韓國數學教育學會誌〈數學教育〉
1988. 6, Vol. XXVI, No. 2.

Journal of the Korea Society of Mathematical Education
Jun. 1988, Vol. XXVI, No. 2.

On the Ideals in the Space of Bounded Operators

by Myeong Hwan Kim

Kangweon National University, Chooncheon, Korea

1. Introduction

The theory of operator ideals in Hilbert spaces have been developed by J. Calkin(2) and R. Schatten (7), and in the period from 1965 to 1975, this theory was developed into a separate discipline of functional analysis, which provides powerful methods for other branches of mathematics. Important applications have been made to the geometry of Banach space, to Brownian motion and, in particular, to eigenvalue problem (5).

J. Calkin (2) showed that the space $F(H)$ of all finite rank operators on a Hilbert space is the minimal and the space $C(H)$ of all compact operators on a Hilbert space is the maximal two - sided ideal of the space $B(H)$ of all bounded operators on a Hilbert space, that is, any two-sided ideal $M(H)$ of the $B(H)$ is contained in $C(H)$ and contains $F(H)$; $F(H) \subset M(H) \subset C(H)$.

In this paper we make clear the two-sided ideal $M(H)$ of $B(H)$ and investigate the inclusion relations of various subspaces in $B(H)$ in terms of the ideal structure.

Our major concepts are developed by three stage sequential processes; In section 2, we summarize somewhat familiar with the rudiments of functional analysis which are needed in the later section. In section 3, we discuss the ideals $F(H)$ and $C(H)$ of $B(H)$. Section 4 includes the ideals $N(H)$ (the space of all nuclear operators) and $S(H)$ (the space of all Hilbert-Schmidt operators) of $B(H)$. Finally we characterize the ideals of $B(H)$ as our conclusion.

2. Notations and Some general results

This section gives notations and some general results of functional analysis which are needed in the later sections.

Assumming that we are familiar with these results, we summarize only some results together with notations without proofs. The elementary details reffered to here can be founded in the indicated references or another standard text on functional analysis.

Throughout this paper, H and Hi $(i=1, 2)$ denote separable Hilbert spaces over the complex field C. A linear operator T is said to be bounded if the norm of T, $\|T\|= sup\{\|Tx\| : \|x\| \leq 1\} < \infty$ for $x \in D(T)$ (the domain of T). We denote by $B(H_1, H_2)$ the

Banach space of all bounded linear operators T of H_1 into H_2. In case $H=H_1=H_2$ we write $B(H)$ in place of $B(H_1, H_2)$.

The adjoint operator $T^* \in B(H_2, H_1)$ of $T \in B(H_1, H_2)$ is determined by $\langle x, T^*y\rangle = \langle Tx, y\rangle$ for all $x \in H_1$, and $y \in H_2$. *If* $T^*=T$ then T is called a self-adjoint operator and if it commutes with its adjoint, i.e., if $T^*T=TT^*$ then T is said to be normal. An operator $T \in B(H_1, H_2)$ is said to be of finite rank if its range $R(T)$ is a finite-dimensional subspace of H_2.

The space of finite rank operators from H_1 into H_2 is denoted by $F(H_1, H_2)$ and $F(H) = F(H, H)$.

An orerator $T \in B(H_1, H_2)$ is called compact if $T(U)$ has compact closure in H_2 where U is the unit ball of H_1. Often it is convenient to use the equivalent definition which asserts that T is compact if and only if, for each bounded sequence $\{x_n\}$ in H_1, there exists a subsequence $\{x_{n_k}\}$ and an element $y \in H_2$ such that $T(x_{n_k}) \rightarrow y$. Another equivalent formulation is that the image, under T, of a bounded set in H is totally bounded in H_2. We denote by $C(H_1, H_2)$ the space of all compact operators in $B(H_1, H_2)$ and write $C(H, H)$ as $C(H)$. We summarize the properties of $F(H)$ and $C(H)$ as follows;

Proposition 1 (1, 4) *(a) If* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *are an orthonormal basis (ONB) of* H_1 *and* H_2 *respectively and* $\{a_n\} \subset C$, *then every operator* $T \in F(H_1, H_2)$ *is represented by*

$$T = \sum_{n=1}^{\infty} a_n \langle \cdot, x_n\rangle y_n = \sum_{n=1}^{\infty} a_n y_n \otimes \overline{x_n}$$

$$T^* = \sum_{n=1}^{\infty} a_n \langle \cdot, y_n\rangle x_n = \sum_{n=1}^{\infty} a_n x_n \otimes \overline{y_n}$$

with rank $(T) \leqq n$.

(b) Every operator $T \in F(H_1, H_2)$ *is bounded and compact.*

(c) If $\dim H < \infty$ *(*$\dim H$ *denotes the dimension of* H*), then every operator* $T \in F(H_1, H_2)$ *is compact.*

Proposition 2 (3, 6) *(a) An operator* $T \in C(H_1, H_2) \Leftrightarrow Tx_n \rightarrow 0$ *for every weak null-sequence* $\{x_n\}$ *from* H_1.

(b) An operator $T \in C(H_1, H_2) \Leftrightarrow$ *there exists a sequence* $\{Tn\}$ *of finite rank operators from* $B(H_1, H_2)$ *for which* $\|Tn - T\| \rightarrow 0$.

(c) If $\{Tn\}$ *is a sequence of compact operators from* $B(H_1, H_2)$ *and* $\|Tn - T\| \rightarrow 0$ *for some* $T \in B(H_1, H_2)$, *then* $T \in C(H_1, H_2)$.

(d) $T \in C(H_1, H_2) \Leftrightarrow T^* \in C(H_2, H_1) \Leftrightarrow T^*T \in C(H_1)$.

An operator $T \in B(H)$ is called positive if $\langle Tx, x\rangle \geqq 0$ for all $x \in H$. We write $T \geqq 0$ if T is positive and $T \leqq S$ if $S - T \geqq 0$. If $T \in B(H)$ and $T \geqq 0$, then there is a unique $S \in B(H)$ with $S \geqq 0$ and $S^2 = T$. Thus if $T \in B(H)$, then $|T| = \sqrt{T^*T}$.

An operator $U \in B(H)$ is called an isometry if $\|Ux\| = \|x\|$ for all $x \in H$. U is called a partial isometry if U is an isometry when restricted to the closed subspace $(Ker\ U)^{\perp}$. If $T \in B(H)$, then there is a partial isometry U such that $T = U|T|$. U is uniquely determind by the condition that $Ker\ U = Ker\ T$ where $Ker\ T$ and $Ker\ U$ denote the kernel of T and U respectively. Thus it follows that $|T| = U^*T = U^{-1}T$. This representation, $T = U|T|$ is called the polar decomposition of the operator T. Let $\{x_n\}_{n=1}^{\infty}$ by an ONB (orthonormal basis) on H. Then for any positive operator $T \in B(H)$ we define $tr(T) = \sum_{n=1}^{\infty} \langle x_n, Tx_n \rangle$. The number $tr(T)$ is called the trace of T and is independent of the ONB chosen. An operator $T \in B(H)$ is called nuclear if and only if $tr|T| = \sum_{n=1}^{\infty} \langle x_n, |T|x_n \rangle < \infty$. The set of all nuclear operators on H is denoted by $N(H)$ and the norm of $T \in N(H)$ is defined by $\|T\|_1 = tr|T|$. We can summarize the properties of $N(H)$ and $tr(T)$ as follows:

Proposition 3 (1, 4, 6) *(a) Every* $T \in N(H)$ *is compact.*

(b) $T \in N(H) \Leftrightarrow \sum_{n=1}^{\infty} \lambda_n < \infty$, *where* $\lambda_n = S_n(T)$ *are the singular values (or s-numbers) of* T.

(c) Every operator $T \in F(H)$ *is* $\|\cdot\|_1$*-dense in* $N(H)$.

(d) If $T, S \in C(H)$, *then* $tr(T+S) = Tr(T) + tr(S)$, $tr(\alpha T) = \alpha tr(T)$ *for all* $\alpha \geqq 0 (\in R)$. $tr(UTU^{-1}) = tr(T)$ *for any unitary operator* U.

(e) If $T \in N(H)$ *and* $S \in B(H)$, *then* $tr(ST) = tr(TS)$.

An operator $T \in B(H)$ is called a Hilbert-Schmidt if and only if $tr(T^*T) < \infty$. The space of all Hilbert-Schmidt operators is denoted by $S(H)$ and the Hilbert-Schmidt norm, $\|T\|_2$ is defined by $\|T\|_2 = (\sum_j \|Tx_j\|^2)^{\frac{1}{2}}$. Thus it follows that $\|T\|_2 = (tr(T^*T))^{\frac{1}{2}}$, $\|T\| \leqq \|T\|_2 \leqq \|T\|_1$, $\|T\|_2 = \|T^*\|_2$. We can summarize the elementary properties of $S(H)$ as follows:

Proposition 4 (1, 3, 6) *(a) Every* $T \in S(H)$ *is compact.*

(b) $T \in S(H) \Leftrightarrow \sum_{n=1}^{\infty} \lambda_n^2 < \infty$, *where* $\lambda_n = S_n(T)$ *are the singular values (or s-numbers) of* T.

(c) Every operator $T \in F(H)$ *is* $\|\cdot\|_2$*-dense in* $S(H)$

(d) $S(H)$ *is not* $\|\cdot\|$*-closed.*

3. The ideals F(H) and C(H) of B(H)

In this section we discuss the property of $F(H)$ and $C(H)$ in the context of the ideal of $B(H)$. From proposition 1 and 2 in the section 2, we see that $F(H) \subset C(H) \subset B(H)$. The fundamental property of the ideal in $B(H)$ was given by J. Calkin (2) as follows:

Proposition 5 (2, 4) *Any two-sided ideal* $M(H)$ *of* $B(H)$ *is contained in* $C(H)$ *and contains* $F(H)$: $F(H) \subset M(H) \subseteq C(H)$, *that is,* $F(H)$ *is the minimal and* $C(H)$ *the maximal closed two-sided ideal of* $B(H)$.

The basic properties of $F(H)$ are given in the following lemma:

Lemma 6. *$F(H)$ is the minimal two-sided $*$-ideal in $B(H)$, that is,*

(a) $F(H)$ *is a linear space*

(b) If $T \in F(H)$, *then* $T^* \in F(H)$.

(c) If $T \in F(H)$ *and* $S \in B(H)$ *then* $TS \in F(H)$ *and* $ST \in F(H)$.

(d) $F(H)$ is the minimal ideal in $B(H)$.

proof. (a) If $T,\ S\in F(H)$ then

$$(T+S)\ (H)\subset T(H)+T(S).$$

The inclusion,

$$rank\ (T+S)\subset rank\ T+rank\ S$$

implies that $T+S$ is finite rank. Thus $F(H)$ is a liner space.

(b) Let $\{x_j\}$ and $\{y_j\}$ be ONB's of H and let $\{\alpha_j\}\subset C$. From proposition 1 an operator $T\in F(H)$ is represented by $T=\sum_{j=1}^{n}\alpha_j y_j\otimes\overline{x_j}$ with rank $(T)\leqq n$. Thus the adjoint operator T^* of $T\in F(H)$ is represented by

$$T^*=\sum_{j=1}^{n}\alpha_j x_j\otimes y_j.$$

Thus we have $T^*\in F(H)$. For another method, see J. Weidman (6, pp. 129).

(c) If $T\in F(H)$ with rank $(T)\leqq n$ and $S\in B(H)$, then from proposition 1

$$ST=\sum_{j=1}^{n}\alpha_j\langle\cdot,x_j\rangle Sy_j=\sum_{j=1}^{n}Sy_j\otimes\overline{x_j}\in F(H)$$

$$TS=\sum_{j=1}^{n}\alpha_j\langle\cdot,S^*x_j\rangle y_j=\sum_{j=1}^{n}y_j\otimes S^*x_j\in F(H),$$

giving (C). Alternatively the inclusion $rank(TS)\subset rank(T)$ shows that $F(H)$ is a left ideal in $B(H)$ and T^* is in $F(H)$ from above (b) which implies that S^*T^* is in $F(H)$ and hence that $TS=(S^*T^*)^*$ is in $F(H)$.

(d) This follows from the proposition 5 or Gohberg and Krein(4, pp. 66).

Therefore $F(H)$ is the minimal two-sided $*-ideal$ in $B(H)$.

By arguments analogous to those we used for $F(H)$ we can prove the following lemma;

Lemma 7 *$C(H)$ is the maximal closed two-sided $*$-ideal in $B(H)$, that is,*

(a) $C(H)$ is a linear space.

(b) If $T\in C(H)$, then $T^*\in C(H)$

(c) If $T\in C(H)$ and $S\in B(H)$, then $TS\in C(H)$ and $ST\in C(H)$

(d) $C(H)$ is the maximal closed ideal in $B(H)$

Proof. (a) Let $T_1,\ T_2\in C(H)$ and $a,\ b\in C$. Then from proposition 2, if $x_n\to 0$ in H, then $T_1x_n\to 0$ and $T_2x_n\to 0$. Hence $(aT_1+bT_2)x_n\to 0$. Thus $C(H)$ is a linear space.

(b) This follows from proposition 2. (d).

(c) We use the fact that $T\in B(H)$ (, that is, $x_n\to x$ implies $Tx_n\to Tx$) is equivalent to that $x_n\overset{\omega}{\to}x$(weakly convergent) implies $Tx_n\overset{\omega}{\to}Tx$(weakly convergent). Now let $T\in C(H)$ and $S\in C(H)$.

If $\{x_n\}$ is a weak null-sequence in H, then the sequence $\{Sx_n\}$ is also a weak null-sequence. As $T\in C(H)$, then $TSx_n\to 0$: hence $TS\in C(H)$. On the other hand if $T\in C(H)$, then $Tx_n\to 0$ for every weak null-sequence $\{x_n\}$ from H (see, proposition 2). Since $S\in B(H)$ is continuous, we also have $STx_n\to 0$, Therefore $ST\in C(H)$.

(d) This follows from proposition 5 or Gohberg and Krein(4, pp. 66-67).

4. The ideals S(H) and N(H) in B(H)

In this section we discuss the ideals $S(H)$ and $N(H)$ in $B(H)$.

By the definion or proposition 3 and 4, we see that $N(H) \subset S(H)$, and that $N(H)$ and $S(H)$ is not closed subset of $C(H)$. We have the Properties of $N(H)$ as follows.

Lemma 8 *$N(H)$ is a two-sided $*$-ideal in $B(H)$, that is,*

(a) $N(H)$ is a linear space

(b) If $T \in N(H)$ and $S \in B(H)$, then $TS \in N(H)$ and $ST \in N(H)$

(c) If $T \in N(H)$, then $T^* \in N(H)$

Proof. (a) Since $|aT| = |a||T|$ for $a \in C$, $N(H)$ is closed under scalar multiplication. Now suppose that $T_1, T_2 \in N(H)$. We wish to prove that $T_1 + T_2 \in N(H)$.
Let U, V and W be the partial isometries arising from the polar decomposition $T_1 + T_2 = U|T_1 + T_2|$, $T_1 = V|T_1|$, $T_2 = W|T_2|$. Then, for an ONB$\{x_n\}$ of H.

$$\sum_{n=1}^{N} \langle x_n, |T_1 + T_2| x_n \rangle = \sum_{n=1}^{N} \langle x_n, U^*(T_1 + T_2) x_n \rangle$$

$$\leqq \sum_{n=1}^{N} |\langle x_n, U^* V |T_1| x_n \rangle| + \sum_{n=1}^{N} |\langle x_n, U^* W |T_2| x_n \rangle|$$

However

$$\sum_{n=1}^{N} |\langle x_n, U^* V |T_1| x_n \rangle| \leqq \sum_{n=1}^{N} \| |T_1|^{\frac{1}{2}} V^* U x_n \| \cdot \| |T_1|^{\frac{1}{2}} x_n \|$$

$$\leqq (\sum_{n=1}^{N} \| |T_1|^{\frac{1}{2}} V^* U x_n \|^2) \quad \cdot (\sum_{n=1}^{N} \| |T_1|^{\frac{1}{2}} x_n \|^2)^{\frac{1}{2}}$$

Thus, if we can show $\sum_{n=1}^{N} \| |T_1|^{\frac{1}{2}} V^* U x_n \|^2) \cdot \leqq tr|T_1|$, *we can*

$$\sum_{n=1}^{N} \langle x_n, |T_1 + T_2| x_n \rangle \leqq tr|T_1| + tr|T_2| \langle \infty$$

and thus $T_1 + T_2 \in N(H)$. To show $\sum_{n=1}^{N} \| |T_1|^{\frac{1}{2}} V^* U x_n \|^2 \leqq tr|T_1|$, we need only prove that

$$tr(U^* V |T_1| V^* U) \leqq tr|T_1|.$$

Picking an ONB, $\{x_n\}$ with each x_n in $KerU$ or $(KerU)^{\perp}$ we see that $tr(U^*(V|T_1|V^*)U) \leqq tr(V|T_1|V^*)$. Similarly, picking an ONB, $\{y_m\}$, with each y_m in $Ker\ V^*$ or $(Ker\ V^*)^{\perp}$ we find $tr(V|T_1|V^*) \leqq tr|T_1|$. Alternatively, it follows from proposition 3(d) that the sum of two nuclear operators is a nuclear operator. Thus $N(H)$ is a linear space.

(b) Since each $T \in B(H)$ can be written as a linear combination of unitary operators, we need only show that $T \in N(H)$ implies $UT \in N(H)$ and $TU \in N(H)$ if U is unitary. But $|UT| = |T|$ and $|TU| = U^{-1}|T|U$, so by part(e) of proposition 3, TU and UT are in $N(H)$.

(c) Let $T = U|T|$ and $T^* = V|T^*|$ be the polar decomposition of T and T^*. Then $|T^*| = V^*|T|U^*$. If $T \in N(H)$, then $|T| \in N(H)$, so by part (b) above $|T^*| \in N(H)$ and $T^* = V|T^*| \in N(H)$.

By argument analogous to those we used for $N(H)$ we also have the following lemma;

Lemma 9 *$S(H)$ is the two-sided ※-ideal in $B(H)$, that is,*

(a) S(H) is a linear space
(b) If $T\in S(H)$, then $T^*\in S(H)$
(c) If $T\in S(H)$ and $S\in B(H)$, then $TS\in S(H)$ and $ST\in S(H)$.
(d) $S(H)\subset C(H)$.

Proof. (a) Let T_1, $T_2\in S(H)$ and $\alpha,\beta\in C$, and let $\{x_n\}$ be a *CONB* (complete orthonormal base) on H. Then

$$\sum_{n=1}^{\infty}\|\alpha T_1x_n+\beta T_2x_n\|^2\leqq 2\sum_{n=1}^{\infty}(|\alpha|^2\|T_1x_n\|^2+|\beta|^2\|T_2x_n\|^2)<\infty.$$

Thus $\alpha T_1+\beta T_2$ is a Hilbert-Schmidt operator, and hence $S(H)$ is a linear space.

(b) Let $T\in S(H_1, H_2)$ and let $\{x_n\}$ be an ONB of H_1 such that

$\sum_{n=1}^{\infty}\|Tx_n\|^2<\infty$. If $\{y_n\}$ is an arbitrary ONB of H_2, then

$$\sum_{n=1}^{\infty}\|Tx_n\|^2=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|\langle y_m, Tx_n\rangle|^2=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|\langle x_n, T^*y_m\rangle|^2=\sum_{m=1}^{\infty}\|T^*y_m\|^2<\infty.$$

Thus T^* is a Hilbert-Schmidt operator.

(c) Let $T\in S(H)$ and $S\in B(H)$, and let $\{x_n\}$ be an ONB of H. Then

$$\sum_{n=1}^{\infty}\|STx_n\|^2\leqq\|S\|^2\sum_{n=1}^{\infty}\|Tx_n\|^2<\infty.$$

Thus $ST\in S(H)$. On the other hand, from the above (b), T^* is in $S(H)$, which implies that S^*T^* is in $S(H)$ and hence that $TS=(S^*T^*)^*$ is in $S(H)$.

(d) Let $\{x_n\}$ and $\{y_m\}$ be CONB of H. Then for $x\in H$,

$$\|Tx\|^2=\sum_{m=1}^{\infty}|\langle Tx, y_m\rangle|^2=\sum_{m=1}^{\infty}|\langle x, T^*y_m\rangle|^2$$

$$\leqq\sum_{m=1}^{\infty}\|x\|^2\|T^*y_m\|^2=\|x\|^2\sum_{n=1}^{\infty}\|Tx_n\|^2$$

$$\therefore\ \|T\|\leqq\Big(\sum_{n=1}^{\infty}\|Tx_n\|^2\Big)^{\frac{1}{2}}$$

The $x=\sum_{n=1}^{\infty}\langle x, x_n\rangle x_n$ is strongly convergent, and hence we may be written by

$Tx=\sum_{n=1}^{\infty}\langle x, x_n\rangle Tx_n$ (strongly convergent). If we define the operater T_m *as* $T_mx=\sum_{n=1}^{m}\langle x, x_n\rangle$

$Tx_n(x\in H)$, then $\sum_{n=1}^{\infty}\|(T-T_m)x_n\|^2=\sum_{n=m+1}^{\infty}\|Tx_n\|^2<\infty$

Thus $T-T_m\in S(H)$ and $\|T-T_m\|\leqq\Big(\sum_{n=m+1}^{\infty}\|Tx_n\|^2\Big)^{\frac{1}{2}}$ and thus

$u-\lim_{m\to\infty}T_m=T$(uniformly convergent). It follows from proposition 1 that $T\in C(H)$

5. Conclusion

In this paper we start with the fact that $F(H)\subset M(H)\subset C(H)$, and then we have lemma 6, 7, 8 and 9.

From these results we can characterize the fact that if the operator $T \in B(H)$ belongs to a two-sided ideal of $B(H)$, then the operator T^* also belong to this ideal; that is, every two-sided ideal in $B(H)$ is selfadjoint, in other words, two-sided $*$-ideal. We also can cite the fact that the ideal $C(H)$ is the only closed two-sided $*$-ideal of $B(H)$.

If we synthesize our results on the properties of $F(H)$, $N(H)$, $S(H)$ and $C(H)$, We have the following theorem as conclusion.

Theorem 10. *(a)* $F(H) \subset N(H) \subset S(H) \subset C(H) \subset B(H)$

(b) $F(H)$, $N(H)$, $S(H)$, and $C(H)$ are all the two sided $*$-ideal of $B(H)$

(c) $F(H)$ is the minimal two-sided $*$-ideal and $C(H)$ is the closed maximal two-sided $*$-ideal of $B(H)$.

Abstract : In this paper we discuss various subpaces of $B(H)$, the space of bounded linear operators in a Hilbert space, and then investigate the inclusion relations of these subspaces in terms of the ideal structure of $B(H)$.

References

1. N. I. Akhiezer and I. M. Glazman, *Theory of linear operators in Hilbert space*, Pitman Advanced pub. program, (1981).
2. J. Calkin Two-sided ideals and congruences in the ring of bounded operators in Hilbert space, *Ann. of Math*, (2) 42. (1941) 839-873.
3. J. B. Conway, *A course in functional analysis*, Springer-Verlag Berlin, (1985).
4. I. C. Gohberg and M. G Krein, *Introduction to the theory of linear nonself-adjoint operators*, A. M. S. Trans. Math. Mono,. Vol. 18(1964).
5. A Pietsch, *Operator ideals*", North-Holland. (1980).
6. J. Weidmann, *Linear operator in Hilbert spaces*, Springer-Verlag, Berlin (1980).
7. R. Schatten, *Norm ideals of completely continuous operators*, Springer-Verlag, Berlin (1960).