⟨Research Paper⟩

Journal of the Korean
Statistical Society
Vol. 17, No.1, 1988

On Two-Piece Double Exponential Distribution

G.S. Lingappaiah*

ABSTRACT

Two-piece double exponential distribution (TPDE) with one piece ($X \leq 0$) having the scale parameter θ_1 while the other piece ($X > 0$) having θ_2 is considered here. Distribution of the sum of n-independent variables from such a distribution is obtained. Special cases of this distribution are also treated. Next, distribution of the ratio of two independent (TPDE) variables is derived. As an extension, distribution of x_1/x_2x_3 is expressed terms of hypergeometric functions. A small table gives the power of the test regarding double exponential against (TPDE).

1. Introduction

Two-piece distributions are well discussed in Kimber (1985) where their uses and applications are given. John (1982) deals with two-piece normal (TPN) distribution where first half ($x \leq 0$) has σ_1^2 as its variance and second half($x > 0$) has σ_2^2 and few characteristics of TPN such as mean, mode, median, fitting and test of symmetry are discussed. Kimber (1985) also deals with TPN and in particular, estimation in a truncated TPN. Lingappaiah (1987) starts with TPN and obtains TP-χ^2 and TP-F type distribution. Also as an extension, Lingappaiah (1987) gives two-piece generalized Dirichlet type distribution. In this paper, two-piece double exponential (TPDE) distribution with different scale parameters in two pieces is considered. First, distribution of the sum of n-independent (TPDE) variables is

*Department of Mathematics, Sir George Williams Campus Concordia University, Montreal, Canada

obtained under very general conditions where 2n parameters are all different. Special cases where some of these 2n parameters are the same are also dealt with, which includes the distribution of the sum of n-independent double exponential variables. Next, distribution of the ratio of two independent TPDE variables is obtained. As an extension, distribution of x_1/x_2x_3 is also derived. By using the above distributions, one can test whether the sample is from a double exponential or from a TPDE. A small table gives the power of such a test.

Motivation, here is to see how the two-piece exponential (TPDE) situation compares with the simple double exponential (DE) distribution. As noted above, simple and two-piece (TPN) variables and their applications are well described in Kimber (1985) and John (1982) Similarly, it is of interest to discuss double exponential and TPDE variables. It is well known that the exponential model plays a very important role in life tests. TPDE deals with a case when two halves have different averages. That is, $(-\infty,0)$ has θ_1 and $(0,\infty)$ has θ_2 as the averages. Also, sum of the exponential variables, which is distributed as gamma distribution is also of much interest, in the experiments. As can be seen in (12) in Section 2, we have here, in a sense, a generalization of the gamma distribution. Coming to the ratios and products, there is quite a large number of works dealing with the distribution of the products of independent uniform variables, independent beta variables and the like.

Pederzoli (1985) and Gupta and Onukogu (1983) are such two references. Similarly, the distribution of the product of independent TPDE variables is of much interest. Hence, the importance of the distribution of the ratios follow. In general, two-piece situations arise when two halves have different parameters. As such their importance in applications. TPDE, in particular is one such.

2. Distribution of the sum

2a: Case When all 2n parameters $\theta_{1i}, \theta_{2i}, i=1,2,\cdots,n$ are different. Let $x_i, i=1,2,\cdots,n$ be n independent variables, where x_i is a TPDE variable which can be expressed as

$$f(x_i) = \begin{cases} A_i \exp(x_i/\theta_{1i}), & \text{if } x_i \leq 0 \\ A_i \exp(-x_i/\theta_{2i}), & \text{if } x_i > 0 \end{cases} \tag{1}$$

where $A_i=1/(\theta_{1i}+\theta_{2i})$. Let $a_i=1/\theta_{1i}$, $b_i=1/\theta_{2i}$, $a_i>0$, $b_i>0$, $i=1,2,\cdots,n$.
Now $A_i=a_ib_i/(a_i+b_i)$. Now from (1), characteristic function of x_j is given by

$$\phi_j(t)=A_j\left[\frac{1}{(a_j+it)}\,\frac{1}{(b_j-it)}\right] \tag{2}$$

$$= a_jb_j/(a_j+it)(b_j-it)\quad . \tag{2a}$$

Consider the sum $y=x_1+\cdots+x_n$ and the characteristic function of y is

$$\phi_y(t)=\prod_{j=1}^{n}[a_jb_j/(a_j+it)(b_j-it)]\quad . \tag{3}$$

By inverting (3), we get the distribution of y as

$$f_1(y)=\prod_{i=1}^{n}(a_ib_i)\left[\sum_{i=1}^{n}\left\{e^{a_iy}\Big/\prod_{\substack{j=1\\j\neq i}}^{n}(a_j-a_i)\prod_{j=1}^{n}(b_j+a_i)\right\}\right],\quad y<0 \tag{4}$$

$$f_2(y)=\prod_{i=1}^{n}(a_ib_i)\left[\sum_{i=1}^{n}\left\{e^{-b_iy}\Big/\prod_{\substack{j=1\\j\neq i}}^{n}(b_j-b_i)\prod_{j=1}^{n}(a_j+b_i)\right\}\right],\quad y>0 \tag{5}$$

$\int_{-\infty}^{0}f_1(y)dy+\int_{0}^{\infty}f_2(y)dy$ gives, from (4) and (5)

$$\prod_{i=1}^{n}(a_ib_i)\left[\sum_{i=1}^{n}\left\{\frac{1}{a_i\prod_{\substack{j=1\\j\neq i}}^{n}(a_j-a_i)\prod_{j=1}^{n}(b_j+a_i)}+\frac{1}{b_i\prod_{\substack{j=1\\j\neq i}}^{n}(b_j-b_i)\prod_{j=1}^{n}(a_j+b_i)}\right\}\right] \tag{6}$$

and (6) is equal to one for all $a_i,b_i>0$ such that $a_i\neq a_j$, $b_i\neq b_j$, $i,j=1,2,\cdots,n$.

Special case 2b: $a_i=b_i$, $i=1,2,\cdots,n$, $a_i\neq a_j$, $i,j=1,2,\cdots,n$. Now we have n-independent double exponential variables where x_i has the parameters (a_i,a_i). That is n variables with parameters $\theta_1',\theta_2',\cdots,\theta_n'$ respectively.

Now, (4) and (5) reduce to

$$f_1(y)=\prod_{i=1}^{n}(a_i^2)\sum_{i=1}^{n}\left[e^{a_iy}\Big/\prod_{\substack{j=1\\j\neq i}}^{n}(2a_i)(a_j^2-a_i^2)\right],\quad y<0 \tag{7a}$$

$$f_2(y)=\prod_{i=1}^{n}(a_i^2)\sum_{i=1}^{n}\left[e^{-b_iy}\Big/\prod_{\substack{j=1\\j\neq i}}^{n}(2b_i)(b_j^2-b_i^2)\right],\quad y>0 \tag{7b}$$

with $a_i=b_i$, $i=1,2,\cdots,n, a_i\neq a_j$, we get from (7a), (7b),

$$\int_{-\infty}^{0} f_1(y)dy+\int_{0}^{\infty} f_2(y)dy \text{ as}$$

$$a_1^2\cdots a_n^2 \sum_{i=1}^{n} \frac{1}{a_i^2 \prod_{\substack{j=1\\ j\neq i}}^{n} (a_j^2-a_i^2)} \tag{8}$$

and (8) is equal to 1. Incidentally, this author has shown in Lingappaiah (1986), that for any real e_i, $i=1,2,\ldots,n$, $e_i \neq e_j$, $i,j=1,2,\ldots,n$,

$$\sum_{i=1}^{n} \frac{(-1)^{n-1}}{e_i \prod_{\substack{j=1\\ j\neq i}}^{n} (e_i-e_j)} = \frac{1}{e_1 e_2 \cdots e_n}. \tag{9}$$

Eq. (8) is a special case of (9) where all e_i's are positive. Actually Lingappaiah (1986) gives a much more general result, that for any r=0 or a positive integer,

$$\{\prod_{i=1}^{n} e_i\} \sum_{i=1}^{n} \frac{(-1)^{n-1}}{e_i^{r+1} \prod_{\substack{j=1\\ j\neq i}}^{n} (e_i-e_j)} = \sum_{\substack{i_1,\cdots,i_n\\ i_1+\cdots+i_n=r}} \left[\frac{1}{\prod_{j=1}^{n} (e_i^{i_j})}\right]. \tag{10}$$

The sum on RHS of (10) is on all permutations of $i_1,\cdots,i_n$ such that $\sum_{j=1}^{n} i_j=r$. Eq. (9) is a special case of (10) for r=0. For r=1, one gets from (10),

$$(e_1\cdots e_n) \sum_{i=1}^{n} \frac{(-1)^{n-1}}{e_i^2 \prod_{\substack{j=1\\ j\neq i}}^{n} (e_i-e_j)} = \left(\frac{1}{e_1}+\cdots+\frac{1}{e_n}\right). \tag{10a}$$

Similarly for r=1,2,3, etc.

Special case 2c: Now, let all n variables have the same two parameters (a,b). Now $a_i=a_j$, $i,j=1,2,\ldots,n$. That is, we have n-independent TPDE variables all having the same parameter tuple (θ_1, θ_2) where $a=1/\theta_1$, $b=1/\theta_2$. Now because of $a_i=a_j$, (4), (5) are not valid. Still, we can get the distribution of y in a direct way. Now, the characteristic function of $y=x_1+\cdots+x_n$, similar to (3) is,

$$\phi_y(t)= a^n b^n/(a+it)^n(b-it)^n \qquad a,b>0. \tag{11}$$

Inverting (11), we get the distribution of y as

$$f_1(y)=(ab)^n \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{e^{ay}(-y)^{n-r-1}(n)_r}{(a+b)^{n+r}(n-1)!}, \quad y<0$$
$$f_2(y)=(ab)^n \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{e^{-by}y^{n-r-1}(n)_r}{(a+b)^{n+r}(n-1)!}, \quad y>0 \qquad (12)$$

with $(n)_r=n(n+1)\cdots(n+r-1)$. From(12), it follows

$$\int_{-\infty}^{0} f_1(y)dy+\int_{0}^{\infty} f_2(y)dy=\sum_{r=0}^{n-1}\binom{n+r-1}{r}(q^n p^r+p^n q^r) \qquad (13)$$

where $p=a/(a+b)$, $p+q=1$ and (13) is equal to 1.

Special case 2d: $a=b$, now TPDE reduces to double exponential distribution. Now y represents the sum of n-independent double exponential variables each with scale parameter $\theta=(1/a)=(1/b)$. Now $p=q=1/2$ and (12) reduces to

$$f_1(y)=\sum_{r=0}^{n-1}\binom{n+r-1}{r}\frac{e^{ay}a^{n-r}(-y)^{n-r-1}}{2^{n+r}(n-r-1)!}, \quad y<0$$
$$f_2(y)=\sum_{r=0}^{n-1}\binom{n+r-1}{r}\frac{e^{-by}a^{n-r}y^{n-r-1}}{2^{n+r}(n-r-1)!}, \quad y>0 \qquad (14)$$

from (14), we get

$$\int_{-\infty}^{0} f_1(y)dy+\int_{0}^{\infty} f_2(y)dy=\sum_{r=0}^{n-1}\binom{n+r-1}{r}\frac{1}{2^{n+r-1}} \qquad (15)$$

and (15) is equal to 1.

TESTING: Now all the above four cases can be used to test various hypotheses. These four cases are,

case 2a: There are 2n parameters here such as θ_{1i}, θ_{2i}, $i=1,2,\cdots\cdots,n$. This is the case of n independent TPDE variables but not identical.

case 2b: Here $\theta_{1i}=\theta_{2i}=\theta'_i$, $i=1,2,\cdots,n$. There are now n independent double exponential variables.

case 2c: We have here only two parameters. $\theta_{1i}=\theta_1$ and $\theta_{2i}=\theta_2$, $i=1,2,\cdots,n$. This represents the case of n, i.i.d. TPDE variables.

case 2d: Now there is only one parameter, $\theta_{1i}=\theta_{2i}=\theta$, $i=1,2,\cdots\cdots,n$. This represents the case of n, i.i.d. DE variables.

(i) Now considering cases 2b and 2d, one can test whether $\theta'_1=\theta'_2\cdots\cdots=\theta'_n$ or not. That is, one can test i.i.d. DE situation against independent only DE situation.

(ii) Cases 2c and 2d can be used to test i.i.d. TPDE variables against DE variables. That is $\theta_1=\theta_2$ against $\theta_1 \neq \theta_2$. Now one uses (12) and (14) to evaluate α and $1-\beta$.

Now From(12), we get

$$\int_{-\infty}^{0} f_1(y)dy + \int_0^t f_2(y)dy = 1-\alpha$$

$$= \sum_{r=0}^{n-1} \binom{n+r+1}{r} \left[q^n p^r + p^n q^r \int_0^t \frac{e^{-by}(by)^{n-r-1}\, b}{\Gamma(n-r)} dy \right] \tag{16}$$

$$\text{Using } \int_0^t \frac{e^{-ax}(ax)^{p-1}}{\Gamma(p)} dx = 1 - \sum_{k=0}^{p-1} \frac{e^{-at}(at)^k}{k!} . \tag{16a}$$

We get from (16)

$$\alpha = p^n \sum_{r=0}^{n-1} \binom{n+r-1}{r} q^r \sum_{r=0}^{n+r-1} \frac{e^{-bt}(bt)^k}{k!} . \tag{17}$$

One can use (17) to get tail probabilities. Following table gives the value of $1-\beta$ for few values of a,b.

Table 1. Double exponential vs TPDE

Eq. (17), n=2, t=1.6;a=b=2, α=.0530

a	b	p	q	(a/b)	$1-\beta$=p(t>1.6)
2	2	.5	.5	1.0	.0530
.6	.4	.6	.4	1.5	.4632
.6	.2	.75	.25	3.0	.7434
.8	.2	.8	.2	4.0	.7993
.5	.1	5/6	1/6	5.0	.8835
.9	.1	.9	.1	9.0	.9387
.98	.02	. 98	.02	49.0	.9636

Eq.(16) can be used to test whether a=b or not. That is, $\theta_1=\theta_2$ against $\theta_1 \neq \theta_2$. This means testing TPDE against DE. Table 1 below gives the tail probabilities when a=b and when a$\neq$b. Last column gives the power. Sample size 2 is chosen just for the simplicity of calculation. One can evaluate power taking sample size as 4,8,12 etc. Now, only computation will be slightly heavy. Last column of Table 1 shows, that the power increases the descrepancy between a and b increases. That is, TPDE quite different from DE. This is quite intuitive.

3. Distribution of ratios

3a. Distribution of x_1/x_2: If x, y are independent variables each having TPDE given in (1), then we have for $z=x/y$, $(x=x_1, y=x_2)$, $[(\theta_1, \theta_2)$ for both x and y$]$,

$$\begin{aligned} &z>0,\ (x>0,\ y>0),\ (x<0,\ y<0) \\ &z\leq 0,\ (x\geq 0,\ y<0),\ (x\leq 0,\ y>0)\ . \end{aligned} \tag{18}$$

Case (i). $z>0$ $(x>0, y>0)$: Now

$$f(x,y)=A^2e^{-b(x+y)} \tag{19}$$

with $A=ab/(a+b)$, and with $z=x/y$, we get

$$f(z)=A^2/b^2(1+z)^2,\ z>0\ . \tag{19a}$$

Similarly for the case $z>0$ $(x<0, y<0)$, we have

$$f(x,y)=A^2e^{-a(x+y)} \tag{20}$$

and

$$f(z)=A^2/a^2(1+z)^2,\ z>0\ . \tag{20a}$$

Case (ii). $z\leq 0$ $(x\leq 0, y\geq 0)$: Now

$$f(x,y)=A^2e^{-(bx+ay)} \tag{21}$$

and

$$f(z)=\left(\frac{A^2}{ab}\right)\left[\frac{(b/a)}{[1+(b/a)z]^2}\right],\ z\leq 0\ . \tag{21a}$$

Similarly for $z\leq 0$ $(x\leq 0, y\geq 0)$, we get

$$f(x,y)=A^2e^{-(ax+by)} \tag{22}$$

$$f(z)=\left(\frac{A^2}{ab}\right)\left[\frac{(a/b)}{[1+(a/b)z]^2}\right],\ z\leq 0\ . \tag{22a}$$

Combining (19a), (20a), (21a) and (22a), we can write

$$f(z)=\begin{cases} A^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\frac{1}{(1+z)^2}, & z>0 \\ \frac{A^2}{ab}\left[\frac{(a/b)}{[1+(a/b)z]^2}+\frac{(b/a)}{[1+(b/a)z]^2}\right], & z\leq 0 \end{cases} \quad (23)$$

It is easy to see from (23),

$$\int_{-\infty}^{0} f(z)\,dz+\int_{0}^{\infty} f(z)\,dz=A^2\left[\frac{a^2+b^2+2ab}{a^2b^2}\right] \quad (23a)$$

and (23a) is equal to 1.

3b. Distribution of x_1/x_2x_3, (θ_1,θ_2, for all x_1,x_2,x_3): Let x_1,x_2,x_3 are three independent TPDE variables with (1) as their p · d · f, then as in 3a section, we have

$$\begin{aligned} u&=z/x_3>0,\ (x_3>0,\ z>0),\ (x_3<0,\ z<0) \\ u&=z/x_3\leq 0,\ (x_3>0,\ z\leq 0),\ (x_3<0,\ z\geq 0) \end{aligned} \quad (24)$$

Case(i). $u>0$, $(z>0, x_3>0)$: Now,

$$f(u,x_3)=CA^3x_3e^{-bx_3}/(1+ux_3)^2 \quad (25)$$

with $C=(1/a^2)+(1/b^2)$.

Now using Erdelyi, *et al* (1953 p. 255) and integrating out x_3 in (25), we get

$$f(u)=\frac{A^3C}{u^2}\psi(2,1;b/u),\ u>0, \quad (25a)$$

where $\psi(a,b;x)$ is the hypergeometric function. Similarly for $u>0$ $(z\leq 0,\ x_3<0)$, we have

$$f(u,x_3)=\frac{A^3}{ab}\left[\frac{(a/b)e^{-ax_3}x_3}{[1+(a/b)ux_3]^2}+\frac{(b/a)x_3e^{-ax_3}}{[1+(b/a)ux_3]^2}\right] \quad (26)$$

and integrating out x_3, it follows

$$f(u)=\frac{A_3}{u^2}\left[\frac{1}{b^2}\psi(2,1;\frac{a^2}{bu})+\frac{1}{a^2}\psi(2,1;b/u)\right],\ u>0 \quad (26a)$$

Case (ii). $u<0$ $(x_3<0, z>0)$: Now,

$$f(u,x_3)=CA^3e^{-ax_3}\,x_3/(1+ux_3)^2 \tag{27}$$

and

$$f(u)=\frac{CA^3}{u^2}\,\psi\,(2,1:a/u),\quad u<0\ . \tag{27a}$$

For the case $u<0$ $(x_3>0, z\leq 0)$, we get

$$f(x,u_3)=A^3\left[\frac{x_3e^{-bx_3}}{b^2[1+(a/b)ux_3]^2}+\frac{x_3e^{-bx_3}}{a^2[1+(b/a)ux_3]^2}\right] \tag{28}$$

and integrating out x_3,

$$f(u)=\frac{A^3}{u^2}\left[\frac{1}{a^2}\,\psi(2,1;\frac{b^2}{au})+\frac{1}{b^2}\,\psi(2,1;a/u)\right],\quad u<0\ . \tag{28a}$$

Combining (25a), (26a), (27a) and (28a), we can write the distribution of $u=z/x_3=x_1/x_2x_3$ as,

$$f(u)=\begin{cases}\dfrac{A^3}{u^2}\left[\dfrac{1}{b^2}\,\psi\,(2,1;\dfrac{a^2}{bu})+\dfrac{1}{a^2}\,\psi\,(2,1;b/u)+C\psi\,(2,1;b/u)\right], & u>0\\[2ex] \dfrac{A^3}{u^2}\left[\dfrac{1}{a^2}\,\psi(2,1;\dfrac{b^2}{au})+\dfrac{1}{b^2}\,\psi(2,1;a/u)+C\psi(2,1;a/u)\right], & u<0\ .\end{cases} \tag{29}$$

For $z=0$, either from (23) or directly from $z=0$ $(x_1=0,\ x_2<0$, or $x_1=0,\ x_2>0)$, we get

$$f(z)=CA^2,\quad z=0\ . \tag{29a}$$

Similarly, for $u=0$, $(z=0,\ x_3<0$ or $z=0,\ x_3>0)$, we get

$$f(u)=C^2A^3,\ u=0. \tag{29b}$$

This can also be seen from (25), (26), (27) and (28).
Now from (29), we get using Erdelyi, *et al* (1953, p. 285),

$$\int_{-\infty}^{0}f(u)\,du=A^3\left[\frac{2}{ba^2}+\frac{C}{b}\right] \tag{30a}$$

and

$$\int_0^{\infty} f(u)\,du = A^3 \left[\frac{C}{a} + \frac{2}{ab^2} \right]. \tag{30b}$$

Combining (30a), (30b), we get

$$A^3 \left[\left(\frac{3}{ba^2} + \frac{1}{b^3} \right) + \left(\frac{3}{ab^2} + \frac{1}{a^3} \right) \right] \tag{31}$$

and (31) is equal to 1.

References

(1) Erdelyi, *et al.* (1953). *Higher Transcendental functions,* Vol. I. McGraw Hill Co., N.Y.

(2) John, S. (1982). The three-parameter two-piece normal family of ditributions and its fitting. *Communications in Statistics (Theory and Methods),* Vol.11, pp.879~886.

(3) Kimber, A. C. (1985). Methods for the two-piece normal distributions. *Communications in Statistics (Theory and Methods),* Vol. 14, pp. 235~245.

(4) Lingappaiah, G. S. (1987). Two-piece Chi-square and F-type distributions. *Publications of the Institute of Statistics* University of Paris. Vol.32. pp. 81~90.

(5) Lingappaiah, G. S. (1986). Variable sample size and inference in samples from gamma populations. *Indian Journal of Pure and Applied Mathematics.* Vol.17, pp. 745~755.

(6) Pederzoli, Giorgio (1985). Some properties of the beta function and the distribution of the product of independent beta random variables. *Trabojos de Estadistica,* Vol.36, pp. 122~128.

(7) Gupta, D. and Onukogu, B. (1983). The distribution of the product of non-central beta variables. *Biometrical Journal,* Vol. 25, pp. 621~624.