韓國數學教育學會誌(數學教育)
1988. 12. Vol. XXⅦ, No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1988. Vol. XXⅦ, No.1

Some Applications of Continuous State Branching Processes to Bartoszyński's Virus Model *

by Song-ho Han and Jae-pil Oh
Kangweon University, Chuncheon, Korea

Ⅰ. Introduction

In the Previons paper [1], Bartoszynski got some interesting, important results concerning with the rabies virus model. Buhler and Keller [3] also, in their short paper, claimed that there is a one to one correspondence between the Bartoszynski's rabies virus model process for the development of rabies in the human organism and the continuous state branching process(CB-process).

The continuous state branching processes are studied first by M. Jirina in 1958, and developed by many probabilities. In 1969, Puri [6] claimed that the discrete state branching processes can be applied in the models of self-reproducing entities such as bacteria or viruses. Also, Pakes and Trajstan [5] established the Bartoszynski process(B-process) as a continuous state branching process and got many limit theorems for B-process of the rabies model. Thus we can image that the properties of continuous state branching processes can also be applied in B-process and will use some results of them.

In this paper, we will give some results about the Bartoszynski's virus model containing the analysis of stochastic models of development of rabies within the human organism. The main objective of this model is to assess the risk of developing the disease in absense of vaccination. The model of this B-process $\{X(t) ; t \geqq 0\}$ is same as the model in [2]. Thus the hypothesis of introduction section of [2] are hold. Let K(t) be the number of jumps of this process as the damage process which denote the number of destroied cells untill time t in the central nurvous system. Also, let

$$u(t) = \int_0^t X(\tau)\, d\tau,\ \tau \geqq 0$$

and $w(t)$ denotes the disappeared volum or quantity of virus untill time t.

In section Ⅱ, we will meet the elementary properties of the processes $X(t)$ $u(t)$, $w(t)$ and $k(t)$ such as the moments and the cumulant generating functions. In section Ⅲ, we will get some limit properties of the cumulant generating functions of the above random variables and the induced random variables. We also get some results concerning with the probability that the disease will not develop at all and with the measure of the total damage sustained by the host.

* 본 연구는 강원대학교 이화학 연구소의 지원으로 이루어졌음

II. Preliminaries

Let $x>0$ be the initial number or quantity of viruses at $t=0$ and $T_1<T_2<\cdots$ be the moments when the 1st, 2nd, $\cdots$ cells of central nerval system break and spill its contents, ravies viruses. Also, assume that $Y_1, Y_2, \cdots$ represent the magnitudes of contents which is spilled out when the cells of spinal cord or brain are break. Then the sequence of random vectors

$(0, x), (T_1, Y_1), (T_2, Y_2), \cdots$

can generate the Bartoszynski virus model process(B-process)

$\{X(t) ; t\geqq 0\}$ as

$$X(t) = \begin{cases} xe^{-\alpha t} \text{ for } 0\leqq t\leqq T \\ [X(T_n)+Y_n]\, e^{-\alpha(t-T_n)} \text{ for } T_n< T< T_{n+1}. \end{cases}$$

We can also define the auxiliary process $Z(t)$ as

$$Z(t)= \begin{cases} 1 \text{ if no symptoms occur before t} \\ 0 \text{ otherwise.} \end{cases}$$

It will be assume that $Z(0)=1$; we now put $Z=\inf\{t|\ Z(t)=0\}$,

So that Z is the latency period.

Because the B-process $\{X(t) ; t\geqq 0\}$ can be regard as a kind of the continuous state branching process, we can use the branching property and get the function $\phi_1(t, s)$ which satisfies

$$E(e^{-sX(t)}|\ X(0)=x)=e^{-x\phi_1(t, s)}$$

Where

$$\frac{\partial}{\partial t}\phi_1(t, s)=\Phi(\phi_1(t, s)),\ \phi_1(0, s)=s,$$

if the regular condition

$$\lim_{t\to 0}\frac{\phi(t, s)-s}{t} \qquad =\Phi(s) \text{ exists for } s\geqq 0$$

is assumed. On the other hand, the integration yields

$$\int_s^{\phi(t, s)}\frac{du}{\Phi(u)}=t$$

Thus we defind the cumulant generating Function of the B-process $\{X(t) ; t\geqq 0\}$ as

$$\phi_1(t, s) =-\frac{1}{X}\log E(e^{-sX(t)}|\ X(0)-x),\ s\geqq 0.$$

From the elemnetary properties of the branching process, we can get the mean and the variance of the random variable $\{X(t)\ |X(0)=x\}$ as

$$E\{X(t)\ |X(0)=x\}=xe^{\alpha t},$$

$$v\{X(t)\ |X(0)=x\}= \begin{cases} -x\beta t, \text{ if } \alpha=0 \\ -x\dfrac{\phi_1''(1, 0+)}{e^{\alpha}(e^{\alpha}-1)}\ e^{\alpha t}(e^{\alpha t}-1), \text{ if } \alpha\neq 0, \end{cases}$$

Where $\alpha=\Phi'(0+)$, $\beta=\lim_{t\to 0}\frac{1}{t}\ E\{(X(t)-1)^2|\ X(0)=1\}$

When $X(0)=1$, we will define the total toxin process by $u(t)=\int_0^t X(\tau)d\tau,\ \tau\geqq 0.$

Here, $u(t)$ can be regard as a measure of the total amount of toxins during time interval $(0, t)$, and from the hypothesis of the B-process (cf. [2]), we see that the probability of no response during $(0, t]$ is given by $exp\{-\lambda \int_0^t X(\tau ; w)\, d\tau\}$.

If we define the cumblant generating function of $u(t)$ as $\phi_2(t, s)=-\log E(e^{-sU(t)})$, $s\geqq 0$. then we can meet many rsults concerning $u(t)$ in [4] and [6]. From the fact

$$E(u(t))= E\int_0^t X(\tau)\, d\tau$$
$$= \int_0^t E(X(\tau)d\tau$$

We also have the results :

$$E(w(t)))=\begin{cases} \dfrac{e^{\alpha t}-1}{\alpha} & , \text{ if } \alpha\neq 0 \\ t, \text{ if } \alpha=0. \end{cases}$$

From the Puri's paper [6]. we can get also the variance of the random variable $u(t)$,

$$v(u(t))=\begin{cases} \dfrac{A}{\alpha^2}[(e^{\alpha t}-1)-2\alpha t e^{\alpha t}], \text{ if } \alpha\neq 0, \\ h''(1)\dfrac{t^3(1-\alpha)}{3} & , \text{ if } \alpha=0, \end{cases}$$

Where $h(s)$ is the generating function of the discrete branching process and

$$A=\frac{h''(1)-h'(1)+1}{h'(1)-1}$$

From the above, if $X(0)=1$, we can also define the cured process $\{w(t) ; t\geqq 0\}$ which denote the disappeared volum or quantity of the rabies virus during $(0, t)$ and the cumulant generating function of $w(t)$ as $\phi_3(t, s)=-\log E(e^{-sW(t)})$, $s\geqq 0$.

[6]), Then we get the expectation and the variance (cf. Puri),

$$E(w(t))=\begin{cases} \dfrac{e^{\alpha t}-1}{e^{\alpha}-1} & , \text{ if } \alpha\neq 0, \\ t, & \text{if } \alpha=0 \end{cases}$$

$$v(w(t))=\begin{cases} \dfrac{e^{\alpha t}-2\alpha t e^{\alpha t}-1}{1-h'(1)}+\dfrac{1+2\alpha t e^{\alpha t}-e^{2\alpha t}}{[1-h''(1)]^2}+\dfrac{h''(1)}{[1-h'(1)]^3}(2\alpha t e^{\alpha t}+1-e^{2\alpha t}), \text{ if } \alpha\neq 0, \\ (1-\alpha)t+h''(1)\dfrac{(1-\alpha)^3t^3}{3} & , \text{ if } \alpha=0. \end{cases}$$

Let $K(t)$ be the random variable of the number of jumps in $(0, t)$, i.e., $K(t)=\#\{n\geqq 1 ; T_n\leqq t\}$.

Because $K(t)$ denotes the number of distroied cells, we will call this process $\{K(t) ; t\geqq 0\}$ as the damage proces,s and also we can define the cumulant generating function of $K(t)$ as

$$k(t, s)=-\frac{1}{x}\log E(s^{K(t)}| X(0)=x),\ 0\leqq s\leqq 1.$$

From the Puri's paper [6] and the Pakes and Trajstman's paper [5], we get the expactation and the variance of $K(t)$. Indeed, if we define the function

$$B(t, s_1, s_2)=-\frac{1}{x}\log E(e^{-s\,X(t)s_2^{K(t)}}| X(0)=x),\ s_1\geqq 0,\ 0\leqq s\leqq 1, \quad \cdots\cdots(1)$$

and letG $(t, s_1, s_2)=1-\rho^{-1}B(t, s_1, s_2)$,

Where $\rho=\lim h(t, 0)$, then we get the relation $B(t, s_1, s_2)=\rho[1-G(t, s_1, s_2)]$.
From the branching property and comparing with the Equation (8) of Puri [6], we obtain the representation.

$$G(t, s_1, s_2)=E[1-\frac{s}{\rho})^{M(t)}s_2^{W(t)}],$$

Where $M(t)$ is a Markov branching process with offspring probability function $f(s)$ and split rate c, $M(0)=1$ (see [5]). Thus we get

$$B(t, s_1, s_2)=\rho[1-E(1-\frac{S_1}{\rho})^{M(t)}s_2^{W(t)} \quad \cdots\cdots (2)$$

If we put $s_1=\rho(1-z)$ *i. e.* $1-s_1/\rho=z$ in the equation (1), then we get by (2),

$$E(e^{-\rho^{(1-z)X(t)}s_2^{K(t)}}|X(0)=x)=e^{-xB(t,\rho(1-z),s_2)}=exp[-x\rho(1-E\{z^{M(t)}s_2^{W(t)}\})]. \quad \cdots\cdots (3)$$

Put $z=1$, *i.e.*, $s_1=0$, $\rho(1-z)=0$ in (3), then

$$E(s_2^{K(t)}|X(0)=x)=exp[-x\rho(1-E\{s_2^{W(t)}\})].$$

From this equation, we get

$$\frac{d}{ds}E(s^{K(t)}|X(0)=x)=E(k(t)s^{K(t)-1}|X(0)=x)|_{s=x}$$
$$=E(k(t)|x(0)=x)$$
$$=exp(-x\rho)exp(x\rho E(s^{W(t)}))x\rho E(w(t)s^{W(t)-1})|_{s=1}$$
$$=x\rho E(w(t)).$$

But we have got the expectation of $w(t)$ already, thus we get

$$E(k(t)|X(0)=x)=\begin{cases} x\rho\frac{e^{\alpha t}-1}{e^{\alpha}-1}, & \text{if } \alpha\neq 0, \\ x\rho t, & \text{if } \alpha=0. \end{cases}$$

We also guess the variance of $k(t)$ from the Puri's paper [6].

$$v(k(t))=x\rho E(w(t))$$
$$=\begin{cases} \frac{1-\alpha}{\alpha}x[r(\frac{1-\alpha}{\alpha})^2(e^{2\alpha t}-1-2\alpha te^{\alpha t})+(\frac{2(1-)}{\alpha}\mu)(\alpha te^{\alpha t}-e^{\alpha t}+1)+e^{\alpha t}-1], & \text{if } \alpha\neq 0, \\ (1-\alpha)xt[\frac{(1-\alpha)^2rt^2}{B}+(1-\alpha)\mu t+1], & \text{if } \alpha=0, \end{cases}$$

where $r=E(Y_1^2)$.

Ⅲ. Some Limit Theorems

Let $\Psi(t, B_1, s_2, s_3)$ denote the Laplace transformation of the process $\{X(t), u(t), w(t)\ ; t \geqq 0\}$ defined by

$\Psi(t, s_1, s_2, s_3)=-\log E(e^{-s_1X(t)}e^{s_2U(t)}e^{-s_3W(t)} \mid X(0)=1)$, for $0\leqq s_1, s_2, s_3\leqq 1$. Then from the branching property of the random variable $X(t)$, we have the semigroup

$$\Psi(t+\tau, s_1, s_2, s_3)=\Psi(t, \Psi(\tau, s_1, s_2, s_3), s_2, s_3).$$

Letting $\tau\to 0$, we have for all t,

$$\pi(s_2, s_3)=\Psi(t, \pi(s_2, s_3), s_2, s_3),$$

thus we get the nonnegative root for x of the equation

$\Psi(t, x, s_2, s_3)=x,\ x\geqq 0,$

Where t may have positive value and $s_2, s_3\geqq 0$.

Theorem 1. *For every fixed* (s_1, s_2, s_3), *such that* $s_1, s_2, s_3\geqq 0$, $\lim_{t\to\infty}\Psi(t, s_1, s_2, s_3)=q(s_2, s_3)$.

Proof. From the semigroup property $\Psi(t+\tau, s_1, s_2, s_3)=\Psi(t, \Psi(\tau, s_1, s_2, s_3), s_2, s_3)$, and the concavity of this function Ψ, we have that for every fixed (s_1, s_2, s_3) such that $s_1, s_2, s_3\geqq 0$ as $t\to\infty$, $\Psi(t, s_1, s_2, s_3)\uparrow q(s_2, s_3)$ and $\Psi(\tau, s_1, s_2, s_3), \downarrow (s_2, s_3)$ according as $s_1<q(s_2, s_3)$ or $s_1>q(s_2, s_3)$, respectively. Thus we get the result #

If we define as the followings

$$R(t)=\frac{X(t)}{E(X(t))},\ s(t)=\frac{u(t)}{E(u(t))},\ M(t)=\frac{W(t)}{E(W(t))},\ \text{and for } u_1, u_2, u_3\geqq 0,$$

$\varphi(t, u_1, u_2, u_3)=-\log E(e^{-u_1R(t)}e^{-u_2S(t)}e^{-u_3M(t)})$,

then we get the relations

$\Psi(t, \frac{u_1}{E(X(t))}, \frac{u_2}{E(u(t))}, \frac{u_3}{E(w(t))}), = \varphi(t, u_1, u_2, u_3)$,and the similar properties of φ as Ψ.

From the properties of the branching process, we can image that if $R=0$ *a.s.*, then

$\lim_{t\to\infty} s(t)=s$ *a.s.*, $\lim_{t\to\infty} M(t)=M$ *a.s.*,

and also get the following in the continuous state.

Lemma 1. *If* $\alpha\leqq 0$ *and* $P(x,t,xe^{\alpha t})<1$, *then*

$\lim_{t\to\infty} R(t)=r=0$ *a.s.*,

If $\alpha>0$ *and there exists a* t_0 *such that* $E[X(t_0)\log X(t_0)]<+\infty$, *then*

$E(R)=X>0$,

and there exists a to sreh that $E[x(to)\log x(to)]=+\infty$ *then*

$E(R)=(0)$.*i.e.* $R=0$ a.s..¢

Theorem 2. *If* $\alpha\leqq 0$ *or if* $\alpha>0$ *and there exists a* t_0 *such that* $E[X(t_0)\log X(t_0)]=+\infty$, *then*

$\lim_{t\to\infty}\varphi(t, u_1, u_2, u_3)=r(u_2, u_3)$, *where* $r(u_2, u_3)=-\log E(e^{-u_2S}e^{-u_3W})$.

Proof. From the Lemma 1, we see that, in our case, $r=0$ *a.s.*. Thus there exist *s.* and *w.* and we get $\varphi(t, u_1, u_2, u_3)\to r(u_2, u_3)$. #

Let $N(t, x)$ be the probabiliny that disease does not occur up to time t, given be bite with serverity x. Then we can denote as

$N(t,x)=p(z(t)=1|\ X(0)=x)=p(z>t|\ X(0)=x)$,

and $N(x)$ is read as the probability that the disease will not develop at all, given the initial value

$X(0)=x$, *i.e.*, $N(x)=\lim_{t\to\infty} N(t, x)=p(z=\infty\ |\ X(0)=x)$.

Let Ω be the probability space for the process $\{X(t);t\geqq 0\}$, then its sample functions may be written as $X_X(t,w)$, where $X(0)=x$. For a fixed w, the function $\lambda X_X(., w)$ is the intensity of transition from 1 to 0 of the process $\{z(t)\}$ (defined on another probability space.) Thus by standard reasoning, we have

$p_X(z(t)\text{-}1|\ w)=p_X(\text{no symptoms occur till } t|w)=e^{-\lambda\int_0^t X_X(s,w)ds}$.

and by integrating, we have

$$p_X(z(t)=1) = p_X(z>t) = N(t,x) = E(e^{-\lambda \int_0^t X_x(s)ds.})$$
$$= E(e^{-\lambda u(t)} | X(0)=x),$$

where λ is Hypothesis I of [2].

From this relation, we have $N(x) = E(e^{-\lambda u} | X(0)=x)$,
where $u = \lim u(t)$.

Lemma 2. *For fixed $t_0 \in (0, \infty)$, we have*

$$\varphi_2(\infty, s) = -\frac{u(t_0)}{X(t_0)} s - \frac{1}{X(t_0)} \log E(e^{-sU} | F_{t_0}),$$

where $F_{t_0} = \sigma\{X(u) | 0 \leqq u \leqq t_0\}$.

Proof. See [4].

Theorem 3. *For each $t_0 \in (0, \infty)$, if we let $N = N(1)$, then*

$$N = exp[\frac{u(t_0)}{X(t_0)} \lambda + \frac{1}{X(t_0)} \log N',],$$

where $N' = E[exp(\lambda u) | f_{t_0})$.

Proof. From the relation, $N(x) = E(e^{-\lambda u} | X(0)=x)$,
we get $\varphi_2(\infty, \lambda) = -\log N$.

$$\log N = \frac{u(t_0)}{X(t_0)} \lambda + \frac{1}{X(t_0)} \log E(e^{-\lambda u} | F_{t_0}). \#$$

Theorem 4. *If K is the total number of jumps of the B-process, then we get*

$$E(k) = \begin{cases} \frac{\rho}{1-e^{\alpha}}, & \text{if } \alpha < 0 \\ \infty, & \text{if } \alpha \geqq 0. \end{cases}$$

Thus, since K can be interpreted as a measure of the total damage sustained by the host, we can image that how much damage have the host when he is bitten by the rabid animals.

References

1. R. Bartoszynai, On te risk of rabies, *Math. Biosci.* 24, (1975a)
2. __, A model for risk of rabies, *ISI invited paper*, No. 9, ISI 1~9, IX, Warsaw (1975b)
3. W.J.Buhler, G.Keller, Some remarks on Bartoszyński's rabies model, *Math. Biosci.* 39, (1978)
4. J.P.Oh, *Total progenies in continuous state branching peocesses*, Doctor thesis (1984)
5. A.G.Pakes, A.C.Trajstman, Some properties of continuous state branching processes, with applications to Bartoszyński's virus model, *Adv. Appl. Prob.* 17, (1985)
6. P.S.Puri, *Some limit theorems on branching processes and certain related processes*, Saṇkhya 31A (1969)