論文 88-25-6-3

An Unified Method of Finding the Inverse of a Matrix with Entries of a Linear Combination of Piecewise Constant Functions

(각 항들이 구간 일정 함수의 선형 결합으로 표현된 행열의 역을 구하는 방법)

李 海 榮* 卞 增 男*

(Hae Young Lee and Zeung Nam Bien)

要 約

본 논문은 각 항들이 구간 일정 함수의 선형 결합으로 표현된 행열의 역을 구하는 방법을 제안한다. 그러한 종류의 행열에 대한 역은 선형 대수 연립방정식의 해를 구함으로서 얻어질 수 있음을 보인다.

Abstract

This paper presents an unified method of obtaining the inverse of a matrix whose elements are a linear combination of piecewise constant functions. We show that the inverse of such a matrix can be obtained by solving a set of linear algebraic equations.

I. Introduction

Chen and Shih[1] analyzed the optimal control problem with quadratic performance index for a class of linear time-varying systems and obtained equally-distributed piecewise-constant gains for the optimal controller via Walsh functions[2]-[7]. According to the result in [1], the time-varying gain K(t) for the controller is given by the following equation:

$$K(t) = -R^{-1}(t)\, B^T(t)\, \lambda_{22}^{-1}(T_f, t)\, \lambda_{21}^{-1}(T_f, t), \quad t \in [0, 1] \qquad (1)$$

Here $\lambda_{22}(T_f, t)$ and $\lambda_{21}(T_f, t)$ are n x n matrices whose elements are a linear combination of Walsh functions. One can easily observe that it is obvious that the inverse of the matrix $\lambda_{22}(T_f, t)$ should be available desirably in an anaytic form in order to use the feedback gains K(t). Also, it is pointed out that as the dimension n of the system and the number m of Walsh functions employed increase, hand calculation of the inverse of $\lambda_{22}(T_f, t)$ is very tedious, and becomes almost impossible for the case when n and m are large. However, in [1], no indication was given con-

*正會員, 韓國科學技術院 電氣 및 電子工學科
(Dept. of Electrical Eng., KAIST)
接受日字：1988年 2月 11日

cerning how to calculate the inverse of λ_{22} (T_f, t), even though they gave an algorithm of finding λ_{22} (T_f,t) or λ_{21} (T_f, t) via Walsh series approximation. Several other piecewise constant functions such as block pulse functions [8]-[19] or delayed unit step functions[16], [20] can be applied to the optimal control problem[1] in order to overcome the difficulties of Walsh functions such as complex operational matrices or the restriction in choosing the number m of subintervals. However, the problem of the inverse of λ_{22} (T_f, t) still exist.

In this paper, we propose an unified method of obtaining an analytic solution for the inverse of λ_{22} (T_f, t) whose elements are a linear combinatin of such piecewise constant functions. By using the special property of each functions used, we will show that the inverse of a matrix each of whose elements is a linear combination of piecewise constant functions can be obtained by solving a set of simultaneous linear algebraic equations. Therefore the proposed algorithm can be easily implemented by computer programming as in the algorithm of finding the state transition matrix [21]-[22]. Thus the result will be very useful not only in solving the optimal control problem of time-varying systems via piecewise constant approximation but also in other areas using piecewise constant approximations.

II. Problem Formulation

Let $\phi_i(t)$, for i=0,1, ... , denote the piecewise constant functions treated such as Walsh functions, block pulse functions, or delayed unit step functions. And also let $\phi_{(m)}(t)$ be the m-vector function defined by

$$\phi^T_{(m)}(t) = [\phi_0(t),\ \phi_1(t), \ldots, \phi_{m-1}(t)],\quad t \in [0, 1] \qquad (2)$$

Here the superscript T represents the transpose.

Let us define the matrix A(t) whose elements are a linear combination of piecewise constant functions as follows:

$$A(t) = \begin{bmatrix} a^T_{11}\phi_{(m)}(t) & a^T_{12}\phi_{(m)}(t) & \cdots & a^T_{1n}\phi_{(m)}(t) \\ a^T_{21}\phi_{(m)}(t) & a^T_{22}\phi_{(m)}(t) & \cdots & a^T_{2n}\phi_{(m)}(t) \\ \vdots & \vdots & & \vdots \\ a^T_{n1}\phi_{(m)}(t) & a^T_{n2}\phi_{(m)}(t) & \cdots & a^T_{nn}\phi_{(m)}(t) \end{bmatrix}, t \in [0, 1] \qquad (3)$$

where for each i,j = 1,2, ..., n, a^T_{ij} is an 1 x m vector of the form

$$a^T_{ij} = a\,[a_{ij0}\ \ a_{ij0}\ \ \ldots a_{ij,m-1}] \qquad (4)$$

Our problem is to find an analytic form solution for the inverse of the matrix A(t).

III. Mathematical Preliminaries

Let there be given the following two arbitrary functions each of which is a linear combination of $\phi_i(t)$, for i = 0,1, ..., m-1;

$$p(t) = p_0\phi_0(t) + p_1\phi_1(t) + \cdots + p_{m-1}\phi_{m-1}(t) \triangleq p^T\phi_{(m)}(t) \qquad (5)$$

and

$$q(t) = q_0\phi_0(t) + q_1\phi_1(t) + \cdots + q_{m-1}\phi_{m-1}(t) \triangleq q^T\phi_{(m)}(t) \qquad (6)$$

where

$$p^T = [p_0\ p_1 \ldots p_{m-1}] \qquad (7)$$

and

$$p^T = [q_0\ q_1 \ldots q_{m-1}] \qquad (8)$$

Let $\Phi_{(mxm)}(t)$ be the product matrix [1] defined by

$$\Phi_{(mxm)}(t) = \phi_{(m)}(t)\,\phi^T_{(m)}(t) \qquad (9)$$

And also let $Q_{(mxm)}$ be the coefficient matrix [1] corresponding to the vector q, q = $[q_0, q_1, \ldots, q_{m-1}]^T$ defined by

$$\Phi_{(mxm)}(t)\,q = Q_{(mxm)}\,\phi_{(m)}(t) \qquad (10)$$

Note that the product matrix and the coefficient matrix as mentioned above play an important role in system analysis and design via algebraic approaches using series approximation. For each piecewise constant function, the product matrix and the coefficient matrix are given as follows:

1) Walsh functions [1]

$$\Phi^{w}_{(m\times m)}(t)=\begin{cases}\phi^{w}_{0}(t) & ,\text{ for } m=1\\ \begin{bmatrix}\Phi^{w}_{(\frac{m}{2}\times\frac{m}{2})}(t) & \Phi^{w}_{+\frac{m}{2}(\frac{m}{2}\times\frac{m}{2})}(t)\\ \Phi^{w}_{+\frac{m}{2}(\frac{m}{2}\times\frac{m}{2})}(t) & \Phi^{w}_{(\frac{m}{2}\times\frac{m}{2})}(t)\end{bmatrix} & ,\text{ for } m\geq 2\end{cases} \tag{11}$$

and

$$Q^{w}_{(m\times m)}=\begin{cases}q^{w}_{0} & ,\text{ for } m=1\\ \begin{bmatrix}Q^{w}_{(\frac{m}{2}\times\frac{m}{2})} & Q^{w}_{+\frac{m}{2}(\frac{m}{2}\times\frac{m}{2})}\\ Q^{w}_{+\frac{m}{2}(\frac{m}{2}\times\frac{m}{2})} & Q^{w}_{(\frac{m}{2}\times\frac{m}{2})}\end{bmatrix} & ,\text{ for } m\geq 2\end{cases} \tag{12}$$

where the superscript w means the Walsh functions, m is an integer with power of 2, and the $\Phi^{w}_{+\frac{m}{2}(\frac{m}{2}\times\frac{m}{2})}$ is the matrix with each element whose subscript is increased by $\frac{m}{2}$ in the $\Phi^{w}_{(\frac{m}{2}\times\frac{m}{2})}$

2) Block pulse functions [14]

A set of m block pulse functions $\phi^{b}_{i}(t)$, i=0,1, ... , m-1, are defined in the time interval $[0, t_f)$ as

$$\phi^{b}_{i}(t)=\begin{cases}1, & \frac{it_f}{m}\leq t<\frac{(i+1)t_f}{m}\\ 0, & \text{otherwise}\end{cases} \tag{13}$$

where m is the number of terms to be used.

The product matrix and the coefficient matrix for the block pulse functions are defined as

$$\Phi^{b}_{(m\times m)}(t)=\begin{bmatrix}\phi^{b}_{0}(t) & 0 & \cdots & 0\\ 0 & \phi^{b}_{1}(t) & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & & \phi^{b}_{m-1}(t)\end{bmatrix} \tag{14}$$

and

$$Q^{b}_{(m\times m)}(t)=\begin{bmatrix}q_0 & 0 & \cdots & 0\\ 0 & q_1 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & & q_{m-1}\end{bmatrix} \tag{15}$$

where the superscript b represents the block pulse functions and m is an integer.

3) Delayed unit step functions

A set of m delayed unit step functions $\phi^{d}_{i}(t)$, i= 0,1, ..., m-1, are defined in the time interval $[0, t_f]$ as

$$\phi^{d}_{i}(t)=\begin{cases}1, & \frac{it_f}{m}\leq t\leq t_f\\ 0, & t<\frac{it_f}{m}\end{cases} \tag{16}$$

Here again m is the number of terms to be used.

Lemma 1

For the m delayed unit step functions, the product matrix, $\phi^{d}_{(m\times m)}(t)$, and the coefficient matrix, $Q^{d}_{(m\times m)}$, corresponding to a coefficient vector $q^T=[q_0, q_1, \ldots, q_{m-1}]$ in (8), are given as follows:

$$\Phi^{d}_{(m\times m)}(t)=\begin{bmatrix}\phi^{d}_{0}(t) & \phi^{d}_{1}(t) & \phi^{d}_{2}(t) & \cdots\phi^{d}_{m-2}(t) & \phi^{d}_{m-1}(t)\\ \phi^{d}_{1}(t) & \phi^{d}_{1}(t) & \phi^{d}_{2}(t) & \cdots\phi^{d}_{m-2}(t) & \phi^{d}_{m-1}(t)\\ \phi^{d}_{2}(t) & \phi^{d}_{2}(t) & \phi^{d}_{2}(t) & \cdots\phi^{d}_{m-2}(t) & \phi^{d}_{m-1}(t)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \phi^{d}_{m-1}(t) & \phi^{d}_{m-1}(t) & \phi^{d}_{m-1}(t)\cdots & \phi^{d}_{m-1}(t) & \phi^{d}_{m-1}(t)\end{bmatrix} \tag{17}$$

and

$$Q^{d}_{(m\times m)}=\begin{bmatrix}q_0 & q_1 & q_2 & \cdots & q_{m-2} & q_{m-1}\\ 0 & q_0+q_1 & q_2 & \cdots & q_{m-2} & q_{m-1}\\ 0 & 0 & q_0+q_1+q_2 & \cdots & q_{m-2} & q_{m-1}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & q_0+q_1+\cdots+q_{m-2} & q_{m-1}\\ 0 & 0 & 0 & \cdots & 0 & q_0+q_1+\cdots q_{m-1}\end{bmatrix} \tag{18}$$

where the superscript d describes the delayed unit step functions and m is an integer.

Proof

According to the definition of the product matrix

$$\Phi_{(m\times m)}(t)=\phi_{(m)}(t)\,\phi^{T}_{(m)}(t)$$

$$=\begin{bmatrix}\phi_0(t)\phi_0(t) & \phi_0(t)\phi_1(t) & \cdots & \phi_0(t)\phi_{m-1}(t)\\ \phi_1(t)\phi_0(t) & \phi_1(t)\phi_1(t) & \cdots & \phi_1(t)\phi_{m-1}(t)\\ \vdots & \vdots & & \vdots\\ \phi_{m-1}(t)\phi_0(t) & \phi_{m-1}(t)\phi_1(t) & & \phi_{m-1}(t)\phi_{m-1}(t)\end{bmatrix} \tag{19}$$

By using the basic property of the delayed unit step function [16], [20]

$$\phi_i^d(t)\phi_j^d(t) = \begin{cases} \phi_i^d(t) & i \geqslant j \\ \phi_j^d(t) & i < j \end{cases} \tag{20}$$

where i,j = 1,2, ... , we obtain (17).
Also,

$$\Phi^d_{(mxm)} q = \begin{bmatrix} \phi_0^d(t) & \phi_1^d(t) & \cdots \phi_{m-1}^d(t) \\ \phi_1^d(t) & \phi_1^d(t) & \cdots \phi_{m-1}^d(t) \\ \vdots & \vdots & \vdots \\ \phi_{m-1}^d(t) & \phi_{m-1}^d(t) & \cdots \phi_{m-1}^d(t) \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_{m-1} \end{bmatrix}$$

$$= \begin{bmatrix} q_0\,\phi_0^d(t) + q_1\,\phi_1^d(t) + \cdots + q_{m-1}\,\phi_{m-1}^d(t) \\ (q_0+q_1)\,\phi_1^d(t) + q_2\,\phi_2^d(t) + \cdots + q_{m-1}\,\phi_{m-1}^d(t) \\ \vdots \\ (q_0+q_1+\cdots+q_{m-1})\,\phi_{m-1}^d(t) \end{bmatrix}$$

$$\triangleq Q^d_{(mxm)}\,\phi_{(m)}(t) \tag{21}$$

Therefore we obtain (18). Q.E.D

According to the property in (10), the multiplication of two functions p(t) and q(t) can be expressed as a linear combination of $\phi_i(t)$, i=0,1,..., m-1. Now we can show that the division of a function p(t) by another q(t) can be similarly expressed as follows.

Lemma 2

Let there be given $p(t) \triangleq p^T\,\phi_{(m)}(t)$ and $q(t) \triangleq q^T\phi_{(m)}(t)$ as in (5) and (6). If the inverse of the coefficient matrix $Q_{(mxm)}$ corresponding to the coefficient vector q exists, the division of the function p(t) by q(t), $q(t) \neq 0$, can be expressed as a linear combination of $\phi_i(t)$, i=0,1, ..., m-1, as:

$$\frac{p(t)}{q(t)} = p^T\,Q^{-1}_{(mxm)}\,\phi_{(m)}(t),\ q(t) \neq 0 \tag{22}$$

Proof

For $q(t) \neq 0$, let

$$\frac{p(t)}{q(t)} = r^T\,\phi_{(m)}(t) \tag{23}$$

where r is a mx1 vector of the form $r = [r_0\ r_1\ \ldots\ r_{m-1}]^T$. Multiply q(t) to $r^T\,\phi(t)$ and use the property of (10), $\Phi_{(mxm)}(t)\,q = Q_{(mxm)}\phi_{(m)}(t)$, to obtain

$$\begin{aligned} p(t) = p^T\phi_{(m)}(t) &= r^T\phi_{(m)}(t)\,q^T\phi_{(m)}(t) \\ &= r^T\phi_{(m)}(t)\,\phi^T_{(m)}(t)\,q \\ &= r^T\Phi_{(mxm)}(t)\,q \\ &= r^T Q_{(mxm)}\,\phi_{(m)}(t) \end{aligned} \tag{24}$$

where $Q_{(mxm)}$ is the coefficient matrix corresponding to the vector q and $\Phi_{(mxm)}(t)$ is the product matrix.

Therefore

$$r^T\ Q_{(mxm)} = p^T \tag{25}$$

Finally, we obtain from the assumption that

$$r^T = p^T\ Q^{-1}_{(mxm)} \tag{26}$$

Q.E.D

Remark 1: Invertability of the coefficient matrix

1. Walsh functions

No method of checking the invertability of $Q_{(mxm)}$ is available at the moment.

2. Block pulse functions

Since the coefficient matrix for the block pulse functions is diagonal one, if q_i, i =0,1, ..., m-1, are not zero, then the inverse of $Q^b_{(mxm)}$ exist.

3. Delayed unit step functions

Since the coefficient matrix for the delayed unit step functions is given as in (18), the determinant of $Q^d_{(mxm)}$ is $q_0\,(q_0+q_1)\ldots(q_0+q_1+\ldots q_{m-1})$. Therefore if $q_0\,(q_0+q_1)\ldots(q_0+q_1+\ldots+q_{m-1})$ is not zero then the inverse of $Q^d_{(mxm)}$ exist.

IV. Main Result

In the following, we present a method of inverting A(t).

Theorem 1

For the matrix A(t) in (3), let the inverse, if it exists, be denoted as

$$A^{-1}(t) = \begin{bmatrix} v_{11} & v_{12} \cdots v_{1n} \\ v_{21} & v_{22} \cdots v_{2n} \\ \vdots & \vdots \quad \vdots \\ v_{n1} & v_{n2} \quad v_{nn} \end{bmatrix} \tag{27}$$

For each i,j =1,2, ... , n, let A^c_{ij} represent the coefficient matrix corresponding to the vector a_{ij} of A(t) in (3), then A^c_{ij} can be obtained by (12), (15), and (18). And define the matrix A^c to be the matrix whose entry in the i-th row and j-th column is the transpose of the matrix A^c_{ij}; that is,

$$A^c = \begin{bmatrix} (A^c_{11})^T & (A^c_{21})^T & \cdots & (A^c_{n1})^T \\ (A^c_{12})^T & (A^c_{22})^T & \cdots & (A^c_{n2})^T \\ \vdots & \vdots & & \vdots \\ (A^c_{1n})^T & (A^c_{2n})^T & \cdots & (A^c_{nn})^T \end{bmatrix} \tag{28}$$

Then, for each i,j= 1,2, ..., n,v_{ij} can be expressed as a linear combination of $\phi_i(t)$, for i = 0,1,..., m-1,; that is,

$$v_{ij} = w_{ij}^T \ \phi_{(m)}(t) \tag{29}$$

where the 1xm vector

$$w_{ij}^T = [\, w_{ij0},\ w_{ij1}, \dots, w_{ij,m-1} \,] \tag{30}$$

can be obtained by solving the following simultaneous algebraic equations:

$$A^c \begin{bmatrix} w_{11} \\ \\ w_{12} \\ \\ \vdots \\ w_{1n} \\ \\ \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \left.\right\} \text{m terms} \\ \\ \left.\right\} \text{m terms} \\ \\ \vdots \\ \\ \left.\right\} \text{m terms} \end{matrix}$$

$$A^c \begin{bmatrix} w_{21} \\ \\ w_{22} \\ \\ \vdots \\ w_{2n} \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \left.\right\} \text{m terms} \\ \\ \left.\right\} \text{m terms} \\ \\ \vdots \\ \\ \left.\right\} \text{m terms} \end{matrix}$$

$$\vdots \qquad \vdots$$

$$A^c \begin{bmatrix} w_{n1} \\ \\ w_{n2} \\ \\ \vdots \\ w_{nn} \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \left.\right\} \text{m terms} \\ \\ \left.\right\} \text{m terms} \\ \\ \vdots \\ \\ \left.\right\} \text{m terms} \end{matrix} \tag{31}$$

Proof

Lemma 2 implies that each entry of the matrix $A^{-1}(t)$, if it exists, can be expressed as a linear combination of $\phi_i(t)$, for i = 0,1, ..., m-1.

Since $A^{-1}(t)\,A(t) = I$, we have

$$\begin{bmatrix} w_{11}^T\phi_{(m)}(t) & w_{12}^T\phi_{(m)}(t) & \cdots & w_{1n}^T\phi_{(m)}(t) \\ w_{21}^T\phi_{(m)}(t) & w_{22}^T\phi_{(m)}(t) & \cdots & w_{2n}^T\phi_{(m)}(t) \\ \vdots & \vdots & & \vdots \\ w_{n1}^T\phi_{(m)}(t) & w_{n2}^T\phi_{(m)}(t) & \cdots & w_{nn}^T\phi_{(m)}(t) \end{bmatrix} \begin{bmatrix} a_{11}^T\phi_{(m)}(t) & a_{12}^T\phi_{(m)}(t) & \cdots & a_{1n}^T\phi_{(m)}(t) \\ a_{21}^T\phi_{(m)}(t) & a_{22}^T\phi_{(m)}(t) & \cdots & a_{2n}^T\phi_{(m)}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}^T\phi_{(m)}(t) & a_{n2}^T\phi_{(m)}(t) & \cdots & a_{nn}^T\phi_{(m)}(t) \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{n} w_{1i}^T\phi_{(m)}(t)\ a_{i1}^T\phi_{(m)}(t) & \sum_{i=1}^{n} w_{1i}^T\phi_{(m)}(t)\ a_{i2}^T\phi_{(m)}(t) & \cdots & \sum_{i=1}^{n} w_{1i}^T\phi_{(m)}(t)\ a_{in}^T\phi_{(m)}(t) \\ \vdots & \vdots & & \vdots \\ \sum_{i=1}^{n} w_{ni}^T\phi_{(m)}(t)\ a_{i1}^T\phi_{(m)}(t) & \sum_{i=1}^{n} w_{ni}^T\phi_{(m)}(t)\ a_{i2}^T\phi_{(m)}(t) & \cdots & \sum_{i=1}^{n} w_{ni}^T\phi_{(m)}(t)\ a_{in}^T\phi_{(m)}(t) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots\cdots & 0 \\ 0 & 1 & \cdots\cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots\cdots & 1 \end{bmatrix} \tag{32}$$

For the lst term in the lst row of (32), we obtain

$$\sum_{i=1}^{n} w_{1i}^T\phi_{(m)}(t)\ a_{i1}^T\phi_{(m)}(t)$$

$$= w_{11}^T\phi_{(m)}(t)\ a_{11}^T\phi_{(m)}(t) + w_{12}^T\phi_{(m)}(t)\ a_{21}^T\phi_{(m)}(t) + \cdots + w_{1n}^T\phi_{(m)}(t)\ a_{n1}^T\phi_{(m)}(t)$$

$$= w_{11}^T\phi_{(m)}(t)\ \phi_{(m)}^T(t)\ a_{11} + w_{12}^T\phi_{(m)}(t)\,\phi_{(m)}^T(t)\ a_{21} + \cdots + w_{1n}^T\phi_{(m)}(t)\ \phi_{(m)}^T(t)\ a_{n1}$$

$$= w_{11}^T \Phi_{(m\times m)}(t)\, a_{11} + w_{12}^T \Phi_{(m\times m)}(t)\, a_{21} + \cdots + w_{1n}^T \Phi_{(m\times m)}(t)\, a_{n1}$$
$$= w_{11}^T A_{11}^c \phi_{(m)}(t) + w_{12}^T A_{21}^c \phi_{(m)}(t) + \cdots + w_{1n}^T A_{n1}^c \phi_{(m)}(t)$$
$$= \begin{cases} [1\ 0 \cdots 0]\ \phi_{(m)}(t), & \text{for walsh or delayed unit step functions} \\ [1\ 1 \cdots 1]\ \phi_{(m)}(t), & \text{for block pulse functions} \end{cases} \tag{33}$$

In the above, the relations of $1=[1\ 0\ \ldots\ 0]\ \phi_{(m)}(t)$ for Walsh or delayed unit step functions and $1=[1\ 1\ \ldots\ 1] = \phi_{(m)}(t)$ for block pulse functions are used. The equation (33) implies

$$w_{11}^T A_{11}^c + w_{12}^T A_{21}^c + \cdots + w_{1n}^T A_{n1}^c = \begin{cases} [1\ 0 \cdots 0]\ \phi_{(m)}(t), & \text{for walsh or delayed unit step functions} \\ [1\ 1 \cdots 1]\ \phi_{(m)}(t), & \text{for block pulse functions} \end{cases} \tag{34}$$

Here, for each $i = 1,2, \ldots, n$, A_{i1}^c denotes the m x m coefficient matrix corresponding to the vector a_{i1}.
Taking the transpose for the above equation, we obtain

$$(A_{11}^c)^T w_{11} + (A_{21}^c)^T w_{12} + \cdots + (A_{n1}^c)^T w_{1n} = \begin{cases} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \text{for walsh or delayed unit step functions} \\ \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} & \text{for block pulse functions} \end{cases} \tag{35}$$

Similarly, for the n-1 terms remained in the lst row of (32), we have

$$(A_{12}^c)^T w_{11} + (A_{22}^c)^T w_{12} + \cdots + (A_{n2}^c)^T w_{1n} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$
$$\vdots$$
$$(A_{1n}^c)^T w_{11} + (A_{2n}^c)^T w_{12} + \cdots + (A_{nn}^c)^T w_{1n} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{36}$$

Let the m x 1 vectors L, M and N be defined as

$$\begin{aligned} L^T &= [\,1\ 1\ \ldots\ 1\,] \\ M^T &= [\,1\ 0\ \ldots\ 0\,] \\ N^T &= [\,0\ 0\ \ldots\ 0\,]. \end{aligned} \tag{37}$$

Then the expressions in (35) and (36) can be put in a more compact form as:

$$\begin{bmatrix} (A_{11}^c)^T & (A_{21}^c)^T \cdots & (A_{n1}^c)^T \\ (A_{12}^c)^T & (A_{22}^c)^T \cdots & (A_{n2}^c)^T \\ \vdots & \vdots & \vdots \\ (A_{1n}^c)^T & (A_{2n}^c)^T \cdots & (A_{nn}^c)^T \end{bmatrix} \begin{bmatrix} w_{11} \\ w_{12} \\ \vdots \\ w_{1n} \end{bmatrix} = \begin{bmatrix} M \\ N \\ \vdots \\ N \end{bmatrix} \tag{38}$$

It is noted that for block pulse functions, we should replace M by L. In a similar manner, for the 2nd row of (32) we obtain

$$\begin{bmatrix} (A_{11}^c)^T & (A_{21}^c)^T \cdots & (A_{n1}^c)^T \\ (A_{12}^c)^T & (A_{22}^c)^T \cdots & (A_{n2}^c)^T \\ \vdots & \vdots & \vdots \\ (A_{1n}^c)^T & (A_{2n}^c)^T \cdots & (A_{nn}^c)^T \end{bmatrix} \begin{bmatrix} w_{21} \\ w_{22} \\ \vdots \\ w_{2n} \end{bmatrix} = \begin{bmatrix} N \\ M \\ N \\ \vdots \\ N \end{bmatrix}, \tag{39}$$

and so forth.
As mentioned above, M should be replaced by L for block pulse functions. Then we can obtain the following a set of n simultaneous algebraic equations.

$$A^c \begin{bmatrix} w_{11} \\ w_{12} \\ \vdots \\ w_{1n} \end{bmatrix} = \begin{bmatrix} M \\ N \\ \vdots \\ N \end{bmatrix}$$
$$A^c \begin{bmatrix} w_{21} \\ w_{22} \\ \vdots \\ w_{2n} \end{bmatrix} = \begin{bmatrix} N \\ M \\ N \\ \vdots \\ N \end{bmatrix}$$
$$\vdots$$
$$A^c \begin{bmatrix} w_{n1} \\ w_{n2} \\ \vdots \\ w_{nn} \end{bmatrix} = \begin{bmatrix} N \\ N \\ \vdots \\ M \end{bmatrix} \tag{40}$$

It is remarked that the vector M should be replaced by L for the block pulse functions. By solving the above simultaneous algebraic equations, we can obtain an analytic solution of the inverse of the matrix A(t). Q.E.D

V. Examples

Let us consider the following matrix A(t) for n,m =2

$$A(t) = \begin{bmatrix} a_{11}^T \phi_{(m)}(t) & a_{12}^T \phi_{(m)}(t) \\ a_{21}^T \phi_{(m)}(t) & a_{22}^T \phi_{(m)}(t) \end{bmatrix}$$

where

$$a_{11}^T = [1 \quad 2] \quad a_{12}^T = [2 \quad 1]$$
$$a_{21}^T = [3 \quad 2] \quad a_{22}^T = [4 \quad 3],$$

and let the inverse of the matrix A(t) be denoted as

$$A^{-1}(t) = \begin{bmatrix} w_{11}^T \phi_{(m)}(t) & w_{12}^T \phi_{(m)}(t) \\ w_{21}^T \phi_{(m)}(t) & w_{22}^T \phi_{(m)}(t) \end{bmatrix}$$

where

$$w_{ij}^T = [w_{ij0}, \quad w_{ij1}], \quad \text{for } i, j = 1, 2.$$

1) Walsh functions case

Equation (40) yields

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} w_{110} \\ w_{111} \\ w_{120} \\ w_{121} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 1 & 2 & 3 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} w_{210} \\ w_{211} \\ w_{220} \\ w_{221} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Solving the above simultaneous algebraic equations, we obtain

$$w_{11}^T = \left[\frac{1}{3} \quad \frac{5}{6} \right] \qquad w_{12}^T = \left[0 \quad -\frac{1}{2} \right]$$
$$w_{21}^T = \left[-\frac{1}{6} - \frac{2}{3} \right] \qquad w_{22}^T = \left[\frac{1}{2} \quad 0 \right]$$

2) Block pulse functions case

Equation (40) yields

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 4 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} w_{110} \\ w_{111} \\ w_{120} \\ w_{121} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 4 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} w_{210} \\ w_{211} \\ w_{220} \\ w_{221} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Solving the above simultaneous algebraic equations, we obtain

$$w_{11}^T = \left[-2 \quad \frac{3}{4} \right] \qquad w_{12}^T = \left[1 \quad -\frac{1}{4} \right]$$
$$w_{21}^T = \left[\frac{3}{2} \quad -\frac{1}{2} \right] \qquad w_{22}^T = \left[-\frac{1}{2} \quad \frac{1}{2} \right]$$

3) Delayed unit step functions case

Equation (40) yields

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & 3 & 2 & 5 \\ 2 & 0 & 4 & 0 \\ 1 & 3 & 3 & 7 \end{bmatrix} \begin{bmatrix} w_{110} \\ w_{111} \\ w_{120} \\ w_{121} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & 3 & 2 & 5 \\ 2 & 0 & 4 & 0 \\ 1 & 3 & 3 & 7 \end{bmatrix} \begin{bmatrix} w_{210} \\ w_{211} \\ w_{220} \\ w_{221} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Solving the above simultaneous algebraic equations, we obtain

$$w_{11}^T = \left[-2 \quad \frac{19}{6} \right] \qquad w_{12}^T = \left[1 \quad -\frac{3}{2} \right]$$
$$w_{21}^T = \left[\frac{3}{2} \quad -\frac{7}{3} \right] \qquad w_{22}^T = \left[-\frac{1}{2} \quad 1 \right]$$

VI. Conclusion

In this paper, we have proposed an unified method of finding the inverse of a matrix whose elements are a linear combination of piecewise constant functions. We have showed that the inverse of such a matrix can be obtained by solving a set of simultaneous linear algebraic equations.

References

[1] W.L. Chen and Y.P. Shin, "Analysis and optimal control of timevarying linear systems via Walsh functions," *Int. J. Contr.*, vol. 27, no. 6, pp. 917-932, 1978.

[2] K.G. Beauchamp, Applications of Walsh and Related Functions, London: Academic Press, 1984.

[3] N. Ahmed and K.R. Rao, Orthogonal Transforms for Digital Signal Processing, Berlin: Springer-Verlag, 1975.

[4] H. Harmuth, Transmission of Information by Orthogonal Functions, New York: Springer-Verlag, 1969.

[5] C.F. Chen and C.H. Hsiao, "A state-space approach to Walsh series solution of linear systems," *Int. J. Systems Sci.*, vol. 6, no. 6, pp. 833-858, 1975.

[6] C.F. Chen and C.H. Hsiao, "Walsh series analysis in optimal control", *Int. J. Contr.*, vol. 21, no. 6, pp. 881-897, 1975.

[7] C.F. Chen and C.H. Hsiao, 'Design of piecewise constant gains for optimal control via Walsh functions," *IEEE Trans. Automat. Contr.*, vol. AC-20, no. 5, pp. 596-603, 1975.

[8] L.S. Shieh, C.K. Yeung and B.C. McInnis, "Solution of state-space equations via block-pulse functions," *Int. J. Contr.*, vol. 28, no. 3, pp. 383-392, 1978.

[9] N.S. Hsu and B. Cheng, "Analysis and optimal control of timevarying linear systems via block-pulse functions," *Int. J. Contr.*, vol. 33, no. 6, pp. 1107-1122, 1981.

[10] C.P. Kwong and C.F. Chen, "The convergence properties of block-pulse series," *Int. J. Systems Sci.*, vol. 12, no. 6, pp. 745-751, 1981.

[11] W.L. Chen and C.L. Lee, "On the convergence of the block-pulse series solution of a linear time-invariant system," *Int. J. Systems* Sci., vol. 13, no. 5, pp. 491-498, 1982.

[12] A.K. Nath and T.T. Lee, "On the multidimensional extension of block-pulse functions and their completeness," *Int. J. Systems Sci.*, vol. 14, no. 2, pp. 201-208, 1983.

[13] K.R. Palanisamy, "A note on block-pulse function operational matrix for integration," *Int. J. Systems Sci.*, vol. 14, no. 11, pp 1287-1290, 1983.

[14] W.L. Chen and C.S. Hsu, "Convergence of the block-pulse series solution of a linear time-varying system," *Int. J. Systems Sci.*, vol. 15, no. 4, pp. 351-360, 1983.

[15] W. Marszalek, "On the nature of block-pulse operational matrices," *Int. J. Systems Sci.*, vol. 15, no. 9, pp. 983-989, 1984.

[16] R.Y. Chang, C.K. Chou and M.L. Wang, "Solution of functional differential equations via generalized block pulse functions," *Int. J. Systems Sci.*, vol. 16, no. 11, pp. 1431-1440, 1985.

[17] W.L. Chen and C.Y. Chung, "Error analysis of block-pulse series solutions," *Int. J. Systems Sci.*, vol., 17, no. 12, pp. 1669-1676, 1986.

[18] C. Hwang and Y.P. Shih, "On the operational matrices of block pulse functions," *Int. J. Systems Sci.*, vol. 17, no. 10, pp. 1489-1498, 1986.

[19] W.L. Chen and C.Y. Chung, "New integral operational matrix in block-pulse series analysis," *Int. J. Systems Sci.*, vol. 18, no. 3, pp. 403-408, 1987.

[20] C. Hwang, "Solution of a functional differential equation via delayed unit step functions," *Int. J. Systems Sci.*, vol. 14, no. 9, pp. 1065-1073, 1983.

[21] A. Jenning, Matrix Computation for Engineers and Scientists, New York: John Wiley & Sons, 1977.

[22] S.D. Conte and C.D. Boor, Elementary Numerical Analysis, London: McGraw-Hill, 1980.