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# Distribution-Free k-Sample Tests for Ordered Alternatives of Scale Parameters

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## ABSTRACT

For testing homogeneity of scale parameters against ordered alternatives, some nonparametric test statistics based on pairwise ranking method are proposed. The proposed tests are distribution-free. The asymptotic distributions of the proposed statistics are also investigated. It is shown that the Pitman efficiencies of the proposed rank tests are the same as those of the corresponding two-sample rank tests in the scale problem. A small-sample Monte Carlo study is also performed. The results show that the proposed tests are robust and efficient.

## 1. Introduction

Statistical inferences in the presence of order conditions have been developed by many investigators in various contexts. But most of them have been developed for the ordered location shifts problem and only a few were concerned with the ordered scale changes problem. The parametric or nonparametric approaches for ordered alternatives of locations were widely investigated by, among others, Jonckheere (1954), Bartholomew (1959a, 1959b, 1961), Chacko(1963) , Puri(1965a), Barlow, et al. (1972), and Fairley and Fligner (1987). On the other hand, the ordered scale changes problem was investigated by Govindarajulu and Haller (1977), Govindarajulu and Gupta (1978), and Rao (1982).

Let $X_{i1}, \cdots, X_{in_i}$ be a random sample from the pupulation with a cumulative distribution function (cdf) $F_i(X)$ for $i=1,2,\cdots,k$. We assume that $F_i(X)=F((x-\theta_i)/\sigma_i)$ for some absolutely continuous cdf $F(x)$ with zero median, where the parameters $\theta_i$ and

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$\sigma_i$ are unknown. Here we may be interested only in scale parameters $\sigma_i$ for $i=1,2,\cdots$, k. In some experimental situations there may be scale changes with the same locations. Moreover, if additional information is available for the ordering of $\sigma_i$, then it will be advantageous to make use of this extra information to obtain more powerful tests. For example, often a technologist may be interested in the homogeneity of products which are made by several machines with different purchase ages. In this case, the variances for the quality of products may have increasing trend according to the years at which the machines are bought.

The above situation can be formulated as follows. Assume that $\theta_1=\theta_2=\cdots=\theta_k=\theta$, where $\theta$ denotes a common location, and that $\sigma_1\leq\sigma_2\leq\cdots\leq\sigma_k$ without loss of generality. The null hypothesis

$$H_0:\sigma_1=\sigma_2=\cdots=\sigma_k \tag{1.1}$$

will be tested against the ordered alternative

$$H_1:\sigma_1\leq\sigma_2\leq\cdots\leq\sigma_k, \tag{1.2}$$

where at least one strict inequality holds. In particular, when $\sigma$'s are restricted to the form $\sigma_i=(1+\delta\cdot c_i)\sigma_0$ with $\sigma_0$ and c's denoting specified constants, some investigators such as Govindarajulu and Haller (1977), Govindarajulu and Gupta (1978) have proposed test statistics for testing

$$H_0^*:\sigma_1=\sigma_2=\cdots=\sigma_k=\sigma_0 \tag{1.3}$$

against

$$H_1^*:\sigma_i=(1+\delta\cdot c_i)\sigma_0, \quad i=1,2,\cdots k, \tag{1.4}$$

where $\delta>0$, and c's are increasing positive constants. The test statistics by Govindarajulu and Gupta (1978) for testing $H_0^*$ against $H_1^*$ will be introduced in later section. The hypotheses $H_0^*$ and $H_1^*$ are more restrictive than the $H_0$ and $H_1$ which are considered in our problem settings.

In this paper we propose some nonparametric tests for testing $H_0$ against $H_1$ based on the pairwise ranking method. The asymptotic distributions of the proposed statistics are investigated by using the results of Puri(1965a) under some regularity conditions. We also compare the small-sample empirical powers of the proposed tests through a Monte Carlo study.

## 2. Linear Rank Statistics

### 2.1. Proposed Test Statistics

In this section we propose a class of rank statistics, based on the pairwise ranking method, for testing $H_0$ against $H_1$ given in (1.1) and (1.2), respectively.

Let $R_{ir}^{(i,j)}$ be the rank of $X_{ir}$ in the combined sample of the $i$-th and $j$-th samples only. Let $R_{js}^{(i,j)}$ be defined similarly. We define

$$S_{ij} = S_i^{(i,j)} - S_j^{(i,j)}, \quad 1 \leq i < j \leq k, \tag{2.1}$$

where

$$S_i^{(i,j)} = n_i^{-1} \sum_{r=1}^{n_i} a_{n_{ij}}(R_{ir}^{(i,j)})$$

and

$$S_j^{(i,j)} = n_j^{-1} \sum_{s=1}^{n_j} a_{n_{ij}}(R_{js}^{(i,j)}) .$$

Here $n_{ij} = n_i + n_j$ and $a_{n_{ij}}(\cdot)$ denotes scores which depend on sample sizes. We also let $N = \sum_{i=1}^{k} n_i$ and assume that $\min(n_1, \cdots, n_k) \to \infty$, with $n_i / N \to \lambda_i$, for $0 < \lambda_i < 1$. If the scores for the two-sample scale problem are applied, the statistic $S_{ij}$ may detect scale changes between the $i$-th and $j$-th populations.

We propose a test statistic of the form

$$S_N = \sum_{i<}^{k} \sum_{j}^{k} n_i n_j S_{ij} \tag{2.2}$$

for testing $H_0$ against $H_1$. The statistic $S_N$ is distribution-free and form a class according to the types of scores. The scores of Ansari-Bradley, Mood, and Klotz, which are commonly used for the two-sample scale problem, will be applied. The statistic $S_N$ has the same form as the statistic suggested by Puri(1965a) for the ordered location shifts problem. The statistic $S_N$ is a linear combination of $S_{ij}$ and hence it can detect scale changes when the scores for the two-sample scale problem are used.

## 2.2. Asymptotic Distributions

Let $F_{n_i}(x)$ be the empirical cdf of $X_{i1},\cdots,X_{in_i}$ and let

$$F_{n_{ij}}(x) = (n_i/n_{ij})F_{n_i}(x) + (n_j/n_{ij})F_{n_j}(x)$$

be the combined empirical cdf of the $i$-th and $j$-th samples. Similarly the combined population cdf of the $i$-th and $j$-th samples is defined by

$$F_{ij}(x) = (n_i/n_{ij})F_i(x) + (n_j/n_{ij})F_j(x).$$

Then the statistic $S_{ij}$ defined in (2.1) is equivalent to the following representation

$$S_{ij} = \int_{-\infty}^{\infty} \phi_{n_{ij}}(F_{n_{ij}}(x))\ d(F_{n_i}(x) - F_{n_j}(x)),$$

where $\phi_{n_{ij}}(.)$ is a function defined on the interval (0,1) such that $\phi_{n_{ij}}(s/n_{ij}) = a_{n_{ij}}(s)$, for $s=1,\cdots,n_{ij}$. While $\phi_{n_{ij}}(.)$ need be befined only at $1/n_{ij},\cdots,n_{ij}/n_{ij}$, we can extend its domain of definition to (0,1) for convenience.

We need some regularity conditions to obtain the asymptotic normality of the statistic $S_N$.

[A1] $\phi(u) = \lim_{N\to\infty} \phi_N(u)$ exists for $0<u<1$ and not a constant

[A2] $\int_{0<F_{n_{ij}}<1} [\phi_{n_{ij}}(F_{n_{ij}}(x)) - \phi(F_{n_{ij}}(x))]\, dF_{n_i}(x) = o_p(N^{-1/2})$

[A3] $\phi_N(1) = o(N^{1/2})$

[A4] $|\phi^{(i)}(u)| = |\frac{d^i\phi}{du^i}| \le M[u(1-u)]^{-1-1/2+\delta}$, for $i=0, 1, 2$, and

( for some $\delta>0$, where M is a constant which may depend on $\phi_N$ but not depend on $F_i(X)$. )

The conditions [A1]~[A4] are called Chernoff-Savage conditions.

**Lemma 2.1.** Assume that the conditions [A1] through [A4] are satisfied. Then under $H_0$ the random vector $N^{1/2}(S_{12}, S_{13}, \cdots, S_{k-1,k})$ has a joint asymptotic normal distribution with zero mean and covariance matrix with elements

$$\sigma_{ij}^2 = (\lambda_i^{-1} + \lambda_j^{-1}) A^2, \tag{2.3}$$

and

$$\sigma_{ij,rs} = \begin{cases} 0, & \text{if } i,j,r,s \text{ are distinct}, \\ \lambda_i^{-1} A^2, & \text{if } i=r,\ j \neq s, \\ \lambda_j^{-1} A^2, & \text{if } i \neq r,\ j=s, \\ -\lambda_i^{-1} A^2, & \text{if } i=s,\ j \neq r, \\ -\lambda_j^{-1} A^2, & \text{if } i \neq s,\ j=r, \end{cases} \tag{2.4}$$

where

$$A^2 = \int_0^1 \phi^2(u)\,du - \{\int_0^1 \phi(u)\,du\}^2. \tag{2.5}$$

**Proof.** The asymptotic normality of $N^{1/2}(S_{12}, S_{13}, \cdots, S_{k-1,k})$ follows from Puri (1965a). The mean vector $N^{1/2}(\mu_{12}, \mu_{13}, \cdots, \mu_{k-1,k})$, where

$$\mu_{ij} = \int_{-\infty}^{\infty} \phi(F_{ij}(x))\, d(F_i(x) - F_j(x)), \tag{2.6}$$

is zero when $H_0$: $\sigma_1 = \sigma_2 = \cdots = \sigma_k$ holds. The elements $\sigma_{ij}^2$ and $\sigma_{ij,rs}$ can also be calculated from Puri (1965a). ■

From Lemma 2.1 we obtain the asymptotic normality of the statistic $S_N$ in the following theorem.

**Theorem 2.1.** Assume that the conditions [A1] through [A4] are satisfied. Then under $H_0$ the statistic $N^{-3/2}S_N$ has an saymptotic normal distribution with zero mean and variance

$$\sigma^2(S_N) = \{(\sum_{i=1}^k \lambda_i)^3 - \sum_{i=1}^k \lambda_i^3\}(A^2/3), \tag{2.7}$$

wher $A^2$ is defined by (2.5).

**Proof.** We note that the statistic $S_N$ is a linear combination of $S_{ij}$ for $i<j$, and hence the asymptotic normality of $N^{-3/2}S_N$ follows from Lemma 2.1. After some algebraic calculations, the asymptotic variance of $N^{-3/2}S_N$ can also be obtained as in (2.7). ■

Next we investigate the asymptotic distribution of the proposed test statistic $S_N$ under a sequence of alternatives of the form

$$H_{1N}: \sigma_i = 1 + N^{-1/2}\delta_i, \quad i = 1, 2, \cdots, k, \tag{2.8}$$

where $\delta_1 \le \delta_2 \le \cdots \le \delta_k$ with at least one strict inequality. The following two additional conditions will be necessary in the remaining of this section.

[A5] The cdf $F(x)$ is differentiable in each of the open intervals $(-\infty, a_1)$, $(a_1, a_2)$, $\cdots, (a_{s-1}, a_s)$, $(a_s, \infty)$, where $a_1, \cdots, a_s$ denote a finite number of reals, and $F'(x)$ is bounded in each of these intervals.

[A6] The function $\frac{d}{dx}\phi(F(x))$ is bounded as $x \to \pm\infty$.

The following theorem, which is a modification of Puri(1965a) for the location shifts model, can be obtained for the scale changes model.

**Theorem 2.2.** For each index N we assume that $n_i = N \cdot \lambda_i$ with $\lambda_i$ a positive integer. If the regularity conditions [A1] through [A6] hold, then under $H_{1N}$ the random vector $N^{-3/2}(V_{12}, V_{13}, \cdots, V_{k-1,k})$, where $V_{ij} = n_i n_j (S_{ij} - \mu n_{ij})$ with $\mu_{ij}$ as given in (2.6), has a joint asymptotic normal distribution with zero mean and covariance matrix with elements $(\lambda_i\lambda_j)^2\sigma_{ij}^2$ and $(\lambda_i\lambda_j\lambda_r\lambda_s)\sigma_{ij,rs}$, where $\sigma_{ij}^2$ and $\sigma_{ij,rs}$ as defined in (2.3) and (2.4), respectively.

**Proof.** The asymptotic normality under $H_{1N}$ also follows from Puri(1965a) since the result of Puri(1965a) can be applied to any models with absolutely continuous cdf $F_i(x)$. We also obtain the covariance terms under the model $F_i(x) = F((x-\theta)/\sigma_i)$ with $\sigma_i = 1 + N^{-1/2}\delta_i$ by using convergence theorem. ■

We now obtain the asymptotic normal distribution of $S_N$ under $H_{1N}$ as follows.

**Theorem 2.3.** For each index N we assume that $n_i = N \cdot \lambda_i$, with $\lambda_i$ a positive integer. Assume also that the regularity conditions [A1] through [A6] are satisfied. Then under $H_{1N}$ the statistic $N^{-3/2}S_N$ has an asymptotic normal distribution with mean

$$\sum_{i<}^{k}\sum_{j}^{k} \lambda_i\lambda_j(\delta_i - \delta_j)\int_{-\infty}^{\infty} x\frac{d}{dx}\phi(F(x))\, dF(x) \tag{2.9}$$

and variance $\sigma^2(S_N)$ given by (2.7).

**Proof.** First, we note that the variance $\sigma^2(S_N)$ may be obtained from Theorem 2.2 in relation to Theorem 2.1 and Lemma 2.1. Next, to derive (2.9) we need two additional conditions

[A5] and [A6]. Let $\sigma^{(N)}$ denote the sequence of $\sigma$'s given by $H_{1N}$, and we denote the asymptotic mean of $S_{ij}$ as $\mu_{ij}(\sigma^{(N)})$, where $\mu_{ij}(\sigma^{(N)})$ is defined by (2.6) with $F_i(x)$ under $H_{1N}$. If we let $\mu_{ij}(1)$ be the asymptotic mean of $S_{ij}$ under $H_0$ for convenience, then $\mu_{ij}(1)=0$.

We may assume that $\theta=0$ to simplify the computation. Thus the cdf is reduced to $F_i(x)=F(x/\sigma_i)$, and we compute $N^{1/2}\mu_{ij}(\sigma^{(N)})$ as

$$N^{1/2}\{\mu_{ij}(\sigma^{(N)})-\mu_{ij}(1)\}=N^{1/2}\{\int \phi(F_{ij}(x))-\int \phi(F(x))\,dF(x)\}$$

$$-N^{1/2}\{\int \phi(F_{ij}(x))\,dF_j(x)-\int \phi(F(x))\,dF(x)\}. \quad (2.10)$$

Following Puri (1965b), it can be shown that the first part of (2.10) converges to

$$\{\lambda_j/(\lambda_i+\lambda_j)\}(\delta_i-\delta_j)\int x\frac{d}{dx}\phi(F(x))\,dF(x).$$

A similar result for the second part of (2.10) can be obtained. Detailed computations may be referred to Jeong (1988). We thus obtain that

$$\lim_{N\to\infty} N^{1/2}\mu_{ij}(\sigma^{(N)})=(\delta_i-\delta_j)\int_{-\infty}^{\infty} x\frac{d}{dx}\phi(F(x))\,dF(x).$$

This completes the proof since the statistic $N^{-3/2}S_N$ is a linear combination of $N^{1/2}S_{ij}$, for $i<j$. ■

## 2.3. Some Rank Tests

In Section 2.1 we proposed a class of test statistics based on the pairwise ranking method. We now consider some special cases of the test statistics. For example, the scores of Ansari-Bradley, Mood, and Klotz, which are commonly used for the two-sample problem, may be applied to the k-sample problem.

When the scores of Ansari-Bradley, Mood, and Klotz are applied to (2.1) and (2.2), we obtain the Ansari-Bradley type statistic $S_{AB}$, the Mood type statistic $S_M$, and the Klotz type statistic $S_K$. The scores $a_n(i)$ and the score functions $\phi(u)$ of these statistics are summarized in the following.

In Klotz scores, $\Phi^{-1}$ denotes the inverse of the standard normal cdf. We would reject $H_0$

| Type | Statistic | Score $a_n(i)$ | Score Function $\phi(u)$ |
|---|---|---|---|
| Ansari-Bradley | $S_{AB}$ | $\frac{n+1}{2n} - \left\lvert \frac{i}{n} - \frac{n+1}{2n} \right\rvert$ | $\frac{1}{2} - \left\lvert u - \frac{1}{2} \right\rvert$ |
| Mood | $S_M$ | $\left(\frac{i}{n} - \frac{n+1}{2n}\right)^2$ | $\left(u - \frac{1}{2}\right)^2$ |
| Klotz | $S_K$ | $\left[\Phi^{-1}\left(\frac{i}{n+1}\right)\right]^2$ | $[\Phi^{-1}(u)]^2$ |

for large values of $S_{AB}$, and for small values of $S_M$ and $S_K$.

Under the sequence of alternatives $H_{1N}$, the asymptotic means of the test statistics can be obtained from (2.9). The asymptotic means of $N^{-3/2}S_{AB}$, $N^{-3/2}S_M$, and $N^{-3/2}S_K$ are given by the following.

$$\mu(S_{AB}) = \sum_{i<j}^{k}\sum^{k} \lambda_i \lambda_j (\delta_i - \delta_j) \left\{ \int_{-\infty}^{0} x f^2(x)\,dx - \int_{0}^{\infty} x f^2(x)\,dx \right\}$$

$$\mu(S_M) = \sum_{i<j}^{k}\sum^{k} \lambda_i \lambda_j (\delta_i - \delta_j) \int_{-\infty}^{\infty} x f^2(x)(2F(x)-1)\,dx$$

$$\mu(S_K) = 2\sum_{i<j}^{k}\sum^{k} \lambda_i \lambda_j (\delta_i - \delta_j) \int_{-\infty}^{\infty} x f^2(x) \{ \Phi^{-1}(F(x)) / \phi[\Phi^{-1}(F(x))] \}\,dx .$$

To compare the proposed tests with others, we now introduce some parametric and nonparametric tests in the following.

Govindarajulu and Gupta (1978) have developed some test statistics for testing $H_0^*$ against $H_1^*$ given in (1.3) and (1.4), respectively. First, the likelihood derivative test statistic is defined by

$$L_{1N} = N^{-1/2} \sum_{i=1}^{k} c_i \sum_{j=1}^{n_i} (X_{ij} - \bar{X})^2,$$

where $\bar{X} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{ij}/N$ and the constants c's depend on $H_1^*$. The statistic $L_{1N}$ is a parametric test statistic. Second, the locally most powerful rank test (LMPRT) statistic for normal alternatives is of the form

$$L_{2N} = N^{-1/2} \sum_{i=1}^{k} c_i \sum_{j=1}^{n_i} E[Z^2_{(R_{is})}],$$

where $R_{is}$ denotes the rank of $X_{is}$ among the combined sample of all k populations, and $Z_{(r)}$ the $r$-th order statistic in a sample of size N drawn from the standard normal distribution. Lastly, Govindarajulu and Gupta (1978) have suggested other class of test statistics of the form

$$L_{3N} = N^{-1/2} \sum_{i=1}^{k} b_i n_i \psi_i ,$$

where

$$\psi_i = \int_{-\infty}^{\infty} \phi_N \left( \sum_{j=1}^{k} \lambda_j F_{n_j}(x) \right) dF_{n_i}(x) .$$

The statistic $L_{3N}$ is a weighted sum of Chernoff-Savage type statistics based on the combined ranking method. For the statistic $L_{3N}$, the optimal weights $b_i = \sum_{j=1}^{k} \lambda_j (c_i - c_j)$ can be chosen to maximize the Pitman efficacy, and the scores of Ansari-Bradley, Mood, and Klotz may be used. The asymptotic normality of the statistics defined above has also been discussed by Govindarajulu and Gupta (1978).

Rao(1982) also proposed a class of test statistics for testing homogeneity against ordered alternatives of scale parameters. Rao (1982) compared his tests with the tests by Govindarajulu and Gupta (1978) in terms of asymptotic relative efficiencies (ARE). But we do not include the tests of Rao in this paper because of their complexity in computation. We note that our proposed statistics for testing $H_0$ against $H_1$ have more availability than the statistics by Govindarajulu and Gupta since little is known about $\sigma$'s beyond their monotonicity in general. Notice also that the statistics $L_{1N}$, $L_{2N}$ and $L_{3N}$ depend on c's.

### 2.4. Asymptotic Relative Efficiencies

The efficacies of the rank tests proposed in Section 2.1 will be computed under a sequence of alternatives of the form

$$H_{1N}^{*} : \sigma_i = 1 + N^{-1/2} \delta \cdot c_i , \quad i = 1, 2, \cdots, k .$$

The $H_{1N}$ can be considered as a particular case of $H_{1N}$ given in (2.8) by letting $\delta_i = \delta \cdot c_i$. To compute the efficacy of the statistic $S_N$ defined in (2.2), we let $\xi = N^{-1/2}\delta$ and $S_N' = N^{-3/2} S_N$. Then from (2.9) the asymptotic mean of $S_N'$ can be represented as

$$\mu(S_N') = \sum_{i<j}^{k}\sum^{k} \lambda_i\lambda_j(c_i - c_j)\{\int_{-\infty}^{\infty} x\frac{d}{dx}\phi(F(x))\, dF(x)\}\cdot\delta$$

$$= N^{1/2}\sum_{i<j}^{k}\sum^{k} \lambda_i\lambda_j(c_i - c_j)\xi\cdot I,$$

where

$$I = \int_{-\infty}^{\infty} x\frac{d}{dx}\phi(F(x))\, dF(x).$$

As a result of routine computation the efficacy of $S_N'$ is given by

$$\mathrm{eff}(S_N') = \frac{\{\sum_{i<j}^{k}\sum^{k} \lambda_i\lambda_j(c_i - c_j)\}^2\cdot I^2}{\{(\sum_{i=1}^{k}\lambda_i)^3 - \sum_{i=1}^{k}\lambda_i^3\}(A^2/3)},$$

where $A^2$ is defined by (2.5). In particular for the case of equal sample sizes and equally spaced constants, that is, when $\lambda_i = 1/k$ and $c_i = i$ for $i = 1, 2, \cdots, k$, the $\mathrm{eff}(S_N')$ could be reduced after some algebraic calculations to

$$\mathrm{eff}(S_N') = (k^2-1)\cdot I^2/(12A^2).$$

The values of $A^2$ and $I^2$ can be evaluated for the scores and score functions of Ansari-Bradley, Mood, and Klotz. We thus obtain the efficacy expressions of the tests based on $S_{AB}$, $S_M$ and $S_K$ as follows.

$$\mathrm{eff}(S_{AB}) = 4(k^2-1)\{\int_{-\infty}^{0} xf^2(x)\,dx - \int_{0}^{\infty} xf^2(x)\,dx\}^2 \tag{2.11}$$

$$\mathrm{eff}(S_M) = 15(k^2-1)\{\int_{-\infty}^{\infty} xf^2(x)(2F(x)-1)\,dx\}^2 \tag{2.12}$$

$$\mathrm{eff}(S_K) = ((k^2-1)/6)\{\int_{-\infty}^{\infty} xf^2(x)\{\Phi^{-1}(F(x))/\phi[\Phi^{-1}(F(x))]\}\,dx\}^2. \tag{2.13}$$

By comparing the expressions in (2.11), (2.12) and (2.13) with those given in Table 1 of Govindarajulu and Gupta (1978), we notice that our proposed rank tests have the same efficacies as the corresponding competitors of $L_{3N}$ when the Ansari-Bradley, Mood, and Klotz scores are used under the settings of this section. Furthermore we note that the Klotz type test $S_K$ has the same efficacy as the LMPRT $L_{2N}$.

We also compare the efficiencies of $S_{AB}$, $S_M$ and $S_K$ in terms of ARE. The ARE's are given by

$$ARE(S_{AB}, S_M) = \left(\frac{4}{15}\right) \frac{\{\int_{-\infty}^{0} x f^2(x)\,dx - \int_{0}^{\infty} x f^2(x)\,dx\}^2}{\{\int_{-\infty}^{\infty} x f^2(x)(2F(x)-1)\,dx\}^2}$$

$$ARE(S_M, S_K) = 90 \frac{\{\int_{-\infty}^{\infty} x f^2(x)(2F(x)-1)\,dx\}^2}{\{\int_{-\infty}^{\infty} x f^2(x) \frac{\Phi^{-1}(F(x))}{\phi[\Phi^{-1}(F(x))]}\,dx\}^2}.$$

We note that the ARE's of $S_{AB}$, $S_M$ and $S_K$ with respect to another statistic coincide with those of the corresponding tests of Ansari-Bardley, Mood, and Klotz for the two-sample problem. The ARE's for the two-sample problem are given in, for example, Table 1 of Klotz (1962).

## 3. Rank-Like Statistics

### 3.1. Proposed Test Statistics

A class of distribution-free rank-like statistics for testing scale changes between two populations has been proposed by Fligner and Killeen (1976), and extended later by Fligner (1979). Our interest is to construct rank-like test statistics for ordered alternatives of scale changes in the k-sample porblem.

Let $\hat{\theta} = \hat{\theta}(X_{11}, \cdots, X_{kn_k})$ be an estimator of the median $\theta$ whose distribution is symmetric in the combined sample $X_{11}, \cdots, X_{kn_k}$. Throughout this section we assume that $\hat{\theta} = \hat{\theta}(X_{11}, \cdots, X_{kn_k})$ is a combined sample median of all k populations if not mentioned otherwise. Thus the distribution of $\hat{\theta}$ is symmetric. Let $Z_{ir} = |X_{ir} - \hat{\theta}|$ for $r = 1, \cdots, n_i$ and $i = 1, \cdots, k$. We also consider the pairwise ranking procedure on the aligned samples $Z_{i1}, \cdots, Z_{in_i}$ for $i = 1, \cdots, k$. That is, we let $Q_{ir}^{(i,j)}$ denote the rank of $Z_{ir}$ among the pairwise combined sample $Z_{i1}, \cdots, Z_{in_i}, Z_{j1}, \cdots, Z_{jn_j}$. Then by Theorem of Fligner, Hogg and Killeen (1976) the rank vector

$$(Q_{i1}^{(i,j)}, \cdots Q_{in_i}^{(i,j)}, Q_{j1}^{(i,j)}, \cdots, Q_{jn_j}^{(i,j)})$$

are equally likely under $H_0$ for each $i < j$.

Let

$$T_{ij}(\hat{\theta}) = n_i^{-1} \sum_{r=1}^{n_i} a_{n_{ij}}(Q_{ir}^{(i,j)}) - n_j^{-1} \sum_{r=1}^{n_j} a_{n_{ij}}(Q_{jr}^{(i,j)})$$

and we define a class of rank-like statistics of the form

$$T_N(\hat{\theta}) = \sum_{i<}^{k} \sum_{j}^{k} n_i n_j T_{ij}(\hat{\theta}) \tag{3.1}$$

The test statistic $T_N(\hat{\theta})$ is distribution-free for testing $H_0$ against the ordered alternative $H_1$. We note that the statistic $T_N(\hat{\theta})$ is of the same form as the rank statistic defined in (2.2), but $T_N(\hat{\theta})$ is not a rank statistic since it is based on the aligned samples $Z_{ir} = | X_{ir} - \hat{\theta} |$. In the remaining of this section $T_{ij}(\hat{\theta})$ and $T_N(\hat{\theta})$ will be abbreviated as $T_{ij}$ and $T_N$ respectively if no confusion occurs.

The following scores will be used in the rank-like statistic $T_N$. That is,

$$a_n(i) = i/n \tag{3.2}$$

$$a_n(i) = (i/n)^2 \tag{3.3}$$

$$a_n(i) = [\Phi^{-1}((n+1+i)/2(n+1))]^2 \tag{3.4}$$

for $i = 1, \cdots, n$. The scores given in (3.2), (3.3), and (3.4) are considered by Fligner and Killeen (1976) for the two-sample rank-like statistics as analogs of the Ansari-Bradley, Mood, and Klotz scores, respectively. We note that the scores in (3.2), (3.3), and (3.4) are strictly increasing for $i = 1, \cdots, n$.

By Lemma 2.1 the random vector $N^{1/2}(T_{12}, T_{13}, \cdots, T_{k-1,k})$ has a joint asymptotic normal distribution, under $H_0$, with zero mean and covariance matrix given by (2.3) and (2.4). We thus can obtain the following theorem.

**Theorem 3.1.** Suppose that the regularity conditions [A1]~[A4] are satisfied. Then under $H_0$ the statistic $N^{-3/2}T_N$ has an asymptotic normal distribution with zero mean and variance $\sigma^2(T_N)$ which is defined by the righthand side of (2.7).

Using the scores given in (3.2), (3.3), and (3.4), three types of rank-like statistics are obtained and will be denoted by $T_{AB}$, $T_M$, and $T_K$, respectively. The asymptotic variances of the statistics can be easily obtained by evaluating $A^2$ in (2.7).

We also define another test statistic as follows.

$$U_N^*(\hat{\theta}) = \sum_{i<j}^{k}\sum^{k} (\text{number of } Z_{ir} \text{ smaller than } Z_{js})$$

$$= \sum_{i<j}^{k}\sum^{k} \sum_{r=1}^{n_i} \sum_{s=1}^{n_j} \Psi(Z_{ir}, Z_{js}), \qquad (3.5)$$

where $\Psi(x,y) = 1$ if $x < y$ and 0 otherwise. The statistic $U_N^*(\hat{\theta})$ is an analog of the well known Jonckheere statistic for testing ordered alternatives of location parameters. Since the statistic $U_N^*(\hat{\theta})$ is distribution-free, the null mean and variance of $U_N^*(\hat{\theta})$ are given by

$$(N^2 - \sum_{i=1}^{k} n_i^2)/4 \qquad (3.6)$$

and

$$\{N^2(2N+3) - \sum_{i=1}^{k} n_i^2 (2n_i+3)\}/72, \qquad (3.7)$$

respectively. The null distribution of $U_N^*(\hat{\theta})$ is asymptotically normal with mean and variance given in (3.6) and (3.7), respectively. The fact that the table (e.g. Table A.8 of Hollander and Wolfe (1973)) for the Jonckheere statistic can be used in determining critical regions is an advantage of $U_N^*(\hat{\theta})$ when sample sizes are small.

### 3.2. Extension to the Case of Unequal Locations

Let $F_i(x)$ be a cdf of the $i$-th population such that $F_i(x) = F((x-\theta_i)/\sigma_i)$ for $i=1, \cdots, k$, where F has zero median and the parameters $\theta_1, \cdots, \theta_k$ are not necessarily equal. We note that the rank-like statistics proposed in the previous section heavily depend on the assumption of a common median for all distributions. Let $\hat{\theta}_i = \hat{\theta}_i(X_{i1}, \cdots, X_{in_i})$ be an estimator of $\theta_i$ based on the $i$-th sample $X_{i1}, \cdots, X_{in_i}$ for each $i=1, \cdots, k$. Throughout this section we assume that $\hat{\theta}_i$ is the $i$-th sample median.

Define

$$U_{ij}(\hat{\theta}_i, \hat{\theta}_j) = (n_i n_j)^{-1} \sum_{r=1}^{k} \sum_{s=1}^{k} \Psi(|X_{ir} - \hat{\theta}_i|, |X_{js} - \hat{\theta}_j|),$$

where $\Psi(x,y)$ as defined in (3.5). We now propose a test statistic of the form

$$U_N(\hat{\boldsymbol{\theta}}) = \sum_{i<j}^{k}\sum^{k} n_i n_j U_{ij}(\hat{\theta}_i, \hat{\theta}_j), \tag{3.8}$$

where $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \cdots, \hat{\theta}_K)'$. The statistic $U_N(\hat{\boldsymbol{\theta}})$ can be considered as a special case of the class of test statistics defined by (3.1) except only that each $\hat{\theta}_i$ is estimated separately by its sample median for each $i=1, \cdots, k$. We note that the statistic $U_N(\hat{\theta})$ is not distribution-free.

To investigate the asymptotic distribution of $U_N(\hat{\theta})$, we let $U_N(\boldsymbol{\theta})$ be defined as in (3.8) with $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \cdots, \hat{\theta}_K)'$ denoting the vector of true parameters $\hat{\theta}_1, \cdots, \hat{\theta}_K$. Then, under $H_0$, $U_N(\hat{\boldsymbol{\theta}})$ is distribution-free and has the same distribution as the well known Jonckheere statistic for ordered alternatives of location shifts. For the two-sample scale problem, Sukhatme (1958) and Raghavachari(1965) have proved the asymptotic normality of their statistics with estimated parameters. More generally we may refer to Randles (1982, 1984), Boos (1986), and De Wet and Randles (1987).

By using the results of Sukhatme(1958) and Randles (1984), we can show that $U_N(\hat{\boldsymbol{\theta}})$ and $U_N(\boldsymbol{\theta})$ have the same asymptotic null distribution. More detailed discussions can be found in Jeong (1988). We thus obtain the following theorem.

**Theorem 3.2.** Suppose that the cdf $F(x)$ is symmetric about zero. Then under $H_0$ the statistic $N^{-3/2}U_N(\hat{\boldsymbol{\theta}})$ has an asymptotic normal distribution with mean

$$N^{1/2}\left(1 - \sum_{i=1}^{k} \lambda_i^2\right)/4$$

and variance

$$\left(1 - \sum_{i=1}^{k} \lambda_i^3\right)/36.$$

## 4. Small Sample Monte Carlo Study

### 4.1. Design of the Experiment

In this section we compare the empirical powers and significance levels of the tests discussed in this paper by a Monte Carlo study. Two kinds of test statistics, rank and rank-like statistics, are proposed. $S_{AB}$, $S_M$, and $S_K$ are rank statistics, and $T_{AB}$, $T_M$, $T_K$, $U_N^*(\hat{\theta})$ and $U_N(\hat{\boldsymbol{\theta}})$ (abbreviated simply as $U_N^*$ and $U_N$, respectively) are rank-like statistics.

We consider various underlying distributions such as the uniform, normal, double exponential, Cauchy and contaminated normal distributions. The cdf of an $\varepsilon$-contaminated normal distribtuion is given by

$$F(x) = (1-\varepsilon)\Phi(x)+\varepsilon\Phi(x/\sigma) .$$

The computations in this Monte Carlo study are carried out on VAX780. We use the subroutine GGUBT in IMSL to generate the uniform random variates. The normal random variates with and without contamination are also generated by using the subroutine GGNPM. The inverse integral transformation is applied to generate the double exponential and Cauchy random variates.

The hypotheses $H_0^*$ and $H_1^*$ given in (1.3) and (1.4), respectively, are assumed in order that our proposed tests and the other tests may be compared under the same settings. The specified constants $\sigma_0$ can be set to be one. For the Cauchy distribution, the value $F^{-1}(0.84)-F^{-1}(0.5)=1.8326$ is used instead of the value of standard deviation. Since the magnitude of scales is usually denoted in terms of ratios, it seems to be appropriate to take c's so that $c_i= i^2$ for $i=1, \cdots, k$. We also take k=3 and equal sample sizes of $n_i=10$. We choose three $\delta$ values, $\delta=0$, and $\delta=0.2$, and $\delta=0.4$. The simulated proportions of rejecting $H_0$ at significance level $\alpha=0.05$ in 1000 replications are summarized in Table 1. The proportions for $\delta=0$ are estimates of the true significance level, and the proportions for other $\delta$ values denote the empirical powers.

To investigate the behaviour of the statistics when the assumption of equal medians is violated, we define a difference in locations in terms of a parameter d defined by

$$d=\int_{\theta_1}^{\theta_2} dF((x-\theta_1)/\sigma_1)$$

where $\theta_1$ and $\theta_2$ denote medians of the first and second populations, respectively. Then d can be interpreted as a fixed amount of probability placed between $\theta_1$ and $\theta_2$ as measured by $F((x-\theta_1)/\sigma_1)$. The vales d=0 and d=0.1 are considered. The value d=0 denotes the case of equal medians.

### 4.2. Simulation Results

The tests based on $S_{AB}$, $S_M$, and $S_K$ seem to perform well in general with respect to significance levels and powers when population medians are equal. The parametric test based on $L_{1N}$ has higher empirical powers than any other tests for the uniform and normal distributions. The test based on $L_{1N}$ is very conservative in controlling Type I error for the uniform distribution which has sort tails. For the Cauchy and contaminated normal distributions, the test based on $L_{1N}$ is out of control in its significance levels. For example, the empirical significance level of $L_{1N}$ for the Cauchy distribution is greater than 0.90 when $\alpha$ is assumed to be 0.05.

The rank-like statistics $T_{AB}$ and $U_N^*$ perform equally well in their powers and significance levels. But the statistics $T_M$ has slightly greater empirical significance levels than nominal level $\alpha$. The tests based on $L_{3N}$ are worse in their powers than any other statistics. We may recommend the statistics $S_{AB}$, $S_M$, $S_K$, $T_{AB}$, and $U_N^*$ when the populations have a common median. Even when there are small failures in the assumption of a common median, the proposed rank statistics perform well in significance levels and powers. But the rank-like statistics are out of control in their significance levels even with a slight violation of the assumption of a common median. This contradicts to the previous work for the two-sample scale problem by Fligner (1979). The test statistic $U_N$ is robust in failures of the assumption of a common median.

**Table 1. Empirical Powers for Tests on Scale**

| | d=0 | | | d=0.1 | | |
|---|---|---|---|---|---|---|
| | $\delta=0$ | $\delta=0.2$ | $\delta=0.4$ | $\delta=0$ | $\delta=0.2$ | $\delta=0.4$ |
| (a) Uniform Distribution | | | | | | |
| $S_{AB}$ | 0.063 | 0.765 | 0.907 | 0.060 | 0.746 | 0.916 |
| $S_M$ | 0.056 | 0.857 | 0.957 | 0.062 | 0.826 | 0.951 |
| $S_K$ | 0.055 | 0.901 | 0.966 | 0.051 | 0.874 | 0.960 |
| $T_{AB}$ | 0.069 | 0.775 | 0.920 | 0.062 | 0.740 | 0.941 |
| $T_M$ | 0.079 | 0.867 | 0.966 | 0.068 | 0.855 | 0.968 |
| $U_N^*$ | 0.056 | 0.758 | 0.909 | 0.055 | 0.720 | 0.932 |
| $U_N$ | 0.048 | 0.592 | 0.809 | 0.048 | 0.572 | 0.844 |
| $L_{1N}$ | 0.005 | 1.000 | 1.000 | 0.006 | 1.000 | 1.000 |
| (AB) | 0.057 | 0.069 | 0.063 | 0.047 | 0.046 | 0.060 |
| $L_{3N}$ (M) | 0.048 | 0.066 | 0.077 | 0.049 | 0.052 | 0.056 |
| (K) | 0.027 | 0.055 | 0.060 | 0.033 | 0.029 | 0.041 |

Table 1. (continued)

| | d=0 | | | d=0.1 | | |
|---|---|---|---|---|---|---|
| | $\delta=0$ | $\delta=0.2$ | $\delta=0.4$ | $\delta=0$ | $\delta=0.2$ | $\delta=0.4$ |
| (b) Normal Distribution | | | | | | |
| $S_{AB}$ | 0.054 | 0.535 | 0.816 | 0.045 | 0.560 | 0.755 |
| $S_M$ | 0.058 | 0.619 | 0.884 | 0.049 | 0.634 | 0.803 |
| $S_K$ | 0.056 | 0.655 | 0.885 | 0.049 | 0.629 | 0.813 |
| $T_{AB}$ | 0.060 | 0.582 | 0.854 | 0.067 | 0.679 | 0.851 |
| $T_M$ | 0.067 | 0.672 | 0.911 | 0.071 | 0.771 | 0.922 |
| $U_N^*$ | 0.053 | 0.552 | 0.846 | 0.061 | 0.660 | 0.834 |
| $U_N$ | 0.047 | 0.477 | 0.767 | 0.047 | 0.528 | 0.740 |
| $L_{1N}$ | 0.052 | 0.999 | 1.000 | 0.070 | 0.998 | 1.000 |
| (AB) | 0.046 | 0.062 | 0.049 | 0.061 | 0.043 | 0.027 |
| $L_{3N}$(M) | 0.044 | 0.067 | 0.061 | 0.059 | 0.044 | 0.027 |
| (K) | 0.024 | 0.050 | 0.044 | 0.031 | 0.019 | 0.019 |
| (c) Double Exponential Distribution | | | | | | |
| $S_{AB}$ | 0.045 | 0.412 | 0.639 | 0.049 | 0.415 | 0.627 |
| $S_M$ | 0.052 | 0.466 | 0.720 | 0.045 | 0.471 | 0.667 |
| $S_K$ | 0.050 | 0.471 | 0.720 | 0.038 | 0.446 | 0.660 |
| $T_{AB}$ | 0.050 | 0.467 | 0.695 | 0.074 | 0.552 | 0.761 |
| $T_M$ | 0.060 | 0.551 | 0.774 | 0.080 | 0.615 | 0.807 |
| $U_N^*$ | 0.043 | 0.447 | 0.678 | 0.065 | 0.528 | 0.742 |
| $U_N$ | 0.037 | 0.406 | 0.658 | 0.037 | 0.427 | 0.647 |
| $L_{1N}$ | 0.108 | 0.991 | 1.000 | 0.127 | 0.995 | 1.000 |
| (AB) | 0.057 | 0.051 | 0.046 | 0.048 | 0.025 | 0.022 |
| $L_{3N}$(M) | 0.060 | 0.056 | 0.059 | 0.049 | 0.031 | 0.025 |
| (K) | 0.037 | 0.035 | 0.043 | 0.029 | 0.019 | 0.015 |

Table 1. (continued)

| | d=0 | | | d=0.1 | | |
|---|---|---|---|---|---|---|
| | $\delta=0$ | $\delta=0.2$ | $\delta=0.4$ | $\delta=0$ | $\delta=0.2$ | $\delta=0.4$ |
| (d) Cauchy Distribution | | | | | | |
| $S_{AB}$ | 0.050 | 0.300 | 0.458 | 0.051 | 0.296 | 0.462 |
| $S_M$ | 0.050 | 0.296 | 0.463 | 0.052 | 0.296 | 0.462 |
| $S_K$ | 0.048 | 0.261 | 0.392 | 0.048 | 0.248 | 0.403 |
| $T_{AB}$ | 0.056 | 0.337 | 0.514 | 0.055 | 0.335 | 0.509 |
| $T_M$ | 0.059 | 0.367 | 0.529 | 0.055 | 0.356 | 0.544 |
| $U_N^*$ | 0.046 | 0.321 | 0.498 | 0.046 | 0.312 | 0.480 |
| $U_N$ | 0.054 | 0.311 | 0.461 | 0.054 | 0.290 | 0.465 |
| $L_{IN}$ | 0.917 | 0.999 | 1.000 | 0.917 | 0.999 | 1.000 |
| (AB) | 0.047 | 0.060 | 0.059 | 0.042 | 0.053 | 0.054 |
| $L_{3N}$(M) | 0.054 | 0.053 | 0.060 | 0.055 | 0.055 | 0.052 |
| (K) | 0.031 | 0.031 | 0.040 | 0.033 | 0.039 | 0.036 |
| (e) Contaminated Normal Distribution ($\varepsilon=0.1$, $\sigma=5$) | | | | | | |
| $S_{AB}$ | 0.044 | 0.508 | 0.717 | 0.044 | 0.411 | 0.581 |
| $S_M$ | 0.050 | 0.544 | 0.753 | 0.050 | 0.416 | 0.612 |
| $S_K$ | 0.050 | 0.489 | 0.708 | 0.053 | 0.376 | 0.538 |
| $T_{AB}$ | 0.050 | 0.552 | 0.760 | 0.083 | 0.638 | 0.824 |
| $T_M$ | 0.061 | 0.610 | 0.818 | 0.100 | 0.690 | 0.869 |
| $U_N^*$ | 0.047 | 0.537 | 0.740 | 0.076 | 0.620 | 0.811 |
| $U_N$ | 0.038 | 0.469 | 0.676 | 0.038 | 0.457 | 0.641 |
| $L_{IN}$ | 0.192 | 0.891 | 0.997 | 0.208 | 0.944 | 1.000 |
| (AB) | 0.050 | 0.044 | 0.054 | 0.036 | 0.023 | 0.025 |
| $L_{3M}$(M) | 0.049 | 0.050 | 0.053 | 0.036 | 0.024 | 0.023 |
| (K) | 0.026 | 0.031 | 0.037 | 0.033 | 0.015 | 0.014 |

Table 1. (continued)

| | d=0 | | | d=0.1 | | |
|---|---|---|---|---|---|---|
| | $\delta=0$ | $\delta=0.2$ | $\delta=0.4$ | $\delta=0$ | $\delta=0.2$ | $\delta=0.4$ |
| (f) Contaminated Normal Distribution($\varepsilon=0.1$, $\sigma=10$) | | | | | | |
| $S_{AB}$ | 0.060 | 0.448 | 0.685 | 0.059 | 0.245 | 0.340 |
| $S_M$ | 0.064 | 0.460 | 0.711 | 0.061 | 0.232 | 0.320 |
| $S_K$ | 0.068 | 0.384 | 0.610 | 0.063 | 0.199 | 0.278 |
| $T_{AB}$ | 0.063 | 0.482 | 0.734 | 0.218 | 0.775 | 0.889 |
| $T_M$ | 0.075 | 0.533 | 0.769 | 0.246 | 0.811 | 0.897 |
| $U_N^*$ | 0.057 | 0.468 | 0.719 | 0.208 | 0.764 | 0.875 |
| $U_N$ | 0.059 | 0.403 | 0.635 | 0.059 | 0.413 | 0.616 |
| $L_{1N}$ | 0.188 | 0.658 | 0.874 | 0.206 | 0.723 | 0.978 |
| (AB) | 0.053 | 0.049 | 0.046 | 0.027 | 0.009 | 0.005 |
| $L_{3N}$(M) | 0.055 | 0.051 | 0.045 | 0.032 | 0.008 | 0.007 |
| (K) | 0.034 | 0.033 | 0.031 | 0.022 | 0.004 | 0.008 |

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