〈Research Paper〉

Journal of the Korean
Statistical Society
Vol. 17, No.2, 1988

A Laplacian Autoregressive Time Series Model+

Young Sook Son* and Sinsup Cho**

ABSTRACT

A time series model with Laplacian(double-exponential) marginal distribution, NLAR(2), was proposed by Dewald and Lewis(1985). The special cases of NLAR(2) process and their properties are considered. Extensions to the NLAR(p) is discussed. It is shown that the NLAR(1) satisfies the *strong-mixing* condition, hence the model-free prediction interval using the sample quantiles can be obtained.

1. Introduction

Most of the current time series analysis which are based on the simple linear models such as ARMA models by Box and Jenkins(1976), the filtering method and etc., are started from the Gaussian assumption for the distribution of innovations. Although under the normality assumption many statistically optimal techniques are developed, there are many other cases which are not satisfied by Gaussian time series modelling even after suitable transformations. For example, the study of 21 economic time series by Nelson and Granger (1979) has found that in more than two thirds of the cases the hypotheses of normality were heavily rejected after the application of the Box-Cox transformation.The non-Gaussianity may be due to the fact that the phenomena are inherently positive-valued and highly skewed, or are distributed with a large kurtosis or a longer(or shorter) tails than is exhibited by Gaussian variates.

The first cases are well illustrated in the river flow studies by Lawrance and Kottegoda

+ This work is supported by Korea Science and Engineering Foundation under Grant 873-0105-003-1.

* Korea Development Institute, Seoul 130-010, Korea.

** Department of Computer Science and Statistics, Seoul National University, Seoul 151-742, Korea.

(1977) and the wind velocity studies by Hugus(1982), Brown, Katz and Murphy(1984). The time series models which well incorporate such features have been constructed by many authors during a recent decade; for example, exponential time series models such as EARMA (Exponential Autoregressive Moving Average) and NEAR(New Exponential Autoregressive) with exponential marginal distribution by Jacobs and Lewis(1977); Lawrance and Lewis(1977, 1980, 1981, 1982, 1985) and gamma time series models such as GAR(Gamma Autoregressive) and BGARMA(Beta-Gamma Autoregressive Moving Average) with gamma marginal distribution by Gaver and Lewis(1980); Lewis, McKenzie and Hugus (1985). Among positive valued and highly skewed distributions the exponential distribution is most widely used and analytically most tractable, for it has many useful properties like the Gaussian distribution and also can be transformed very easily into other distributions which are either more skewed or less skewed than itself. It is shown in Lewis(1985) and Raftery(1982) that the *Weibull, Gamma, Uniform, Erlang,* and *Beta* process can be directly converted from exponential process.

The second cases occur in the random fluctuations study of response rate by McGill(1963), pseudo-voice signal study by Linde and Gray(1978) and Uddenfeldt and Zettenberg(1976), position errors study in navigation by Hsu(1979), and two dimensional discrete cosine transform coefficients study for images by Reininger and Gibson(1983). McGill(1963) showed that the intervals between responses which are expressed as deviates from excitation period are well described by Laplacian distribution. In the pseudo-voice signal study, Linde and Gray(1978) considered the autoregressive source using Laplacian variables, i.e.

$$X_t = cX_{t-1} + \eta_t\sqrt{1-c^2},$$

where $0.8 \leq c \leq 0.9$ and $\{\eta_t\}$ is a sequence of i.i.d. Laplacian variables. Hsu(1979) used a mixture of two Laplacian distributions to explain better features of navigational position errors, and showed empirically its better adequacy over other models. Reininger and Gibson (1983) showed that for a two-dimensional discrete cosine transform image coding system the statistics of transformation coefficients is best approximated by a Laplacian distribution. To describe a Laplacian process, Dewald and Lewis(1985) developed a family of New Laplacian Autoregressive(NLAR) processes which is very similar to the family of New Exponential Autoregressive(NEAR) processes studied by Lawrance and Lewis(1985). Dewald and Lewis studied the properties of NLAR process and showed that it satisfies the Yule-Walker type difference equations, thus, the second-order moment dependency aspects are indistinguishable from those for Gaussian AR process. The estimation methods and the forecasting procedures were not studied, however.

In this paper, we study the modelling and forecasting procedures in NLAR processes.

Estimation procedures are currently under study and will be reported elsewhere. In Section 2, we review the NLAR(2) process of Dewald and Lewis(1985) and its properties. Also the LAR class, the special case of NLAR class and extension to the NLAR(p) are studied. It is shown that the NLAR(1) process satisfies the *strong-mixing* condition in Section 3. Finally, in Section 4 the forecasting procedures in the NLAR process are considered and the simulation results are presented.

2. New Laplacian Autoregressive Process

NLAR(2) process

Following the terminology in Gaver and Lewis(1977, 1980), Jacobs and Lewis(1977), and Lawrance and Lewis(1977, 1980, 1981, 1982, 1985), Dewald and Lewis(1985) proposed a time series model called the second-order New Laplacian Autoregressive(NLAR(2)) which describe the stationary Laplacian process with i.i.d. Laplacian innovations as follows.

Theorem. (Dewald and Lewis) Let $\{X_t\}$ be a stationary process with standard Laplacian marginal distribution. Note that for all t

$$X_t = \phi_1 V_{1,t} X_{t-1} + \phi_2 V_{2,t} X_{t-2} + \varepsilon_t, \tag{2.1}$$

where $\{V_{1,t}, V_{2,t}\}$ is a sequence of i.i.d. bivariate random variables with distribution,

$$\{V_{1,t}, V_{2,t}\} = \begin{cases} (1,0) & \text{w.p.} \quad p_1 \\ (0,1) & \text{w.p.} \quad p_2 \\ (0,0) & \text{w.p.} \quad 1-p_1-p_2, \end{cases} \tag{2.2}$$

for $t = 0, \pm 1, \pm 2, \cdots$; $\{\varepsilon_t\}$ is an i.i.d. innovation sequence, and $\{\varepsilon_t\}$ and $\{V_{1,t}, V_{2,t}\}$ are independent. For all t, let (2.1) and (2.2) hold with

$$0 < |\phi_i| < 1,\ 0 < p_i < 1 \text{ for } i = 1,2 \text{ and } p_1 + p_2 < 1. \tag{2.3}$$

Then

$$\varepsilon_t = U_t E_t, \tag{2.4}$$

where $\{E_t\}$ is a sequence of i.i.d. standard Laplacian variates; $\{U_t\}$ is a sequence of i.i.d. random variables with distribution,

$$\{U_t\} = \begin{cases} 1 & \text{w.p.} \quad \pi_1 \\ b_2 & \text{w.p.} \quad \pi_2 \\ b_3 & \text{w.p.} \quad \pi_3, \end{cases} \tag{2.5}$$

and is independent of $\{E_t\}$, $\{V_{1,t}, V_{2,t}\}$ for all t. Here,

$$\begin{aligned}\pi_1 &= \{(1-\phi_1^2)(1-\phi_2^2)\} / \{(1-\phi_1^2)(1-\phi_2^2)+ p_1\phi_1^2(1-\phi_2^2)+ p_2\phi_2^2(1-\phi_1^2)\},\\ \pi_2 &= \{(p_1\phi_1^2+ p_2\phi_2^2) b_2^2-(p_1+ p_2)\phi_1^2\phi_2^2\} / \{(b_2^2-b_3^2)(1-b_2^2)\},\\ \pi_3 &= \{(p_1+ p_2)\phi_1^2\phi_2^2-(p_1\phi_1^2+ p_2\phi_2^2) b_3^2\} / \{(b_2^2-b_3^2)(1-b_3^2)\},\end{aligned} \tag{2.6}$$

where $0<b_3^2= (u-\sqrt{u^2-4v})/2<b_2^2= (u+\sqrt{u^2-4v})/2<1$ (2.7)

with $u=(1-p_1)\phi_1^2+ (1-p_2)\phi_2^2$

and $v=(1-p_1-p_2)\phi_1^2\phi_2^2$.

proof. The proof of Theorem directly follows the one in Lawrance and Lewis(1985) for the NEAR(2) process. Also, the proof for the stationarity of NLAR(2) is an immediate consequence of the work of Nicholls and Quinn(1982). ■

The stationary NLAR(2) process has four parameters, Laplacian marginal distribution for $\{X_t\}$, second-order autoregressive Markovian dependence, fully time irreversibility defined by Weiss(1975), and partially time reversibility in the sense of third-order moments that

$$E(X_t^2X_{t-l})= E(X_tX_{t-l}^2), \tag{2.8}$$

for all l. It can be easily shown that the autocorreleation function(a.c.f.) of NLAR(2) process, $\rho_k= \mathrm{Corr}(X_t,X_{t-k})$, satisfies the Yule-Walker type difference equation,

$$\rho_k= \phi_1p_1\rho_{k-1}+ \phi_2p_2\rho_{k-2}, \tag{2.9}$$

for k=1,2,$\cdots$, which is the same as the Yule-Walker equation for Gaussian AR(2) process and the admissible regions for parameters and for ρ_1 and ρ_2 are $|\phi_1p_1|+ |\phi_2p_2|<1$ and $|\rho_1|<(1+\rho_2)/2$ with $|\rho_1|, |\rho_2|<1$, respectively. For more details, see Dewald and Lewis (1985).

Now, we consider several special cases of the NLAR(2) which are similar to EAR(Exponential Autoregressive) considered as special cases of the NEAR(2) in Lawrance and Lewis(1985).

LAR(2) process

If we let $p_1= 1-\phi_2^2$, $p_2= \phi_2^2$ in NLAR(2) of (2.1), we obtain (2.10) which we call the second-order Laplacian Autoregressive(LAR(2)) process,

$$X_t = \left\{ \begin{array}{lll} \phi_1 X_{t-1} & \text{w.p.} & 1-\phi_2^2 \\ \phi_2 X_{t-2} & \text{w.p.} & \phi_2^2 \end{array} \right\} + \varepsilon_t, \tag{2.10}$$

where

$$\varepsilon_t = \left\{ \begin{array}{lll} E_t & \text{w.p.} & \pi_1' = (1-\phi_1^2)(1-\phi_2^2)/(1-\delta^2) \\ |\delta| E_t & \text{w.p.} & \pi_2' = (1-\phi_2^2)(\phi_1^2-\phi_2^2)^2/(1+\phi_1^2-\phi_2^2)(1-\delta^2) \\ 0 & \text{w.p.} & \pi_3' = \phi_1^2/(1+\phi_1^2-\phi_2^2), \end{array} \right. \tag{2.11}$$

with $\delta^2 = \phi_2^2(1+\phi_1^2-\phi_2^2)$. The a.c.f. for LAR(2) is

$$\rho_k = \phi_1(1-\phi_2^2)\rho_{k-1} + \phi_2^3\rho_{k-2}, \quad k>3, \tag{2.12}$$

which is easily obtained if we set $p_1 = 1-\phi_2^2$ and $p_2 = \phi_2^2$ in (2.9).

NLAR(1) process

The NLAR(1) process is the first order autoregressive version of NLAR(2) with $\phi_2 = 0$ and is given by

$$X_t = \left\{ \begin{array}{lll} 0 & \text{w.p.} & 1-p \\ \phi X_{t-1} & \text{w.p.} & p \end{array} \right\} + \varepsilon_t, \tag{2.13}$$

where

$$\varepsilon_t = \left\{ \begin{array}{lll} E_t & \text{w.p.} & 1-\pi \\ \sqrt{1-p}\,|\phi|\,E_t & \text{w.p.} & \pi = \phi^2 p/(1-(1-p)\phi^2). \end{array} \right. \tag{2.14}$$

The random coefficient form of process is convenient for the analysis of a process with random coefficients which switches terms included in the process. The first-order Random Coefficient Autoregressive(RCA(1)) form for NLAR(1) is

$$X_t = \phi V_t X_{t-1} + U_t E_t, \tag{2.15}$$

where $\{V_t\}$ is a sequence of i.i.d. Bernoulli(p) variables and $\{U_t\}$ is a sequence of i.i.d. discrete random variables with distribution,

$$\{U_t\} = \left\{ \begin{array}{lll} 1 & \text{w.p.} & 1-\pi \\ \sqrt{1-p}\,|\phi| & \text{w.p.} & \pi. \end{array} \right. \tag{2.16}$$

The a.c.f. for NLAR(1) is

$$\rho_k = (\phi p)^k,\ k \geq 1. \tag{2.17}$$

LAR(1) process

As LAR(2) is obtained from NLAR(2), LAR(1) process in (2.18) is obtained from NLAR (1) with p=1;

$$X_t = \phi X_{t-1} + \varepsilon_t \quad , \text{where } \varepsilon_t = \begin{cases} E_t & \text{w.p.} \quad 1-\phi^2 \\ 0 & \text{w.p.} \quad \phi^2. \end{cases} \tag{2.18}$$

The a.c.f. for LAR(1) is

$$\rho_k = \phi^k \quad , k>1. \tag{2.19}$$

From the above model constructions, we know that the NLAR(2) model is a general model including LAR(2), NLAR(1), and LAR(1) models. Similarly, the NLAR(1) model includes the LAR(1) model. The NLAR model provides great flexibility to model building because of the broad range of correlations and dependence structure which can be obtained with the use of more parameters than the order, while the LAR class is simpler in manipulation than the NLAR class. But its innovation variable has a zero component often called *zero defect* which makes the maximum likelihood estimation impossible and is an unrealistic feature in most practical situation; the zero innovation in the LAR(1) implies that $X_t = \phi X_{t-1}$ and ϕ can be determined exactly from the sample path of the process.

NLAR(p) process

An NLAR(p) process, p-th order Laplacian($p \geqq 3$), can be constructed analogously to the NLAR(2) process and may be written,

$$X_t = \begin{cases} \phi_1 X_{t-1} & \text{w.p.} \quad a_1 \\ \phi_2 X_{t-2} & \text{w.p.} \quad a_2 \\ \cdots \quad & \cdots \\ \phi_1 X_{t-p} & \text{w.p.} \quad a_p \\ 0 & \text{w.p.} \quad 1-p^* \end{cases} + \varepsilon_t \tag{2.20}$$

for $t = 0, \pm 1, \pm 2, \cdots$, where $p^* = \Sigma_{i=1}^{p} a_i$. The LAR(p) process can be obtained if we set

$$\begin{cases} a_1 = 1-\phi_2^2 \\ a_j = (1-\phi_{j+1}^2)\prod_{i=2}^{j}\phi_i^2, \quad j=2,3,\cdots,p-1 \\ a_p = \prod_{i=2}^{p}\phi_i^2 \end{cases}$$

in NLAR(p) process.

In NLAR(p) process, the structure of error ε_t which gives a stationary standard Laplacian marginal distribution of $\{X_t\}$ would be very difficult to derive, because it is algebraically complicated to solve the defining equation for the distribution of ε_t's if it exists. However, letting $\phi_1 = \phi_2 = \cdots = \phi_p = \phi$, the same error structure as that of NLAR(1) is obtained. In this case, NLAR(p) takes the following form,

$$X_t = \left\{ \begin{array}{lll} \phi X_{t-1} & \text{w.p.} & a_1 \\ \phi X_{t-2} & \text{w.p.} & a_2 \\ \cdots & \cdots & \\ \phi X_{t-p} & \text{w.p.} & a_p \\ 0 & \text{w.p.} & 1-p^* \end{array} \right\} + \varepsilon_t, \tag{2.22}$$

for $t=0,\pm1,\pm2,\cdots$, where $p^* = \Sigma_{i=1}^{p} a_i$ and

$$\varepsilon_t = \begin{cases} E_t & \text{w.p.} \quad 1-\pi \\ \sqrt{1-p^*}\,|\phi|\,E_t & \text{w.p.} \quad \pi = \phi^2 p^* / (1-(1-p^*)\phi^2). \end{cases} \tag{2.23}$$

Also, the simplified LAR(p) process in case of $p^* = \Sigma_{i=1}^{p} a_i = 1$ in NLAR(p) of (2.22) is given by

$$X_t = \left\{ \begin{array}{lll} \phi X_{t-1} & \text{w.p.} & a_1 \\ \phi X_{t-2} & \text{w.p.} & a_2 \\ \cdots & \cdots & \\ \phi X_{t-P} & \text{w.p.} & a_p \end{array} \right\} + \left\{ \begin{array}{lll} E_t & \text{w.p.} & 1-\phi^2 \\ 0 & \text{w.p.} & \phi^2, \end{array} \right. \tag{2.24}$$

for $t = 0,\pm1,\pm2,\cdots$.

The properties of NLAR(p) process in (2.22) are similar to that of NLAR(2); NLAR(p) has a Laplacian marginal distribution and is p-th order Markovian and strictly stationary. The a.c.f. for the NLAR(p) of (2.22) is

$$\rho_k = \phi \sum_{i=1}^{p} a_i \rho_{k-i}, \quad k \geq 1. \tag{2.25}$$

The equation (2.25) is the same as Yule-Walker type equation for Gaussian AR(p), and so its general solution is

$$\rho_k = \sum_{i=1}^{p} A_i G_i^k, \tag{2.26}$$

where $G_1^{-1}, G_2^{-1}, \cdots, G_p^{-1}$ are the roots of characteristic equation $1 - \phi \Sigma_{i=1}^{p} a_i B^i = 0$.

3. Strong-mixing property

It was shown by Cho and Miller(1987) that the LAR(1) of (2.18) is *strong-mixing*. Raftery (1980) showed that the NEAR(1) in case of $\phi = p$ is *strong-mixing*. Following Raftery (1980), we will show the mixing property of $\{X_t\}$ in the Laplacian process. The NLAR(1) in (2.13) can be rewritten as

$$X_{t+k} = \begin{cases} \sum_{j=0}^{k-1} \phi^j \varepsilon_{t+k-j} + \phi^k X_t & \text{w.p} \quad p^k \\ \sum_{j=0}^{i} \phi^j \varepsilon_{t+k-j} & \text{w.p.} \quad p^i(1-p), \ i = 0, 1, 2, \cdots, k-1. \end{cases} \tag{3.1}$$

From (2.13) we have

$$M_{\varepsilon_j}(s) = E[\exp(s\varepsilon_j)] = \frac{1-\phi^2 s^2}{(1-s^2)\{1-(1-p)\phi^2 s^2\}}, \quad \text{for any } j, \tag{3.2}$$

and

$$E[\exp(s \sum_{j=0}^{k-1} \phi^j \varepsilon_{t+k-j})] = \prod_{j=0}^{k-1} M_{\varepsilon_{t+k-j}}(s\phi^j)$$

$$= \frac{\phi^{2k} + (1-\phi^{2k})/(1-s^2)}{\prod_{j=1}^{k}\{1-\phi^{2j}(1-p)s^2\}}. \tag{3.3}$$

The moment generating function (3.3) of the distribution of $Y = \Sigma_{j=0}^{k-1} \phi^j \varepsilon_{t+k-j}$ gives the following relation,

$$Y=\sum_{j=1}^{k} |\phi|^{j}\sqrt{1-p}\,L_j + \begin{cases} 0 & \text{w.p.} \quad \phi^{2k} \\ L_0 & \text{w.p.} \quad 1-\phi^{2k}, \end{cases} \tag{3.4}$$

where $\{L_j: j= 0,1,2,\cdots,k\}$ is a sequence of i.i.d. standard Laplacian variables. $L=\Sigma_{i=1}^{n}\lambda_i L_i$ can be written as the difference between $\Sigma_{i=1}^{n}\lambda_i E_{1i}$ and $\Sigma_{i=1}^{n}\lambda_i E_{2i}$ assuming 2n i.i.d. standard exponential variables, E_{1i}, E_{2i}, $i= 1,2,\cdots,n$. Following Johnson and Kotz(1970, pp222) we have the probability density of $L= \Sigma_{i=1}^{n}\lambda_i E_{1i}- \Sigma_{i=1}^{n}\lambda_i E_{2i}$

$$f_L(l)=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\lambda_i^{n-1}\lambda_j^{n-1}\exp(-|l|/\lambda_i)}{(\lambda_i+\lambda_j)\prod_{u\neq i}^{n}(\lambda_i-\lambda_u)\prod_{v\neq j}^{n}(\lambda_j-\lambda_v)} \tag{3.5}$$

for $\lambda_i \neq \lambda_j$. Thus the probability density of Y in (3.4) is directly obtained from (3.5);

$$\frac{dF_Y(y)}{dy}=\exp(-|y|)\{U_1+\sum_{j=1}^{k}U_{2j}(y)+\sum_{i=1}^{k}\sum_{j=1}^{k}V_{1ij}(y) + \sum_{i=1}^{k}\sum_{j=1}^{k}V_{2ij}(y)+\sum_{i=1}^{k}V_i(y)\}, \tag{3.6}$$

where

$$\begin{aligned} U_1 &= 2^{-1}(1-\phi^{2k})\prod_{u=1}^{k}(1-|\phi|^{u}\sqrt{1-p})^{-2} \\ &\leq 2^{-1}(1-\phi^{2k})\prod_{u=1}^{\infty}(1-|\phi|^{u}\sqrt{1-p})^{-2} \\ &\leq 2^{-1}(1-\phi^{2k})\prod_{u=1}^{\infty}(1-|\phi|^{u})^{-2} \\ &\leq \{1+\sum_{u=1}^{\infty}|\phi|^{u}(1-|\phi|)^{-1}(1-|\phi|^{2})^{-1}\cdots(1-|\phi|^{u})^{-1}\}^{2} \\ &= Cu_1, \text{ a constant, from Erdelyi and et al. (1955),} \end{aligned} \tag{3.7}$$

$$\begin{aligned} U_{2j}(y) &= (1-\phi^{2k})\sqrt{1-p}\{\prod_{u=1}^{k}(1-|\phi|^{u}\sqrt{1-p})^{-1}\} \\ &\times\{\prod_{v\neq j}^{k}(|\phi|^{j}-|\phi|^{v})^{-1}\}\{\phi^{2j}(1-p)-1\}^{-1}|\phi|^{jk}, \end{aligned} \tag{3.8}$$

$$V_{1ij}(y) = (1-\phi^{2k})\sqrt{1-p}\left\{\prod_{u\neq i}^{k}(|\phi|^{i}-|\phi|^{u})^{-1}\right\}(|\phi|^{i}\sqrt{1-p}-1)$$

$$\times\left\{\prod_{v\neq j}^{k}(|\phi|^{j}-|\phi|^{v})^{-1}\right\}(|\phi|^{j}\sqrt{1-p}-1)^{-1}|\phi|^{k(i+j)}$$

$$\times(|\phi|^{i}+|\phi|^{j})^{-1}\exp\left[-|y|\left\{(|\phi|^{i}\sqrt{1-p})^{-1}-1\right\}\right], \tag{3.9}$$

$$V_{2ij}(y) = \phi^{2k}\sqrt{1-p}^{-1}\left\{\prod_{u\neq i}^{k}(|\phi|^{i}-|\phi|^{u})^{-1}\right\}$$

$$\times\left\{\prod_{v\neq j}^{k}(|\phi|^{j}-|\phi|^{v})^{-1}\right\}|\phi|^{(i+j)(k-1)}$$

$$\times(|\phi|^{i}+|\phi|^{j})^{-1}\exp\left[-|y|\left\{(|\phi|^{i}\sqrt{1-p})^{-1}-1\right\}\right], \tag{3.10}$$

and

$$V_{i}(y) = (1-\phi^{2k})\sqrt{1-p}\left\{\prod_{v=1}^{k}(1-|\phi|^{v}\sqrt{1-p})^{-1}\right\}\left\{\prod_{u\neq i}^{k}(|\phi|^{i}-|\phi|^{u})^{-1}\right\}$$

$$\times\left\{\phi^{2i}(1-p)-1\right\}^{-1}|\phi|^{ik}\exp\left[-|y|\left\{(|\phi|^{i}\sqrt{1-p})^{-1}-1\right\}\right]. \tag{3.11}$$

Here, we have

$$\sum_{i=1}^{k}V_{i}(y) \leq \sum_{i=1}^{k}|V_{i}(y)| \leq \sum_{i=1}^{\infty}|V_{i}(y)| = C_{V}, \tag{3.12}$$

a constant, by ratio test. Similarly, $\Sigma_{j=1}^{k}U_{2j}(y) < Cu_2$, $\Sigma_{i=1}^{k}\Sigma_{j=1}^{k}V_{1ij}(y) < Cv_1$, and $\Sigma_{i=1}^{k}\Sigma_{j=1}^{k}V_{2ij}(y) \leqq Cv_2$, where Cu_2, Cv_1, and Cv_2 are constants. Thus,

$$dF_{Y}(y) < Ce^{-|y|}dy, \tag{3.13}$$

where $C = Cu_1 + Cu_2 + Cv_1 + Cv_2 + Cv$ is a constant.

Suppose F_t and G_{t+k} are sigma fields generated by $(\cdots, X_{t-1}, X_t)$ and $(X_{t+k}, X_{t+k+1}, \cdots)$, respectively, and consider $f \in L^2(F_t)$ and $g \in L^2(G_{t+k})$ in the sense that $E(f^2) < \infty$ and $E(g^2) < \infty$. For the NLAR(1) process of (2.13) we have

$$E(f \cdot g) - E(f)E(g) = E[f \cdot \{E(g|X_t) - E(g)\}]. \tag{3.14}$$

Consider the event $A = \{ X_{t+k} = \Sigma_{j=0}^{k-1} \phi^j \varepsilon_{t+k-j} + \phi^k X_t \}$ with $Pr(A) = p^k$ and the events $A_i = \{X_{t+k} = \Sigma_{j=0}^{i} \phi^j \varepsilon_{t+k-j}\}$ with $Pr(A_i) = p^i(1-p)$ for $i = 0,1,2,\cdots,k-1$. Then

$$E(g|X_t) = \sum_{i=0}^{k-1} p^i(1-p) E[(g|X_t)|A_i] + p^k E[(g|X_t)|A] \tag{3.15}$$

and

$$E(g) = \sum_{i=0}^{k-1} p^i(1-p) E(g|A_i) + p^k E(g|A). \tag{3.16}$$

For $i = 0,1,2,\cdots,k-1$,

$$E[(g|X_t)|A_i] = E(g|A_i). \tag{3.17}$$

Hence

$$E(g|X_t) - E(g) = p^k\{E[(g|X_t)|A] - E(g|A)\}. \tag{3.18}$$

Assuming, $\phi^k x_t > 0$,

$$\begin{aligned}
|E[(g|X_t)|A]| &= \left| \int_{-\infty}^{\infty} E(g|X_{t+k} = y)\, dF_Y(y - \phi^k x_t) \right| \\
&\le C \int_{-\infty}^{\infty} E(|g||X_{t+k} = y) \exp(-|y - \phi^k x_t|)\, dy \\
&= C\{\exp(-\phi^k x_t) \int_{-\infty}^{\phi^k x_t} E(|g||X_{t+k} = y)\, e^y\, dy \\
&\quad + \exp(\phi^k x_t) \int_{\phi^k x_t}^{\infty} E(|g||X_{t+k} = y)\, e^{-y}\, dy\} \\
&\le C \cdot \exp(\phi^k x_t) \int_{-\infty}^{\infty} E(|g||X_{t+k} = y)\, e^{-|y|}\, dy \\
&= 2C \cdot \exp(\phi^k x_t)\, E(|g|).
\end{aligned} \tag{3.19}$$

Also,

$$|E(g|A)| = \left|\int_{-\infty}^{\infty} E[(g|X_t)|A]\cdot 2^{-1}\cdot \exp(-|X_t|)\,dx_t\right|$$

$$\leq C\cdot E(|g|)\int_{-\infty}^{\infty} \exp(\phi^k x_t - |x_t|)\,dx_t, \quad \text{from (3.19)}$$

$$= 2C\cdot E(|g|)(1-\phi^{2k})^{-1}.$$

From (3.14), (3.19), and (3.20)

$$\begin{aligned}
&|E(f\cdot g)-E(f)E(g)| \leq p^k E[|f|\{|E(g|X_t)+|E(g)|\}]\\
&\leq 2Cp^k E[|f|E(|g|)\{\exp(\phi^k x_t)+(1-\phi^{2k})^{-1}\}]\\
&\leq 2Cp^k\sqrt{E(|f|^2)E(|g|)}\ \sqrt{E[\exp(\phi^k x_t)+(1-\phi^{2k})^{-1}]^2}\\
&\leq C'p^k\sqrt{E(|f|^2)E(|g|)} \qquad \text{, for } k\geq -\log k'/\log|\phi|,
\end{aligned} \tag{3.21}$$

where C' is a constant and k' is much larger than 2. For any measurable sets $B\in F_t$ and $D\in G_{t+k}$, let $f=1_B(\cdots,X_{t-1},X_t)$ and $g=1_D(X_{t+k},X_{t+k+1},\cdots)$, in case of which $f\in L^2(F_t)$ and $g\in L^2(G_{t+k})$. Now, from (3.21) and the fact that $Pr(B), Pr(D)<1$

$$\sup_{\substack{B\in F_t\\ D\in G_{t+k}}} |Pr(B\cap D)-Pr(B)Pr(D)| \leq C'p^k \to 0 \quad \text{as} \quad k\to\infty. \tag{3.22}$$

Alternatively, assuming $\phi^k x_t<0$, the same result as (3.22) is obtained. Thus the NLAR(1) process is *strong-mixing* with $\alpha(k)=C'p^k$

4. Forecasting

4.1. Point forecasting

An observation X_{t+k} generated by the Laplacian autoregressive process defined in Section 2 may be expressed as a randomly weighted infinite sum of current and previous innovations;

$$X_{t+k} = \sum_{j=0}^{\infty} \psi_{j,t+k}\,\varepsilon_{t+k-j} \tag{4.1}$$

where $\psi_t=(\psi_{0,t},\psi_{1,t},\cdots)'$; $\{\psi_{j,t}: t=0,\pm1,\pm2,\cdots\}$ is a sequence of multiples of random

coefficients with fixed parameters, $\{\psi_t\}$ and $\{\varepsilon_t\}$ are independent, and $\{\varepsilon_t\}$ is a sequence of innovations with mean 0 and variance σ_ε^2.

We assume that a forecast at origin t for lead time k is a linear function of current and previous innovations $\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, \cdots$ and denote a sigma field generated by $\{X_1, X_2, \cdots, X_t\}$ by H_t. Then the minimum mean square error (m.m.s.e.) forecast at origin t for lead time k is

$$\hat{X}_t(k) = E(X_{t+k} | H_t) = \sum_{j=k}^{\infty} E(\psi_{j,t+k} | H_t) \varepsilon_{t+k-j} \tag{4.2}$$

The forecast error for lead time k is

$$e_t(k) = \sum_{j=0}^{k-1} \psi_{j,t+k} \varepsilon_{t+k-j} + \sum_{j=k}^{\infty} \{\psi_{j,t+k} - E(\psi_{j,t+k} | H_t)\} \varepsilon_{t+k-j}. \tag{4.3}$$

Since $E(e_t(k) | H_t) = 0$, the m.m.s.e. forecast is unbiased and the variance of forecast error is

$$Var(e_t(k) | H_t) = \sigma_\varepsilon^2 \sum_{j=0}^{k-1} E(\psi_{j,t+k}^2 | H_t) + \sum_{j=k}^{\infty} Var(\psi_{j,t+k} | H_t) \varepsilon_{t+k-j}^2. \tag{4.4}$$

It is easily shown that that any linear function $\Sigma_{k=1}^{m} W_k X_t(k)$ is a m.m.s.e. forecast of the corresponding linear function $\Sigma_{k=1}^{m} W_k X_{t+k}$ of future observations.

At time t+k the NLAR(1) of (2.13) may be written

$$X_{t+k} = \varepsilon_{t+k} + \sum_{j=1}^{k-1} \phi^j \left(\prod_{i=0}^{j-1} V_{t+k-i}\right) \varepsilon_{t+k-j} + \phi^k \left(\prod_{i=0}^{k-1} V_{t+k-i}\right) X_t, \tag{4.5}$$

or

$$X_{t+k} = \sum_{j=0}^{\infty} \psi_{j,t+k} \varepsilon_{t+k-j}, \tag{4.6}$$

where $\psi_{0,t+k} \equiv 1$ and $\psi_{j,t+k} = \phi^j \Pi_{i=0}^{j-1} V_{t+k-i}$ $j = 1, 2, \cdots$. Taking conditional expectations at origin t, the m.m.s.e. forecast for lead time k, $\hat{X}_t(k) = \phi^k p^k x_t$, which decays to zero as k gets larger since $|\phi| < 1$, is obtained. Exponential decay of $\hat{X}_t(k)$ to zero is smooth if ϕ is positive and oscillatory if ϕ is negative. The forecast function in this case is determined entirely by the single observation, x_t, at origin t. The forecast error and the forecast variance for lead time k are, respectively,

$$e_t(k) = \varepsilon_{t+k} + \sum_{j=1}^{k-1} \phi^j \left(\prod_{i=0}^{j-1} V_{t+k-i}\right) \varepsilon_{t+k-j} + \phi^k \left(\left(\prod_{i=0}^{k-1} V_{t+k-i}\right) - p^k\right) X_t \tag{4.7}$$

and

$$\mathrm{Var}(e_t(k) \mid H_t) = (\phi^2 p)^k (1-p^k) x_t^2 + \sigma_\varepsilon^2 (1 - (\phi^2 p)^k) / (1 - \phi^2 p), \tag{4.8}$$

where $\sigma_\varepsilon^2 = 2(1-\pi + \pi(1-p)\phi^2)$ with $\pi = \phi^2 p / (1-(1-p)\phi^2)$. We see that the forecast variance for this NLAR(1) process converges to a constant value $\sigma_\varepsilon^2 / (1-\phi^2 p)$ as k tends to infinity. Also

$$\begin{aligned} E(e_t(k) \cdot X_t) &= \phi^k E\left[\left(\prod_{i=0}^{k-1} V_{t+k-i}\right) - p^k\right) X_t^2 \,] \\ &= \phi^k E\left[\left(\prod_{i=0}^{k-1} V_{t+k-i}\right) - p^k\right] E(X_t^2) = 0, \end{aligned} \tag{4.9}$$

which implies that there is not any more information about X_t's left that can be obtained from the forecast error $e_t(k)$ up to time t.

Suppose we make a series of forecasts for different lead times k and $k+l$ from the same origin t. Then we have two forecast errors, $e_t(k)$ of (4.7) and

$$\begin{aligned} e_t(k+l) &= \varepsilon_{t+k+l} + \sum_{j=1}^{k+l-1} \phi^j \left(\prod_{i=0}^{j-1} V_{t+k+l-i}\right) e_{t+k+l-j} \\ &\quad + \phi^{k+l} \left(\left(\prod_{i=0}^{k+l-1} V_{t+k+l-i}\right) - p^{k+l}\right) X_t, \end{aligned} \tag{4.10}$$

for $l = 1,2,\cdots$ The covariance between the t-origin forecasts at lead times k and $k+l$ is

$$\mathrm{Cov}(e_t(k), e_t(k+l) \mid H_t) = (\phi^2 p)^{k+l} (1-p^k) x_t^2 + \frac{\sigma_\varepsilon^2 \phi^l p^l (1-(\phi^2 p)^k)}{(1-\phi^2 p)} \tag{4.11}$$

and the correleation coefficient between t-origin forecast errors at lead time k and $k+l$ is

$$\begin{aligned} &\mathrm{Corr}(e_t(k), e_t(k+l) \mid H_t) \\ &= \frac{(\phi^2 p)^{k+l} (1-p^k) x_t^2 + \sigma_\varepsilon^2 \dfrac{\phi^l p^l (1-(\phi^2 p)^k)}{1-\phi^2 p}}{\sqrt{\left\{(\phi^2 p)^k (1-p^k) x_t^2 + \sigma_\varepsilon^2 \dfrac{1-(\phi^2 p)^k}{1-\phi^2 p}\right\} \left\{(\phi^2 p)^{k+l} (1-p^{k+l}) x_t^2 + \sigma_\varepsilon^2 \dfrac{1-(\phi^2 p)^{k+l}}{1-\phi^2 p}\right\}}}. \end{aligned} \tag{4.12}$$

We note that the correlations goes to zero as l tends to infinity.

The m.m.s.e. forecast at origin t for lead time k in the NLAR(2) process of (2.1) is

$$\begin{cases} \hat{X}_t(1) = \phi_1 p_1 x_t + \phi_2 p_2 x_{t-1} \\ \hat{X}_t(2) = \phi_1 p_1 \hat{X}_t(1) + \phi_2 p_2 x_t \\ \hat{X}_t(k) = \phi_1 p_1 \hat{X}_t(k-1) + \phi_2 p_2 \hat{X}_t(k-2) \quad , k = 3,4,\cdots \end{cases} \tag{4.13}$$

It is analogous to forecasting a Gaussian AR(2) process, since the forecast function for lead time k is the solution of $\varphi(B)\hat{X}_t(k) = 0$, where $\varphi(B) = 1 - \phi_1 p_1 B - \phi_2 p_2 B^2$. This fact implies that the general forecast function for NLAR(2) process is a mixture of exponentials and damped sines. The forecast function at origin t is determined entirely by the last two observations x_{t-1} and x_t.

Similarly, the m.m.s.e. forecast at origin t for lead time k in the NLAR(p) process of (2.22) is

$$\hat{X}_t(k) = \begin{cases} \phi \sum_{i=1}^{k-1} a_i \hat{X}_t(k-i) + \phi \sum_{i=k}^{p} a_i x_{t+k-i} \quad , k = 1, 2, \cdots, p \\ \phi \sum_{i=1}^{p} a_i \hat{X}_t(k-i), \quad k = p+1, p+2, \cdots . \end{cases} \tag{4.14}$$

and is the soultion of $\varphi(B)\hat{X}_t(k) = 0$, where $\varphi(B) = 1 - \phi \Sigma_{i=1}^{p} a_i B^i$, which applies for all lead times and passes through the last p available values of the series.

4.2. Interval forecasting

Prediction interval can be obtained using the probability limits of forecasts at any time. Since in NLAR(1) of (2.13) the conditional probability distribution of X_{t+k} given information up to time t is

$$f_{x_{t+k}|x_t,\, x_{t-1}\cdots}(y|x_t, x_{t-1}, \ldots) = p^k f_Y(y - \phi^k x_t) + (1-p) \sum_{i=0}^{k-1} p^i f_Y(y), \tag{4.15}$$

where $f_Y(y)$ is defined in (3.6), the $(1-\alpha)$ probability limits for X_{t+k}, $X_{t+k}(\pm)$, can be obtained by solving

$$\int_{X_{t+k}(-)}^{X_{t+k}(+)} f_{x_{t+k}|x_t,\, x_{t-1}\cdots}(y|x_t, x_{t-1}, \ldots)\, dy = 1 - \alpha . \tag{4.16}$$

The form of $f_{x_{t+k}|x_t,x_{t-1},\cdots}(y|x_t,x_{t-1},\cdots)$ is complicated, however, and it is not easy to obtain, in practice, such prediction interval as in the Gaussian AR case. This leads us to consider other interval forecasting method, i.e., the robust model-free prediction method to be followed.

Butler(1982) proposed a nonparametric prediction interval for a future observation X_{n+1} given a random sample $X_1, X_2, \cdots, X_n$ using the sample quantiles. Cho and Miller(1987) showed that, for dependent processes, the sample quantiles can also be used to from a prediction interval if certain conditions are satisfied. Since the *strong-mixing* coefficient, $\alpha(k)$, for the NLAR(1) satisfies the condition $\alpha(k) = O(\exp(-\theta k))$,for some θ, model-free prediction interval can be obtained for the NLAR(1) by theorem of Cho and Miller(1987). Simulation is perforemd to see how this model-free method works for small samples. Coverage percentages of one step ahead model-free P.I.'s for the NLAR(1) with $\phi = -0.9$ to 0.9 and p= 0.1 to 0.9 in increments of 0.1, respectively, are obtained when sample size are n=20 and 100. To obtain the P.I.'s for n=100(20) we generate 101(21) observations from the NLAR(1) using an IMSL function, GGUBFS. Using the first 100(20) observations 90% P.I. is obtained. We use the 5th and the 95th, $[X_{(5)},X_{(95)}]$, observations for n=100 and the minimum and the maximum, $[X_{(1)},X_{(20)}]$, observations for n=20. 500 replications are made for each(ϕ,p) pairs. Simulation results are summarized in Table 4.1 and 4.2. For n=100, it is observed that the coverage percentages of the 101th observation by the 90% P.I.'s range from 85 to 93% which is the similar result reported in Cho and Miller(1987) for the LAR(1) case. The 90% P.I.'s capture also 85 to 93% of the 21th observation except the case of (0.9, 0.9) for (ϕ,p). From the above simulation result we can see that for interval prediction the model-free method using the sample quantiles is not only easy to apply but also performs well for small samples as well as for large samples, while the model based method using (4.16) is hard to use in practice because of the complication of the distribution function. Raftery(1982) stated that the NEAR(p) is Φ-*mixing* with $\Phi_n(k) = (a_1 + a_2 + \cdots + a_p)^k$. Based on his statement we may be able to extend the model-free method to the NLAR(p). Although we have not been able to prove our conjecture, however, to support this we report our simulation result for the NLAR(2) in Table 4.3 where that 90% P.I.'s capture 85 to 92% of the 101th observation for each ϕ_2 and p_2 with fixed $\phi_1 = -0.6$ and $p_1 = 0.3$.

Table 4.1. Coverage percentages of one step-ahead 90% P.I.'s for NLAR(1) process of size n=100 using 500 simulation trials

ϕ \ p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
−0.9	0.8660	0.8880	0.8820	0.9080	0.8960	0.8940	0.8540	0.8960	0.9040
−0.8	0.8860	0.8960	0.8820	0.9080	0.8760	0.8680	0.8860	0.9160	0.8920
−0.7	0.8740	0.8920	0.8880	0.8880	0.9220	0.8700	0.8520	0.8760	0.8900
−0.6	0.8940	0.8940	0.8920	0.8860	0.8680	0.8660	0.8980	0.9040	0.9040
−0.5	0.9120	0.9020	0.9060	0.8900	0.8880	0.8820	0.8900	0.8960	0.8840
−0.4	0.9060	0.8940	0.8920	0.9120	0.8840	0.8840	0.8760	0.8900	0.8720
−0.3	0.8760	0.8780	0.8940	0.9060	0.8920	0.8820	0.8920	0.9120	0.8960
−0.2	0.8880	0.8580	0.8760	0.8860	0.8960	0.8820	0.8720	0.8800	0.9060
−0.1	0.8900	0.8860	0.8740	0.9020	0.9020	0.8940	0.8920	0.9120	0.8980
0.0	0.8980	0.8920	0.8920	0.8740	0.8820	0.8860	0.8980	0.9020	0.8940
0.1	0.8780	0.9220	0.9020	0.8680	0.8980	0.9000	0.9000	0.8760	0.8920
0.2	0.8920	0.9040	0.9060	0.8960	0.8860	0.8880	0.8800	0.9060	0.8780
0.3	0.8740	0.8940	0.8940	0.8600	0.9040	0.8960	0.8800	0.9000	0.8640
0.4	0.8860	0.9080	0.8900	0.9320	0.8940	0.8880	0.9180	0.8960	0.8980
0.5	0.9280	0.8720	0.8860	0.8920	0.8800	0.8820	0.8920	0.9040	0.8800
0.6	0.8980	0.8900	0.8900	0.8920	0.9020	0.8940	0.8860	0.9060	0.8920
0.7	0.8780	0.9100	0.8680	0.8960	0.9140	0.8780	0.8660	0.8880	0.8760
0.8	0.8720	0.8880	0.8980	0.8820	0.8660	0.8960	0.8840	0.8840	0.8520
0.9	0.8980	0.9020	0.9020	0.8940	0.8780	0.8920	0.8860	0.8820	0.8960

Table 4.2. Coverage percentages of one step-ahead 90% P.I.'s for NLAR(1) process of size n=20 using 500 simulation trials.

ϕ \ p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
−0.9	0.9140	0.9220	0.9040	0.9140	0.9200	0.9280	0.9000	0.8900	0.9160
−0.8	0.9100	0.8920	0.9080	0.8800	0.8920	0.8720	0.9140	0.8960	0.8980
−0.7	0.9220	0.9020	0.9060	0.8980	0.9020	0.9020	0.8960	0.9100	0.8920
−0.6	0.9200	0.9080	0.9160	0.9080	0.9000	0.9100	0.8900	0.9080	0.9100
−0.5	0.9140	0.9200	0.9220	0.9060	0.9040	0.9340	0.9020	0.9060	0.9080
−0.4	0.9060	0.9080	0.8740	0.8720	0.9180	0.8960	0.8940	0.9060	0.9100
−0.3	0.9020	0.9040	0.8940	0.9200	0.9000	0.9220	0.9240	0.9020	0.9200
−0.2	0.8860	0.8900	0.9100	0.9360	0.9200	0.9120	0.8920	0.8920	0.9040
−0.1	0.9020	0.9180	0.8900	0.9300	0.8960	0.8940	0.8820	0.9260	0.9240
0.0	0.9320	0.9100	0.9080	0.9100	0.9160	0.8860	0.9280	0.8980	0.9320
0.1	0.9060	0.8920	0.9160	0.8960	0.8860	0.9340	0.9280	0.8940	0.9140
0.2	0.9100	0.9100	0.9280	0.8840	0.8960	0.9120	0.8920	0.9140	0.9240
0.3	0.9100	0.9140	0.8800	0.9200	0.9000	0.8940	0.8900	0.9140	0.8820
0.4	0.9160	0.9240	0.9040	0.9200	0.9000	0.8940	0.8900	0.9140	0.8820
0.5	0.9020	0.9180	0.9220	0.9060	0.8980	0.9060	0.8960	0.8880	0.9160
0.6	0.8880	0.8860	0.9140	0.8920	0.9160	0.9040	0.8880	0.8840	0.8700
0.7	0.9120	0.9000	0.9220	0.9220	0.8800	0.9080	0.8640	0.8980	0.8840
0.8	0.9180	0.9020	0.9080	0.8940	0.8660	0.8820	0.8700	0.8780	0.8500
0.9	0.8800	0.8800	0.8820	0.8580	0.8920	0.8680	0.8880	0.8540	0.8280

Table 4.3. Coverage percenages of one step-ahead 90% P.I.'s for NLAR(2) process with fixed $\phi_1 = -0.6$, $p_1 = 0.3$ of size n=100 using 500 simulation trials

ϕ_2 \ p_2	0.1	0.2	0.3	0.4	0.5	0.6
−0.9	0.8840	0.8900	0.8880	0.9040	0.8940	0.9080
−0.8	0.8820	0.8720	0.8760	0.9080	0.8980	0.8940
−0.7	0.8820	0.8820	0.8840	0.9020	0.9080	0.8960
−0.6	0.9000	0.8920	0.8980	0.8680	0.8880	0.9020
−0.5	0.9120	0.8940	0.8720	0.8840	0.8900	0.9020
−0.4	0.9060	0.8960	0.8980	0.8920	0.8860	0.8960
−0.3	0.8760	0.8800	0.8980	0.8920	0.8840	0.8760
−0.2	0.8880	0.8580	0.8840	0.8880	0.8860	0.8840
−0.1	0.8900	0.8860	0.8780	0.8940	0.9000	0.8900
0.0	0.8980	0.8920	0.8920	0.8740	0.8820	0.8860
0.1	0.8780	0.9220	0.9040	0.8740	0.8980	0.9000
0.2	0.8920	0.9040	0.9120	0.8840	0.8820	0.9020
0.3	0.8760	0.8940	0.8940	0.8660	0.9280	0.8820
0.4	0.8880	0.9140	0.8900	0.9140	0.8800	0.9020
0.5	0.9280	0.8640	0.8860	0.8840	0.8960	0.8640
0.6	0.8900	0.8920	0.8940	0.8620	0.8880	0.8600
0.7	0.8880	0.9120	0.8860	0.8660	0.8620	0.8800
0.8	0.8760	0.8740	0.8860	0.8590	0.8980	0.9000
0.9	0.8680	0.9080	0.8760	0.8840	0.8780	0.8900

References

(1) Anderson, T.W.(1970), *The Statistical Analysis of Time Series,* New York: John Wiley and Sons, Inc.

(2) Box, G.E.P. and Jenkins, G.M.(1976), *Time Series Analysis Forecasting and Control, San Francisco*: Holden-Day, Inc.

(3) Brown, B.G., Katz, R.W. and Murphy, A.H.(1984), Time series models to simulate and forecast wind speed and wind power, *Journal of Climate and Applied Meteorology,* Vol.23, 1184-1195.

(4) Butler, R.W.(1982), Nonparametric interval and point prediction using data trimmed by a Grubbs-type outlier rule, *The Annals of Statistics,* Vol.10, 197-204.

(5) Cho, S.S. and Miller, R.B.(1987), Model free one-step-ahead prediction intervals: Asymptotic theory and small sample simulations, *The Annals of Statistics,* Vol.15, 1064-1078.

(6) Dewald, L.S. and Lewis, P.A.W.(1985), A new Laplace second-order autoregressive time series model-NLAR(2), *IEEE Transactions Information Theory,* IT-31, No.5, 645-651.

(7) Erdelyi, A., Magnus, W., Oberhettinger, F., and Tricani, F.G.(1955), *Higher Transcendental Function,* Vol.3, McGraw-Hill, New York, pp.176.

(8) Gaver, D.P. and Lewis, P.A.W. (1977), First-order autoregressive gamma sequences and point processes, *Advances in Applied Probabiliy* Vol. 12(1980), pp.727-745.

(9) Gaver, D.P. and Lewis, P.A.W.(1980), First-order gamma sequences and point processes, *Advances in Applied Probability,* Vol.12, No.3, 727-745.

(10) Hsu, D.A.(1979), Long-tailed distributions for position errors in navigation, *Journal of the Royal Statistical Society, Series* C, Vol.28, 62-72.

(11) Hugus, D.K.(1982), Extensions of some models for positive valued time series, Ph. D. *Thesis, Naval Postgraduate School, Monterey,* CA.

(12) Jacobs, P.A. and Lewis, P.A.W.(1977), A mixed autoregressive-moving average exponential sequence and point process, (EARMA(1,1)), *Advances in Applied Probability,* Vol.9, 87-104.

(13) Johnson, N.L. and Kotz, S.(1970), *Continuous Univariate Distributions,* Vol.1, John Wiley and Sons, New York, pp.222.

(14) Lawrance, A.J. and Lewis, P.A.W.(1977), An exponential moving average sequence and point process, (EMA(1)), *Journal of Applied Probability,* Vol.14, 98-113.

(15) Lawrance, A.J. and Lewis, P.A.W. (1980), The exponential autoregressive-moving average EARMA(p,q) process, *Journal of the Royal Statistical Society, Series B,* Vol. 42, No.2, 150-161.

(16) Lawrance, A.J. and Lewis, P.A.W.(1981), A new autoregressive time series model, in exponential variables (NEAR(1)), *Advances in Applied Probability,* Vol.13, 826-845.

(17) Lawrance, A.J. and Lewis, P.A.W.(1982), A mixed exponential time serices model *Management Science,* Vol.28, 1045-1053.

(18) Lawrance, A.J. and Lewis, P.A.W.(1985), Modelling and residual analysis of nonlinear autoregressive time series in exponential variables, *Journal of the Royal Statistic al Society, Series* B, Vol.47, No.1, 165-202.

(19) Lawrance, A.J. and Kottegoda, N.T.(1977), Stochastic modelling of river flow time series, *Journal of the Royal Statistical Society, Series A,* Vol.140, 1-47.

(20) Lewis, P.A.W.(1985), Some simple models for continuous variate time series, NPS55-85-009, *Naval Postgraduate School, Monterey,* CA.

(21) Lewis, P.A.W., McKenzie, E. and Hugus, D.K.(1985), Gamma processes, *Naval Postgraduate School, Monterey,* CA.

(22) Linde, Y. and Gray, R.M.(1978), Fake process approach to data compression, *IEEE Transation Communications,* COM-26, 840-847.

(23) McGill, W.J.(1962), Random fluctuations of response rate, *Psychometrika,* Vol.27, 3-17.

(24) Nelson, H.L. and Granger, C.W.J.(1979), Experience with using the Box-Cox transformation when forecasting economic time series, *Journal of Econometrics,* Vol.10, 57-70.

(25) Raftery, A.E.(1980), Un processus autorégressif á loi marginal exponentielle : propri étés asymptotiques et estimation de maximum de vraisemblance, *Ann.Sci.Univ.Clermont.*

(26) Raftery, A.E.(1982), Generalised non-normal time series models, *Time Series Analysis: Theory and Practice,* Vol.1, 621-640.

(27) Reininger, R.C. and Gibson. J.D.(1983), Distribution of the two-dimensional DCT coefficients for images, *IEEE Transaction Communications,* COM-31, 835-839.

(28) Uddenfeldt, J. and Zettenberg, L.H.(1976), Algorithm for delayed encoding in delta modulation with speech-like signals, *IEEE Transaction Communications,* Vol. COM24, 652-658.

(29) Weiss, G.(1975), Time reversibility of linear stochastic proceses, *Journal of Applied Probability,* Vol.12, 831-836.