g(\beta, y)$$

which is contained in $X - cl((U | P_1 : y \in P_1$ and $P = (P_1, P_2) \subseteq P_\beta|)$

If $|x_\beta \setminus \beta < \alpha| \rightarrow x \in P_1 \subseteq P_2 \subseteq U$ and $P = (P_1, P_2) \subseteq P_\beta$,

$$J = |\beta < \alpha| \chi_\beta \in \bigcup_{Y \in \mathcal{V}} g(\beta, y)$$

is eventually in $x \in P_1 \subseteq P_2 \subseteq U$ and $P = (P_1, P_2) \subseteq P_\beta$.

(3) $\leftrightarrow$ (1) Let $g$ be a map as is described in (3).

Define a map $S : \alpha \times \mathcal{J} \rightarrow \{\text{closed subsets of } X\}$

by $S(\beta, U) = X - U | g(\beta, \chi) : \chi \in X - U|$.

$S(\beta, U) \subseteq X - (X - U) = U$. Since $\chi \in g(\beta, \chi)$ for all $\beta < \alpha$. Conversely, assume $\chi \in U | S(\beta, U) : \beta < \alpha|$

Then $\chi \in U | g(\beta, y) : y \in X - U$ for all $\beta < \alpha$. This implies there is an $y_\beta \in X - U$ such that $\chi \in g(\beta, y_\beta)$ for each $\beta < \alpha$.

Thus $|y_\beta : \beta < \alpha|$ satisfies the condition (b) of (4), and hence converges to $\chi$. 
Since $X - U$ is closed, we have $\chi \in cl(|\gamma_\beta : \beta < \alpha|) \subset X - U$. Finally, the condition (d) of Definition 2.1 is satisfied by the property (d) of $g$. Thus the proof is completed.

3. Properties of $C^n$-semistratifiable over $\alpha$

**THEOREM 3.1.** Every subspace of a $C^n$-semistratifiable over $\alpha$ is a $C^n$-semistratifiable over $\alpha$.

**Proof.** Let $S$ be an $\alpha$-$C^n$-semistratification of $X$, and $Y$ be a subspace of $X$. Define $S' : \alpha \times \mathcal{P}_\alpha \rightarrow |\text{closed subsets of } Y|$ by the restriction of $S$ to $\mathcal{P}_\alpha$-open subset of $X$. It is easily verified that $S'$ is an $\alpha$-$C^n$-semistratification for $Y$.

Now, we shall prove that a finite product of spaces $C^n$-semistratifiable over the same $\alpha$ is again $C^n$-semistratifiable over $\alpha$.

**LEMMA 3.2.** Let $\alpha$ be an infinite initial ordinal number, and let $|A_\lambda : \lambda \in \Lambda|$ be a family of linearly ordered sets such that $\alpha$ has cardinality strictly greater than that of $A$, and $\alpha$ is cofinal with $A$ for all $\lambda \in \Lambda$. If $\Lambda$ is finite or if $\alpha$ is a regular cardinal, then $A = \bigcap |A_\lambda : \lambda \in \Lambda|$ can be well-ordered so that for every majorized $H \subset A$, we have $Pr(H)$ (i.e., the $\lambda$th projection) is majorized in $A_\lambda$ for all $\lambda \in \Lambda$ and $\alpha$ is cofinal in $A$. Further, if $\alpha$ is the smallest initial ordinal cofinal with each $A_\lambda$, then $\alpha$ is the smallest initial ordinal cofinal with $A$.

**Proof.** See the proof of Lemma 5.19.

**Theorem 3.3.** Let $\alpha$ be an initial ordinal number $\alpha \geq \omega$. Let $X_i$ be $C^n$-semistratifiable over $\alpha$ for each $i < \omega$. Then $\Pi |X_i : i \leq n|$ is $C^n$-semistratifiable over $\alpha$ for all $n < \omega$.

**Proof.** Each $X_i$ has a linearly cushioned $\alpha$-pair-net $P_i$, such that $\alpha$ is cofinal with $P_i$. For each $n < \omega$ and each $Q = (P^1, ..., P^n)$

$\Pi |P_i : i \leq n| \equiv \Pi |\chi = (\chi_i) : \chi_i \in P_i^i$ for $i \leq n|$, and similarly define $\Pi |P_i : i \leq n|$, Set $B_0 = \Pi |P_i : i \leq n|$, and $B_n = |B_0, B_0 : Q \in \Pi |P_i : i \leq n| |$ and order the index set of $B_n$, as Lemma 3.2 so that $\alpha$ is cofinal with $B_n$ clearly $B_n$ is a $\alpha$-pair-net for $\Pi |X_i : i \leq n|$, and if we consider $(\chi_i) \in \Pi |X_i : i \leq n|$, then $B = U |B_n : n < \omega|$ is a $\alpha$-pair-net for $\Pi |X_i : i \leq n|$. We now show that each $B_n$ is a linearly cushioned collection of pairs in $X = \Pi |X_i : i \leq n|$. Suppose $H$ is a majorized subset of $\Pi |P_i : i \leq n|$ and $\chi \in U |B_0 : Q \in H|$.

Let $N_i = X_i -(U |P_i : P = (P_i, P_\alpha) \in P_{\gamma_\beta} (H)$ and $\chi \in P_i |)$. Then $N_i$ is an open neighborhood of $\chi_i$ in $X_i$ because $P_\alpha(H)$ is a majorized subset of $P_i$. Finally, $\Pi N_i$ is a neighborhood of $\chi$ in $X$ which misses $U |B_0 : Q \in H|$. Thus $(U |B_0 : Q \in H|)^C \subset U |B_0 : Q \in H|$, and this completes the proof.

**Lemma 3.4.** Let $X$ be $C^n$-semistratifiable over $\alpha$ and $Y$ be a closed subspace of $X$ with an $\alpha$-$C^n$-semistratification $S$. Then there is an $\alpha$-$C^n$-semistratification $T$ for $X$ such that $S(\beta, V \cap Y) = T(\beta, V) \cap Y$ for every $\beta < \alpha$ and every open $V$ in $X$.

**Proof.** Let $S'$ be any $\alpha$-$C^n$-semistratification for $X$. Define an $\alpha$-$C^n$-semistra-
tification \( T \) for \( X \) as follows:

\[
T(\beta, V) = S(\beta, V \cap Y) \cup S^c(\beta, V^\sim Y)
\]

It is clear that \( T \) is an \( \alpha \)-cn-semistratification.

Now, we show that \( T \) satisfies (d).

Let \( |\chi_\alpha| \) be a net in \( X \) converging to \( \chi \). Given an open set \( U \) of \( X \) containing \( \chi \), if \( \chi \in U \cap Y \). Since \( U \cap Y \) is a relative open subset in \( Y \) there is \( \gamma < \alpha \) such that \( |\chi_\alpha| \) is eventually in \( S(\gamma, U \cap Y) \). Therefore \( |\chi_\alpha| \) is eventually in \( T(\gamma, U) \).

If \( \chi \in U \cap Y \) it is clear.

This completes the proof.

Lemma 3.5. The union of two closed (in the union) subspaces which are cn-semistratifiable over \( \alpha \) is also cn-semistratifiable over \( \alpha \).

Proof. Apply Lemma 3.4 with respect to the common subspace.

Theorem 3.6. If \( X \) is a locally finite union of closed cn-semistratifiable over \( \alpha \), then \( X \) is cn-semistratifiable over \( \alpha \).

Proof. By Lemma 3.5, the proof is verified easily.

4. Net-covering maps

Frank Siewiec introduced the concept of sequence-covering map in [8]. Now, we introduce the extended concept of sequence-covering map.

Definition 4.1. A mapping \( f : X \to Y \) is said to be net covering if given any convergent net \( \chi_\alpha \to \chi \) in \( Y \), there exists a convergent net \( \chi_\alpha \to \chi \) in \( X \) such that \( f(\chi_\alpha) = \chi_\beta \; \beta < \alpha \).

Theorem 4.2. The image of a cn-semistratifiable over \( \alpha \) under a closed continuous net-covering map is cn-semistratifiable over \( \alpha \).

Proof. Let \( f \) be a closed continuous net-covering map from \( \text{cn-semistratifiable over } \alpha : X \to Y \) onto a space \( Y \). Let \( S \) be a \( \alpha \)-cn-semistratification for \( X \). For each open \( V \) of \( Y \) and \( \beta < \alpha \), let \( T(\beta, V) = f(S(\beta, f^{-1}(V)) \) clearly \( T \) is a \( \alpha \)-cn-semistratifiable.

Then there is a convergent net \( \chi_\alpha \to \chi \) in \( X \) such that \( f(\chi_\alpha) = \chi_\beta \) for \( \beta < \alpha \). Since \( X \) is cn-semistratifiable over \( \alpha \), there exists a \( \gamma < \alpha \) such that \( \chi_\alpha | \beta < \alpha \) is eventually in \( S(\gamma, f^{-1}(V)) \) for any open \( V \). Thus, \( \chi_\beta \) is eventually in \( T(\gamma, V) = f(S(\gamma, f^{-1}(V)) \).

W.K.MIN proved that \( K \)-semistratifiable over \( \alpha \) with \( \alpha \)-fundamental system of neighborhoods \( |W_\alpha(\chi) : \beta < \alpha \) and \( W_\beta(\chi) \subseteq W_\gamma(\chi) \) for \( \gamma < \beta < \alpha \) for each \( \chi \in X \) is stratifiable over \( \alpha \).

Theorem 4.3. A cn-semistratifiable over \( \alpha \) with \( \alpha \)-fundamental system of neighborhoods \( |W_\alpha(\chi) : \beta < \alpha \) for each \( \chi \in X \) is stratifiable over \( \alpha \).

Proof. Let \( S \) be an \( \alpha \)-cn-semistratification for \( X \). Suppose that \( P \in V \), where \( V \) is open. Let \( |W_\beta(P) : \beta < \alpha \) be \( \alpha \)-fundamental system of neighborhoods for \( p \) such that \( V \supseteq W_\gamma \supseteq W_\alpha \) for \( \gamma < \beta < \alpha \).

If \( W_\alpha \subseteq S(\beta, V) \) for each \( \beta < \alpha \), choose points \( \gamma_\beta \in W_\beta \subseteq S(\beta, V) \) for each \( \beta < \alpha \). The net convergents to \( p \), and so there is such that \( |y_\beta \beta < \alpha \) is eventually in
\[ S(\gamma, V) \]. Therefore, for some \( \gamma < a \), \( W_\gamma(p) \subset S(\gamma, V) \).

By Lemma 3.4 [6] \( X \) is stratifiable over \( a \).

References


(요)

이 논문에서는 C―Semistratifiable 공간보다 더 일반화된 공간 Cn—Semistratifiable을 정의하여 그에 따른 여러 가지 성질들을 조사하였다.

위상 공간 \((X,\tau)\)에 대하여 \(a \times \tau\)에서 \(X\)의 개집합으로의 함수 \(S\)가 존재하여 다음 조건들을 만족할 때 공간\(X\)는 Cn―Semistratifiable over \(a\)라 정의한다.

\(a\) 임의의 개집합 \(U\)에 대하여 \(U \in S(\beta, U) : \beta < a\)

\(b\) \(U, V\)가 \(X\)의 개집합이고 \(U \subset V\)이면 모든 \(\beta < a\)에 대하여 \(S(\beta, V) \subset S(\beta, V)\)이다.

\(c\) 만약 \(\gamma < \beta < a\) 이라면 임의의 개집합 \(U\)에 대하여 \(S(\gamma, U) \subset S(\beta, U)\)이다.

\(d\) \(X\)의 수렴하는 net \(\{x_n\} \to x\)와 \(X\)를 품는 임의의 개집합 \(U\)에 대하여 적당한 \(\beta < a\)가 존재하여 \(x \in S(\beta, U)\)이고 \(\{x_n\}\)는 \(S(\beta, U)\) 안에 eventual 하게 들어간다.

위의 정의에 의하여 다음과 같은 성질들이 증명되었다.

1. Stratifiable over \(a\) \(\Rightarrow cn―semistratifiable over \)セmistratifiable over \(a\)

2. 어떤 공간이 cn—semistratifiable over \(a\)이기 위한 필요충분 조건은 그것이 linearly cushioned cn—pairnet 를 갖는 것이다.

3. cn—semistratifiable over \(a\)의 부분공간 역시 cn—semistratifiable over \(a\) 한다.

4. cn—semistratifiable over \(a\)의 유한개의 적공간 역시 cn—semistratifiable over \(a\)한다.

5. \(\not\exists\) cn—semistratifiable over \(a\) 부분공간들의 합공간 역시 cn—semitrivial over \(a\)

6. 개연속 net—cevering 함수에 의하여 cn—semistratifiable over \(a\) 성질이 보존된다.
Introduction

In 1972, the concept of a linearly stratifiable space was introduced by J. E. Vaughan [9].

The class of linearly stratifiable space is composed of special sedclasses called $\alpha$-stratifiable spaces (where $\alpha$ is an infinite cardinal number) of which the class of stratifiable spaces is the subclass corresponding to the first infinite cardinal.

An analogous extension of the concept of a semistratifiable space [1] was introduced by K. B. Lee [4].

In this paper, a $C_n$-semistratifiable over $\alpha$ is defined and some results will be given throughout this paper, all spaces will be $T_1$. 