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General Laws of the Iterated Logarithm for Levy Processes[+]

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ABSTRACT

Let $\{X(t);\ 0 \le t < \infty\}$ be a real-valued process with stationary independent increments. In this paper, we obtain necessaty and sufficint condition for there to exist a positive, nondecreasing function $\beta(t)$ so that $0 < \lim\sup |X(t)|/\beta(t) < \infty$ a.s. both as t tends to zero and infinity. When no such $\beta(t)$ exists we give a simple integral test for whether $\lim\sup |X(t)|/\beta(t)$ is zero or infinity for a given $\beta(t)$.

1. Introduction

Let $\{X(t);\ 0 \le t < \infty\}$ be a real-valued process with stationary independent increments whose characteristic funtion is given by

$$E \exp\{iu\,X(t)\} = \exp(t\varphi(u))$$

where $\varphi(u) = ibu - \sigma^2 u^2/2 + \int (e^{iux} - 1 - iux/(1+x^2)^{-1})\, d\nu(x)$ with ν (called the Levy measure) satisfying $\int \min(x^2, 1)\, d\nu(x) < \infty$. In this case, it is said that the process $\{X(t)\}$ has Levy representation (b, σ^2, ν).

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The object of this paper is to obtain necessary and sufficient condition to be able to find $\beta(t)$ so that

$$0 < \lim\sup \frac{|X(t)|}{\beta(t)} < \infty \text{ a.s.} \tag{1.1}$$

both as t tends to zero and infinity.
The only assumption that will be made about $\beta(t)$ is that it is a member of the class $\mathcal{M}$ of positive and nondecreasing functions defined on $[0, \infty]$ with $\beta(0)=0$ and $\beta(\infty)=\infty$.

It is well-known that if $X(t)$ is a normalized Brownian motion then

$$\lim\sup \frac{|X(t)|}{\sqrt{2t \log|\log t|}} = 1 \text{ a.s. as } t \to 0 \text{ and } t \to \infty$$

but that if $X(t)$ is a strictly stable process of exponent α $(0<\alpha<2)$ then

$$\lim_{t\to 0} \sup \frac{|X(t)|}{\beta(t)} = 0 \text{ or } \infty \ (\text{resp. } t \to \infty)$$

according as $\int_0^1 \beta(t)^{-\alpha} dt$ is finite or infinite (resp. $\int_1^\infty$) (see Fristedt (1974) p. 361). Thus there doesn't exist such upper function $\beta(t)$ in strictly stable case.

For the case of general Levy processes, Blumenthal and Getoor (1961) showed that

$$\lim_{t\to 0} \sup \frac{|X(t)|}{t^{1/\alpha}} = \begin{cases} 0, & \alpha > \alpha_0 \\ \infty, & \alpha < \alpha_0 \end{cases}$$

$$\lim_{t\to \infty} \sup \frac{|X(t)|}{t^{1/\alpha}} = \begin{cases} 0, & \alpha < \alpha_\infty \\ \infty, & \alpha > \alpha_\infty \end{cases}$$

where

$$\alpha_0 = \inf\{\alpha > 0 : \int_{|x|<1} |x|^\alpha d\nu(x) < \infty\}$$

$$\alpha_\infty = \sup\{0 < \alpha \le 2 : \int_{|x|>1} |x|^\alpha d\nu(x) < \infty\}.$$

These indices α_0 and α_∞ are consistent with the stable exponent if $X(t)$ is a stable process. But at α_0 or α_∞, the behavior is unknown.

As other results related to our concerns, Fristedt (1967) obtained integral test for subordinator $X(t)$ which show whether the lim sup in (1.1) is zero or infinity for all strictly increasing convex $\beta(t)$ as $t\to 0$ and also as $t\to\infty$ if $EX(1)=\infty$. Also he (1971) studied the upper functions for symmetric processes under some additional assumptions. Recently, Griffin (1985) showed that the law of it erated logarithm for symmetric stable processes holds if we ignore the large jumps.

Analogous problems for sums of i.i.d. random variables were solved by Kesten (1972), Pruitt (1981a), Vvedenskaya (1985). This paper was motiviated by Pruitt's paper(1981a), and we obtain the similar results to the case of sums of i.i.d. random variables by replacing distribution function with Levy measure. Section 2 consists of the basic facts and necessary probability estimates which will be used in later section. In section 3 we obtain necessary and sufficient condition for there to exist $\beta(t)\in\mathfrak{m}$ that (1.1) holds. If it fails we give a simple integral test for whether the lim sup in (1.1) is zero or infinity for a given $\beta(t)$. Finally some examples are given.

In order to state our main result in details, we introduce some notation. First we assume that $X(t)$ has no Gaussian part (i.e. $\sigma^2=0$), for if it did then a result of Khinchine (1939) shows that the process behaves near the origin as if the non-Gaussian part were zero. And as usual, we assume that $X(0)=0$ and that we are dealing with a version which has almost all sample functions right continuous and having left limits. Now we define for $a>o$,

$$G(a)=\int_{|x|>a} d\nu(x)$$

$$K(a)=a^{-2}\int_{|x|\le a} x^2\, d\nu(x)$$

$$M(a)=a^{-1}\left\{b+\int_{|x|\le a} x^3/(1+x^2)\,d\nu(x)-\int_{|x|>a} x/(1+x^2)\,d\nu(x)\right\}$$

$$f(a)=G(a)+K(a).$$

It is easy to verify that f is positive, continuous, decreasing and zero at infinity. Also $a^2 f(a)$ is nondecreasing because

$$a^2 f(a)=\int (x^2\wedge a^2)\, d\nu(x).$$

We assume that $\int d\nu(x)=\infty$ since then f is strictly decreasing and $f(a)\to\infty$ as $a\to 0$.

Our main result is as follows:

Theorem. (a) There is a $\beta(t)\in\mathfrak{m}$ such that

$$0 < \limsup_{t \to 0} \frac{|X(t)|}{\beta(t)} < \infty \quad \text{a.s.}$$

if and only if

$$\liminf_{x \to 0} \frac{G(x)}{f(x) + |M(x)|} = 0 \quad . \tag{1.2}$$

If (1.2) fails and $\beta(t) \in \mathcal{M}$ then

$$\limsup_{t \to 0} \frac{|X(t)|}{\beta(t)} = 0 \text{ or } \infty \quad \text{a.s.}$$

according as $\int_0^1 G(\beta(t))\,dt$ converges or diverges.

(b) The similar statements hold if $t \to 0$, $x \to 0$ and $\int_0^1$ are replaced by $t \to \infty$, $x \to \infty$ and $\int_1^\infty$, respectively.

2. Preliminaries

We start with the decomposition of $X(t)$ as follows: Let

$$E \exp\{iu X_a^j(t)\} = \exp\{t\, \varphi_j(u)\}, \quad j = 1,2.$$

where

$$\varphi_1(u) = iu\{b - \int_{|x|>a} x/(1+x^2)\,d\nu(x)\} + \int_{|x|\le a} (e^{iux} - 1 - iux/(1+x^2))\,d\nu(x)$$

$$\varphi_2(u) = \int_{|x|>a} (e^{iux} - 1)\,d\nu(x) \quad .$$

Then $X(t) = X_a^1(t) + X_a^2(t)$, where the two processes are independent. It is easy to show that by differentiating the characteristic function,

$$E X_a^1(t) = taM(a), \quad \mathrm{Var}\, X_a^1(t) = ta^2K(a) \quad .$$

We note $X_a^2(t)$ is a compound Poisson process with parameter $G(a)$, and so

$$P\{X_a^2(s) = 0 \text{ for all } s \le t\} = e^{-tG(a)} \tag{2.1}$$

Now we derive necessary probability estimates which will be used in later section.

Lemma 2.1 For any $s>0$ and $r>0$,

$$P\{X_a^1(t)-EX_a^1(t)\geq 2^{-1}re^r taK(a)+sar^{-1}\}\leq e^{-s} .$$

Proof. Let $u=ra^{-1}$. Then , for $0<\theta<1$,

$$E\exp\{u(X_a^1(t)-E X_a^1(t))\}$$

$$=\exp\{t\int_{|x|\leq a}(e^{ux}-1-ux)\,d\nu(x)\}$$

$$=\exp\{2^{-1}u^2t\int_{|x|\leq a}x^2e^{\theta ux}\,d\nu(x)\} \tag{2.2}$$

$$\leq\exp\{2^{-1}u^2e^{au}ta^2K(a)\}$$

$$=\exp\{2^{-1}r^2e^r tK(a)\} . \tag{2.3}$$

The result now follows from the elementary inequality

$$P\{X\geq c\}\leq E e^{ux}\cdot e^{-uc} \blacksquare$$

Lemma 2.2 If $tK(a)$ is sufficiently large, there exist $c_1, c_2\in(0,1)$ such that

$$P\{X_a^1(t)-EX_a^1(t)\geq c_1 ta K(a)\}\geq\exp\{-c_2tK(a)\} .$$

Proof. Let $Y(t)=X_a^1(t)-EX_a^1(t)$. Then by (2.2),

$$E\exp\{uY(t)\}\geq\exp\{2^{-1}u^2e^{-au}ta^2K(a)\} .$$

Let $r_0>o$ be fixed so that $2r_0^2e^{2r_0}<1$ and $u_0=r_0a^{-1}$. Then

$$E\exp\{u_0Y(t)\}\geq\exp\{2^{-1}r_0^2e^{-r_0}tK(a)\} . \tag{2.4}$$

Integrating by parts, we obtain

$$E\exp\{u_0Y(t)\}=\int_{-\infty}^{\infty}u_0e^{u_0x}P\{Y(t)\geq x\}dx . \tag{2.5}$$

Now we choose $0<c_1<\frac{1}{2}r_0e^{-r_0}$, $2r_0^2e^{2r_0}<c_2<1$ and let

$$\xi_1 = c_1 \, t a K(a), \quad \xi_2 = c_2 \, t \, u_0^{-1} K(a) \; .$$

Then by (2.4), as $tK(a) \to \infty$,

$$\int_{-\infty}^{\xi_1} u_0 e^{u_0 x} P\{Y(t) \geq x\} \, dx \leq e^{u_0 \xi_1} = o\,(E \exp\{u_0 Y(t)\}) \; . \tag{2.6}$$

Next, using Chebyshev's inequality and (2.3),

$$\int_{\xi_2}^{\infty} u_0 e^{u_0 x} P\{Y(t) \geq x\} \, dx \leq \int_{\xi_2}^{\infty} u_0 e^{u_0 x} E \exp\{2u_0 \, Y(t)\} \, e^{-2u_0 x} \, dx$$
$$= E \exp\{2 \, u_0 Y(t)\} \; e^{-u_0 \xi_2} \leq \exp\{(2 \, r_0^2 \, e^{2r_0} - c_2) \, t \, K(a)\} \; .$$

Since the coefficient of $tK(a)$ is negative, this term is also small compared to (2.5) when $tK(a)$ is large. Thus we have by (2.5) and (2.6),

$$E\exp\{u_0 \, Y(t)\}$$
$$\leq \delta \; E \exp\{u_0 \, Y(t)\} + \int_{\xi_1}^{\xi_2} u_0 e^{u_0 x} \, P\{Y(t) \geq x\} \, dx \tag{2.7}$$

where $0 < \delta < 1$ is a fixed constant.
Using (2.4) and (2.7),

$$1 \leq (1 - \delta) \exp\{2^{-1} r_0^2 e^{-r_0} t K(a)\}$$
$$\leq (1 - \delta) \, E \exp\{u_0 Y(t)\}$$
$$\leq P\{Y(t) \geq \xi_1\} \; e^{u_0 \xi_2} . \quad \blacksquare$$

We conclude this section with the statements of two well-known lemmas which will be used in later section.

Lemma 2.3 (Borel-Cantelli). Let $\{A_n\}$, $\{B_n\}$ be two sequences of events such that the events $\{A_n\}$ are independent and for each n,the pair A_n, B_n are independent. Suppose that $\sum P(A_n) = \infty$ and that $P(B_n) \geq c > 0$ for all n. Then

$$P\{A_n \cap B_n \;\; i.o.\} > 0.$$

Proof. See Pruitt (1981a) p.14 or Kochen and Stone (1964).

Lemma 2.4. (Skorohod). Let $\{Y_n\}$ be a sequence of independent random variables and $S_n = \sum_{j=1}^{n} Y_j$. Suppose that

$$P\{S_n - S_k \geq -\xi\} \geq c > 0 \quad \text{for all } k \leq n .$$

Then

$$P\{\max_{k \leq n} S_k \geq \lambda + \xi\} \leq c^{-1} P\{S_n \geq \lambda\} .$$

Proof. see Gihman and Skorohod (1975) p.324.

3. Main results

In this section, we obtain necessary and sufficient condition for there to exist $\beta(t) \in \mathcal{M}$ so that

$$0 < \limsup_{t \to 0} \frac{|X(t)|}{\beta(t)} < \infty \quad \text{a.s.} .$$

If it fails we give a simple integral test for whether the lim sup is zero or infinity for a given $\beta(t)$. Also if $E|X(1)| < \infty$ then we obtain the similar result for the process $X(t)$-$EX(t)$ as its corollary. Finally some examples are given to apply our main theorem. Here we omit the proof for the case of $t \to \infty$ since its behavior is very similar. Throughout this paper, $f(x) \sim g(x)$ implies $f(x)/g(x) \to 1$ as $x \to 0$.

Lemma 3.1. Suppose that for all sufficiently small x

$$f(x) \leq C\,G(x) . \tag{3.1}$$

For any $\beta(t) \in \mathcal{M}$, let

$$a(t) = t\,\beta(2^{-k})\,M(\beta(2^{-k})), \quad 2^{-k} < t \leq 2^{-k+1} .$$

If $\int_0^1 G\{\beta(t)\}\,dt < \infty$ then

$$\lim_{t \to 0} \frac{X(t) - a(t)}{\beta(t)} = 0 \quad \text{a.s.} . \tag{3.2}$$

On the other hand if $\int_0^1 G\{\beta(t)\}\,dt = \infty$ then

$$\limsup_{t \to 0} \frac{|X(t)|}{\beta(t)} = \infty \quad \text{a.s.} .$$

The divergent case is also valid without (3.1) provided $t^{-\lambda}\beta(t)$ increase for some $\lambda>0$.

Proof. For $2^{-k}< t \leq 2^{-k+1}$, let

$$X^1(t)=X^1_{\beta(2^{-k})}(t),\quad X^2(t)=X^2_{\beta(2^{-k})}(t) .$$

Then for any $\varepsilon>0$, by Kolmogorov's inequality and (2.1)

$$P\{\sup_{2^{-k}<t\leq 2^{-k+1}} |X(t)-a(t)|>\varepsilon\beta(2^{-k})\}$$

$$\leq P\{\sup_{2^{-k}<t\leq 2^{-k+1}} |X^1(t)-E\,X^1(t)|>2^{-1}\varepsilon\beta(2^{-k})\}$$

$$+P\{\sup_{2^{-k}<t\leq 2^{-k+1}} |X^2(t)|>2^{-1}\varepsilon\beta(2^{-k})\}$$

$$\leq 4\varepsilon^{-2}\,2^{-k}K\{\beta(2^{-k})\}+1-\exp\{-2^{-k}G(\beta(2^{-k}))\}$$

$$\leq 4\varepsilon^{-2}\,2^{-k}K\{\beta(2^{-k})\}+2^{-k}G\{\beta(2^{-k})\}$$

$$\leq C_\varepsilon 2^{-k}G\{\beta(2^{-k})\}\quad \text{by (3.1)}$$

where C_ε is constant depending only on ε.

Thus,

$$P\{\sup_{2^{-k}<t\leq 2^{-k+1}} \frac{|X(t)-a(t)|}{\beta(t)}>\varepsilon\}$$

$$\leq P\{\sup_{2^{-k}<t\leq 2^{-k+1}} |X(t)-a(t)|>\varepsilon\beta(2^{-k})\}$$

$$\leq C_\varepsilon 2^{-k}G\{\beta(2^{-k})\}\leq 2C_\varepsilon\int_{2^{-k-1}}^{2^{-k}} G\{\beta(t)\}\,dt .$$

This proves the first assertion.

Now suppose that the integral diverges. Since $x^2f(x)$ is nondecreasing, we have by (3.1) that for any $c>1$

$$x^2G(x)\leq x^2f(x)\leq c^2x^2f(cx)\leq Cc^2x^2G(cx) .$$

Thus,

$$\int_0^1 G\{c\beta(t)\}\,dt=\infty \text{ for all } c>1 \tag{3.3}$$

Note that (3.3) is still true without assuming (3.1) when $t^{-\lambda}\beta(t)$ increases since this implies $c^{\lambda}\beta(t) \le \beta(ct)$ for any $c>1$ and so

$$\int_0^1 G\{c^{\lambda}\beta(t)\}dt \ge \int_0^1 G\{\beta(ct)\}\ dt = \infty \quad .$$

Now, suppose that $\limsup_{t\to 0} \dfrac{|X(t)|}{\beta(t)} < \infty$ a.s.

$$\text{i.e. } P\{\,|X(t)| \le \delta\beta(t) \text{ eventually } t\to 0\,\} = 1 \text{ for some } \delta > 0.$$

Then there is no jump greater than $2\delta\beta(t)$ for sufficiently small t. Let

$$A = \{\,(t,x) : t\in[0,1],\ |x| > 2\,\delta\beta(t)\,\}$$

$$N(A) = \text{the number of } t \text{ for which } (t,\ X(t) - X(t^-)) \in A \quad .$$

Note that the random variable $N(A)$ has Poisson distrbution with paramter $(\mu\times\nu)(A)$ – it being understood that

$$P\{N(A) = \infty\} = 1 \text{ if } (\mu\times\nu)(A) = \infty$$

where μ is Lebesgue measure.
Thus,

$$(\mu\times\nu)\ (A) = \int_0^1 G\{2\delta\beta(t)\}\ dt < \infty \quad .$$

This is a contradiction. ■

Lemma 3.2. Suppose that we have an decreasing sequence of truncation points $\{u_k\}$, and a sequence of times $\{t_k\}$ satisfying

$$K(u_k) \sim t_k^{-1} \log k \text{ and } t_{k+1} < 2^{-1} t_k \ . \tag{3.4}$$

For $t_{k+1} < t \le t_k$, let

$$X^1(t) = X^1_{u_k}(t)$$

$$\beta(t) = u_k \log k \quad . \tag{3.5}$$

Then

$$0 < \limsup_{t\to 0} \frac{X^1(t) - EX^1(t)}{\beta(t)} < \infty \quad \text{a.s.} \quad .$$

Proof. Take $c>1$. For $t_{k+1}<t\leq t_k$ and k sufficiently large,

$$\begin{aligned} &P\{\,|X^1(t)-EX^1(t)| \geq (2c\log k)^{\frac{1}{2}}\,u_k\} \\ &\leq (2c\log k)^{-1} u_k^{-2}\ \mathrm{Var}\,X^1(t) \\ &= (2c\log k)^{-1}\, t\, K(u_k) \\ &\leq 1/2 \quad . \end{aligned}$$

Then by lemmas 2.4 and 2.1,

$$\begin{aligned} &P\{\sup_{t_{k+1}<t\leq t_k} \{X^1(t)-EX^1(t)\} \geq 2^{-1} e\, t_k u_k K(u_k) + 2u_k \log k + (2c\log k)^{\frac{1}{2}} \\ &\leq 2P\{X^1(t_k) - EX^1(t_k) \geq 2^{-1} e t_k u_k K(u_k) + 2u_k \log k\} \\ &\leq 2k^{-2} \quad . \end{aligned}$$

Therefore we have for $t_{k+1}<t\leq t_k$ and k sufficiently large,

$$X^1(t) - EX^1(t) \leq 4u_k \log k = 4\beta(t) \quad . \tag{3.6}$$

For the lower bound, we let $r_k = t_k - t_{k+1}$.
Then $t_{k+1} < ½\, t_k \leq r_k < t_k$, and by (3.6),

$$P\{X^1(t_{k+1}) - EX^1_{u_k}(t_{k+1}) \geq -(2c\log k)^{\frac{1}{2}} u_k\} \geq 1/2 \quad .$$

By lemma 2.2 and (3.4), for large k

$$\begin{aligned} &P\{X^1(t_k) - X^1_{u_k}(t_{k+1}) - EX^1(r_k) \geq c_1 r_k u_k K(u_k)\} \\ &\geq \exp\{-c_2 r_k K(u_k)\} \\ &\geq \exp\{-c_2 t_k K(u_k)\} \\ &\sim k^{-c_2} \quad . \end{aligned}$$

By lemma 2.3 and zero-one law, we have infinitely often with probability one

$$\begin{aligned} X^1(t_k) - EX^1(t_k) &\geq c_1 r_k u_k K(u_k) - (2c\log k)^{\frac{1}{2}} u_k \\ &\geq c_3 u_k \log k \end{aligned}$$

for some $c_3 > 0$. This completes the proof. ■

Lemma 3.3. For an arbitrary Levy measure ν,

$$\liminf_{x\to 0} \frac{G(x)}{f(x)+|M(x)|}=0$$

if and only if at least one of the following conditions holds:

$$\liminf_{x\to 0} \frac{G(x)+|M(x)|}{K(x)}=0$$

or

$$\liminf_{x\to 0} \frac{f(x)}{|M(x)|}=0 \quad .$$

Proof. Note that for $c>1$,

$$\begin{aligned}|M(x)| &= x^{-1}\,\Big|\, b+\int_{|y|\le cx} y^3/(1+y^2)\,d\nu(y)-\int_{x<|y|\le cx} y^3/(1+y^2)\,d\nu(y)\\ &\quad -\int_{|y|>cx} y/(1+y^2)\,d\nu(y)-\int_{x<|y|\le cx} y/(1+y^2)\,d\nu(y)\,\Big|\\ &= c\,|M(cx)|+c\,G(x)\quad .\end{aligned}$$

The remainder of the proof is same as Pruitt's lemma (See Pruitt (1981a) p.8). ■

Theorem 3.4. There is a $\beta(t)\in \mathfrak{m}$ such that

$$0<\limsup_{t\to 0}\frac{|X(t)|}{\beta(t)}<\infty \quad \text{a.s.}$$

if and only if

$$\liminf_{x\to 0} \frac{G(x)}{f(x)+|M(x)|}=0 \quad . \tag{3.7}$$

If (3.7) fails and $\beta(t)\in\mathfrak{m}$ then

$$\limsup_{t\to 0}\frac{|X(t)|}{\beta(t)}=0 \quad \text{or} \quad \infty$$

according as $\int_0^1 G\{\beta(t)\}\,dt$ converges or diverges.

Proof. First suppose that (3.7) fails so that there is a positive constant C such that for all sufficiently small x

$$f(x)+|M(x)|\le C\,G(x) \quad .$$

The divergent case follows immediately from lemma 3.1 and for the convergent case we only need to show that

$$a(t)=o(\beta(t)).$$

For $2^{-k}<t\leq 2^{-k+1}$,

$$\begin{aligned}|a(t)| &\leq 2^{-k+1}|M(\beta(2^{-k}))|\beta(t)\\ &\leq C2^{-k+1}G(\beta(2^{-k}))\beta(t)\\ &\leq 4C\,\beta(t)\int_{2^{-k-1}}^{2^{-k}}G\{\beta(t)\}\,dt\\ &=o(\beta(t)).\end{aligned}$$

Now, we suppose that (3.7) is satisfied. Then by lemma 3.3, we must have

$$\liminf_{x\to 0}\frac{G(x)+|M(x)|}{K(x)}=0 \tag{3.8}$$

or

$$\liminf_{x\to 0}\frac{f(x)}{|M(x)|}=0 . \tag{3.9}$$

If (3.8) is satisfied, we can find $\{u_k\}$ such that

$$K(u_{k+1})>2K(u_k)$$

and

$$\sum_k \log k\,\frac{G(u_k)+|M(u_k)|}{K(u_k)}<\infty . \tag{3.10}$$

We let

$$t_k = \log k/K(u_k)$$
$$\beta(t)=u_k\log k,\ t_{k+1}<t\leq t_k .$$

Then (3.4) is satisfied and lemma 3.2 applies so that

$$0<\limsup_{t\to 0}\frac{X^1(t)-EX^1(t)}{\beta(t)}<\infty \quad \text{a.s.} . \tag{3.11}$$

But for $t_{k+1}<t\leq t_k$,

$$|EX^1(t)| \le t_k u_k |M(u_k)|$$
$$\le \beta(t)\ K(u_k)^{-1} |M(u_k)| = o(\beta(t)) . \tag{3.12}$$

Using (2.1) and (3.10)

$$\sum_k P\{X(t)-X^1(t) \ne 0 \ \text{for some}\ t_{k+1} < t \le t_k\}$$
$$\le \sum_k (1-e^{-t_k G(u_k)}) \le \sum_k t_k G(u_k) < \infty .$$

Thus, we have $X(t)=X^1(t)$ for sufficiently small t and (3.11), (3.12) imply

$$0 < \limsup_{t \to 0} \frac{X(t)}{\beta(t)} < \infty \ \text{a.s.} .$$

Considering $\{-X(t)\}$ instead of $\{X(t)\}$, we conclude

$$0 < \limsup_{t \to 0} \frac{|X(t)|}{\beta(t)} < \infty \ \text{a.s.} .$$

Now, suppose that (3.9) is true. Then we can find $\{u_k\}$ such that

$$|M(u_{k+1})| \ge 2 |M(u_k)|$$

and

$$\sum_k f(u_k) |M(u_k)|^{-1} < \infty . \tag{3.13}$$

We define

$$t_k = 1/|M(u_k)|$$
$$\beta(t) = u_k,\ X^1(t) = X^1_{u_k}(t),\ t_{k+1} < t \le t_k .$$

Then we have by Kolmogorov's inequality

$$P\{\sup_{t_{k+1} < t \le t_k} |X^1(t) - EX^1(t)| \ge \varepsilon\beta(t)\} \le \varepsilon^{-2}\ t_k K(u_k)$$

and

$$P\{X(t)-X^1(t) \ne 0 \ \text{for some}\ t_{k+1} < t \le t_k\} \le t_k G(u_k) .$$

Since both of these are summable by (3.13), we have

$$\lim_{t\to 0} \frac{X(t)-EX^1(t)}{\beta(t)}=0 \quad \text{a.s.} \tag{3.14}$$

But for $t_{k+1}<t\leq t_k$,

$$|EX^1(t)|= tu_k\ |M(u_k)| = t\beta(t)\, t_k^{-1}$$

which, in conjunction with (3.14), show that

$$\limsup_{t\to 0} \frac{|X(t)|}{\beta(t)} = 1 \quad \text{a.s.} \quad ■$$

Corollary 3.5. Suppose that $\int_{|x|>1}|x|\, d\nu(x)<\infty$. Then there is a $\beta(t)\in \mathfrak{m}$ such that

$$0<\limsup_{t\to 0} \frac{|X(t)-EX(t)|}{\beta(t)}<\infty \quad \text{a.s.}$$

if and only if

$$\liminf_{x\to 0} \frac{G(x)}{f(x)+|N(x)|}=0 \tag{3.15}$$

where $N(x)=x^{-1}\int_{|y|>x} y\, d\nu(y)$.
If (3.15) fails and $\beta(t)\in \mathfrak{m}$, then

$$\limsup_{t\to 0} \frac{|X(t)-EX(t)|}{\beta(t)}=0 \text{ or } \infty \quad \text{a.s.}$$

according as $\int_0^1 G\{\beta(t)\}\, dt$ converges or diverges.

Proof. Note that

$$\int_{|x|>1} |x|\, d\nu(x) <\infty \quad \text{iff} \quad E|X(t)|<\infty$$

and this implies $EX(t)=t\{b+\int x^3/(1+x^2)\, d\nu(x)\}$.
Thus if the process $\{X(t)\}$ has Levy representation (b,o,ν) then the process $\{X(t)-EX(t)\}$ has Levy representation (b^1,o,ν), where $b^1= -\int x^3/(1+x^2)\, d\nu(x)$.
This proves our assertions. ■

Example 1. Consider the stable case with $\nu(dy)=|y|^{-1-\alpha}\, dy$.

Then

$$G(x) = 2\alpha^{-1} x^{-\alpha},\ K(x) = 2(2-\alpha)^{-1} x^{-\alpha},\ M(x) = bx^{-1}$$

Thus if b=0 or $1 \le \alpha < 2$, then (3.7) fails. But if $b \neq 0$ and $0 < \alpha < 1$ then (3.9) holds. In latter case, if we define

$$u_k = 2^{-k/(1-\alpha)},\ t_k = |b|^{-1} u_k$$
$$\beta(t) = u_k,\ t_{k+1} < t \le t_k$$

then

$$\limsup_{t \to 0} \frac{|X(t)|}{\beta(t)} = 1 \quad \text{a.s.}$$

In case that ν is non-symmetric, if either $\alpha > 1$ or

$$0 < \alpha < 1 \text{ and } b - \int x/(1+x^2)\, d\nu(x) = 0$$

then (3.7) fails. But if either $\alpha = 1$ or

$$0 < \alpha < 1 \text{ and } b - \int x/(1+x^2)\, d\nu(x) \neq 0$$

then (3.9) holds.

Example 2. Let $\nu(dy) = |y|^{-3} |\log y|^{-2} dy$ for $|y|$ small.
Then

$$G(x) + |M(x)| \sim G(x) \sim x^{-2} |\log x|^{-2}$$
$$K(x) \sim 2x^{-2} |\log x|^{-1},$$

and so (3.8) is satisfied. Thus if we define

$$u_k = k^{-2^k},\ t_k = 2^{k-1} k^{-2^{k+1}} (\log k)^2$$
$$\beta(t) = u_k \log k,\ t_{k+1} < t \le t_k$$

then

$$0 < \limsup_{t \to 0} \frac{X(t)}{\beta(t)} < \infty \text{ a.s.}$$

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