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Study on the Extension of Tonelli's Theorem with respect to Product Measure, $\mu_1 \times \mu_2 \times \cdots \times \mu_n$

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1. Introduction

Let (X_1,S_1,μ_1), $(X_2,S_2,\mu_2),\cdots,(X_n, S_n,\mu_n)$ be σ-finite measure spaces. Assume that $f:X_1 \times X_2 \times \cdots X_n \to [0,\infty]$ is nonnegative $S_1 \times S_2 \times \cdots \times S_n$- measurable function. In this case, I show that

$$fd(\mu_1 + \mu_2 \times \cdots \times \mu_n) = \int_{X_1}(\int_{X_2}(\cdots(\int_{X_{n-1}}(\int_{X_n}(fx_1,x_2,\cdots,x_{n-1})d\mu_n(x_n)x_{n-2}\int_{X_1 \times X_2 \times \cdots \times X_n} d\mu_{n-1}(x_{n-1}))x_{n-3}d\mu_{n-2}(x_{n-2}))\cdots)x_2\text{-}d\mu_3 (x_3))x_1 d\mu_2(x_2))d\mu_n(x_1) = \int_{X_1}d\mu_{m}(x_1)\int_{X_2}d\mu_2(x_2)\int_{X_3}d\mu_3(x_3)\cdots\int_{X_{n-1}}d\mu_{n-1}(x_{n-1})\int_{X_n}f(x_1,x_2,\cdots,x_n)=\int_{X_n}d\mu_n(x_n)=\int_{X_n}d\mu_n(y_n)\int_{X_{n-1}}d\mu_{n-1}(y_n)\int_{X_{n-1}}d\ \mu_{n-1}(y_{n-1})\cdots\int_{X_2}d\mu_2(y_2)\int_{X_1}f(y_1,y_2,\cdots,y_n)d\mu_1(y_1)$$

by Toneli's Theorem and Mathematical Induction, where $x_1 \in X_1, y_1 \in X_1$ and $(x_1,x_2,\cdots,x_{n-1}) \in X_1 \times X_2 \times \cdots \times X_{n-1}, (y_1,y_2,\cdots,y_{n-1}) \in X_1 \times X_2 \times \cdots \times X_{n-1}$.

Theorem 1. *Let (X,S,μ) and (Y,β,υ) be-finite measure Spaces. Then there is a Unique measure $\mu \times \upsilon$ on the σ-algebra $S \times \beta$ such that $(\mu \times \upsilon)(A \times B) = \mu(A)\upsilon(B)$ holds for each $A \in S$ and $B \in \beta$ and'$(\mu \times \upsilon)(E) = \int_X \upsilon(E_x)d\mu(x) = \int_Y \mu (E^y)d\upsilon(y)$ for arbitrary set $E \in S \times \beta$.*

Theorem 2(Tonelli's Theorem) *Set (X, S, μ) and (Y,β,υ) be σ-finite measure spaces, and let $f: X \times Y \to [0,\infty]$ be nonnegative $S \times \beta$-measurable function.*
Then the function $h: X \to [0,\infty]$ is S-measurable and the fuction

$$\begin{matrix} \psi \\ x \mapsto \int_Y f_x d\upsilon \end{matrix}$$

$\kappa: Y \to [0,\infty]$ is β-measurable and f satisfies that $\overset{\psi}{y} \mapsto \int_X f^y d\mu$

$$\int_{X \times Y} fd(\mu \times \upsilon) = \int_Y(\int_X d\mu)d\upsilon(y) = \int_X(\int_Y f_x d\upsilon)d\mu(x).$$

Theorem 3. *Let $(X_1, S_1, \mu_1),(X_2, S_2 , \mu_2),\cdots,(X_n, S_n, \mu_n)$ be σ-finite measure spaces. Then $S_1 \times S_2 \times \cdots \times S_n$ is an σ-algebra on $X_1 \times X_2 \times \cdots \times X_n$.*

Theorem 3. is proven by Mathematical Induction and the Definition of the Product of the σ-algebra Si and Sj($i \neq \alpha$: $i\alpha = 1, 2,\cdots,n$)

Theorem 4. *Let$(X_1,S_1,n_1),(X_2,S_2,\mu_2),\cdots,(X_n,S_n,\mu_n)$be σ-finite measure space.*
Then there is a measure.

Theorem 4 is proved by Mathematical Induction and Theorem 1.

Theorem 5. *Let* $(X_1, S_1, \mu_1), (X_2, S_2, \mu_2), \cdots, (X_n, S_n, \mu_n)$ *be* σ*-finite measure spaces. Let* $f: X_1 \times X_2 \times \cdots \times X_n \to [0,\infty]$ *be nonnegatible* $s_1 \times s_2 \times \cdots \times s_n$*-measurable function. Let* $h: X_1 \times X_2 \times \cdots \times X_{n-1} \to [0,\infty]$ *be a function and*

$$(x_1, x_2, \overset{\Psi}{\cdots}, x_{n-1}) \mapsto \int X_n(f x_1, x_2, \cdots, x_{n-1}) d\mu_n(x_n)$$

let $h: X_n \to [0,\infty] \to f^{y_n} d(\mu_1 \times \mu_2 \times \cdots \times \mu_{3-1\ n-1}) d\mu_n(x_n)$

$$\overset{\Psi}{y_n} \mapsto \int X_1 \times X_2 \times \cdots \times X_{n-1}$$

for $(x_1, x_2 \cdots, x_{n-1}) \in X_1 \times X_2 \times \cdots \times X_{n-1}$, *and* $x_n \in X_n$, $y_1, y_2 \cdots, y_{n-1}) \in X_1 \times X_2 \times \cdots \times X_{n-1}$ *and* $y_n \in X_n$.

Then $\int_{X_1 \times X_2 \times \cdots \times X_n} f d(\mu_1 \times \mu_2 \times \cdots \mu_n)$

$$= \int X_1 (\int X_2 (\cdots (\int X_n (f x_1, x_2, \cdots, x_{n-1}) d\mu_n(x_n)) x_{n-2} d\mu_{n-1}(x_{n-2})) x_{n-3} d\mu_{n-2}(x_{n-2}) \cdots) x_2 \; d\mu_3(x_3)) x_1 d\mu_2(x_2)) d\mu_1(x_1)$$

$$= \int X_1 d\mu_1(x_1) \int X_2 d\mu_2(x_2) \int X_3 d\mu_3(x_3) \cdots \int X_{n-1} d\mu_{n-1}(x_{n-1} \int X_n f(x_1, x_2, \cdots, x_n)\; d\mu_n(x_n)$$

$$= \int X_n d\mu_n(y_n) \int X_{n-1} d\mu_{n-1}(y_{n-1}) \cdots \int X_2 d\mu_2(y_2) \int X_1 f(y_1, y_2, \cdots, y_n) d\mu_1\ (y_1)$$

Proof . Let's prove theorem 5 by Tonelli's Theorem and Mathematical Induction.

i) Let (X_1, S_1, μ_1) and (X_2, S_2, μ_2) be σ-finite measure spaces.

$h(x_1) = \int X_2 f x_1 d\mu_2(x_2)$, $k(y_2) = \int X_1 f^{y_2} d\mu_1(y_1)$ for $x_1 \in X_1, x_2 \in X_2$ and $y_1 \in X_1, y_2 \in X_2$ since $h \geqq 0$, $k \geqq 0$, By Tonelli's theorem h is S_1-measurable function and k is S_2-measurable function and

$$\int X_1 h d\mu_n = \int X_1 \times X_2 f d(\mu_1 \times \mu_2) = \int X_2 k d\mu_2$$

Hence $\int X_1 \times X_2 f d(\mu_1 \times \mu_2) = \int X_1 (\int X_2 f x_1 d\mu_2(x_2) d\mu_1(x)$

$$= \int X_2 (\int X_1 f^{y_2} d\mu_1(y_1)) d\mu_2(y_2)$$

$$= \int X_1 d\mu_1(x_1) \int X_2 (x_1, x_2) d\mu_2(x_2)$$

$$= \int X_2 d\mu_2(y_2) \int X_1 f(y_1, y_2) d\mu_1(y_1),$$

where $f x_1(x_2) = f(x_1, x_2)$, $f^{y_2}(y_1) = f(y_1, y_2)$.

ii) Suppose that

$$\int X_1 \times X_2 \times \cdots \times X_{m-1} f d(\mu_1 \times \mu_2 \times \cdots \times \mu_{m-1})$$

$$= \int X_1 (\int X_2 (\int X_3 (\cdots \int X_{m-2} (\int X_{m-1} (f x_1, x_2, \cdots x_{m-2}) d\mu_{m-1}(x_{m-1})) x_{m-3} d\mu_{m-2}(x_{m-2})) x_{m-4}\ \ d\mu_{m-3})) \cdots) x_2 \; d\mu_3(x_3)) x_1 d\mu_2\ (x_2) x_1 d\mu_2(x_2) d\mu_1(x_1)$$

$$= \int X_1 d\mu_1(x_1) \int X_2 d\mu_2(x_2) \int X_3 d\mu_3(x_3) \cdots \int X_{m-2} d\mu_{m-2}(x_{m-2}) \int X_{m-1} f(x_1, x_2, \cdots, x_{m-1}) d\mu_{m-1}(x_{m-1})$$

$$= \int X_{m-1} d\mu_{m-1}(y_{m-1}) \int X_{m-2} d\mu_{m-2}(y_{m-2}) \cdots \int y_2 d\mu_2(y_2\ ) \int X_1 f(x_1, x_2, \cdots\ , y_{m-1}) d\mu_1(y_1).$$

iii) Let $(X_1 \times X_2 \times \cdots \times X_{m-1}, S_1 \times S_2 \times \cdots \times S_{m-1}, \mu_1 \times \mu_2 \times \cdots \times \mu_{m-1})$ and (X_m, S_m, μ_m) be σ-finite measure spaces.

$$X_1 \times X_2 \times \cdots \times X_{m-1} \overset{\text{let}}{=} X, X_m \overset{\text{let}}{=} Y.$$

Let $h: X_1 \times X_2 \times \cdots \times X_{m-1} \to [0,\infty]$

$$(x_1, x_2, \overset{\Psi}{\cdots}, x_{m-1}) \mapsto \int_m (f x_1, x_2, \cdots, x_{m-1}) d\mu_m(x_m)$$

and let $k: X_m \to [0,\infty]$ be a funtion.

$$\overset{\Psi}{y_m} \mapsto \int_{X_1 \times X_2 \times \cdots \times X_{m-1}} f^{y_m} d(\mu_1 \times \mu_2 \times \cdots \times \mu_{m-1})\ (y_1, y_2, \cdots, y_{m-1})$$

By Tonelli's theorem and Mathematial Induction, $h(h\geqq 0)$ is $S_1\times S_2\times\cdots\times S_{m-1}$- measurable function on $X_1\times X_2\times\cdots\times X_{m-1}$ and $k(k\geqq 0)$ is S_m-measurable function on X_m. Hence By Tonelli's theorem and i), ii),

$$\int_{X_1\times X_2\cdots\times X_m} fd(\mu_1\times\mu_2\times\cdots\times\mu_m)$$
$$=\int X_1(\int X_2(\int X_3(\cdots(\int X_{m-1}(fX_m(fx_1,x_2,\cdots,x_{m-1})d\mu_m\ (x_m))x_{m-2}d\mu_{m-1}(x_{m-1}))x_{m-3}d\mu_{m-2}(x_{m-2}))\cdots$$
$$x_2 d\mu_3\ (x_3))x_1 d\mu_2(x_2))d\mu_1(x_1)$$
$$=\int X_1 d\mu_1(x_1)\int X_2 d\mu_2(x_2)\int X_3 d\mu_3(x_3)\cdots\int X_{m-1}d\mu_{m-1}\ (x_{m-1})\int X_m f(x_1,x_2,\cdots,x_m)d\mu_1(x_m)$$
$$=\int X_m d\mu_m(y_m)\int X_{m-1}d\mu_{m-1}(y_{m-1})\cdots\int X_2 d\mu_2(y_2)\int X_1 f(y_1,y_2,\cdots,y_m)d\mu_1(y_1)$$

$\therefore$ By Mathematical Induction

$$\int_{X_1\times X_2\times\cdots\times X_{\mu_2})} fd(\mu_1\times\mu_2\times\cdots\ \mu_n)$$
$$=\int X_1(\int X_2(\int X_3(\cdots(\int X_{n-1}(\int X_n(\int x_1,x_2,\cdots,x_{n-1})d\mu_{n-1}\ (x_n))x_{n-2}d\mu_{n-1}(x_{n-1}))x_{n-3}d\mu_{n-2}(x_{n-2}))\cdots x_2 d\mu_3$$
$$(x_3))x_1 d\mu_2(x_2))d\mu_1(x_1)$$
$$=\int X_1 d\mu_1(x_1)\int X_2 d\mu_2(x_2)\int X_3 d\mu_3(x_3)\cdots\int X_{n-1}d\mu_{n-1}(x_{n-1})\int X_n f(x_1,x_2,\cdots,x_n)d\mu_n(x_n)$$
$$=\int X_n d\mu_n(y_n)\int X_{n-1}d\mu_{n-1}(y_{n-1})\cdots\int X_2 d\mu_2(y_2)\cdots\int X_1(y_1,y_2,\cdots,y_n)d\mu_1(y_1),$$

where $(fx_1,x_2,\cdots,x_{n-1})(x_n)=f(x_1,x_2,\cdots,x_2)$.

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