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A Contour-Integral Derivation of the Asymptotic Distribution of the Sample Partial Autocorrelations with Lags Greater than p of an AR(p) Model[+]

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ABSTRACT

The asymptotic distribution of the sample partial autocorrelation terms after lag p of an AR(p) model has been shown by Dixon(1944). The derivation is based on multivariate analysis and looks tedious. In this paper we present an interesting contour-integral derivation.

1. Introduction

Consider the autoregressive model of order p, AR(p),

$$\phi(B)y_t = v_t, \tag{1.1}$$

where $\phi(B) = -\phi_0 - \phi_1 B - \dots - \phi_p B^p$, $\phi_0 = -1$, B is the backshift operator and $\{v_t\}$ is a sequence of independent and identically distributed random variables with $E(v_t)=0$, $E(v_t^2)=\sigma^2$ and $E(v_t^4)=3\sigma^4+k_4(<\infty)$. We assume that the process is stationary, i.e., the equa-

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tion $\phi(z)=0$ has all the roots outside the unit circle. Let $\{\sigma(l)\}$ and $\{\rho_l\}$ be the autocovariance function, ACVF, and the autocorrelation function, ACRF, respectively. The stationarity implies that the ACRF satisfies the following Yule-Walker equations:

$$\rho_j = \phi_1 \rho_{j-1} + \cdots + \phi_p \rho_{j-p}, \quad j = 1, 2, \cdots, \tag{1.2}$$

$$\sum_{l=1}^{p} \phi_l \rho_l = -\sigma^2/\sigma(0). \tag{1.3}$$

Henceforth we denote $(x_n,\cdots,x_1)^t$ by $\tilde{\underset{\sim}{x}}$ for any vector $\underset{\sim}{x}=(x_1,\cdots,x_n)^t$. Let $\underset{\sim}{\rho}(k)=(\rho_1,\cdots,\rho_k)^t$. For a positive integer k, we define a k by k Toeplitz matrix B(k) with (l,m) element ρ_{l-m}. Since B(k) is positive definite, we can define some vectors:

$$\begin{aligned} \underset{\sim}{\phi}(k) &= B(k)^{-1} \underset{\sim}{\rho}(k), \\ \theta(k) &= \rho_{k+1} - \tilde{\underset{\sim}{\phi}}(k)^t \underset{\sim}{\rho}(k), \\ \lambda(k) &= 1 - \underset{\sim}{\rho}(k)^t \underset{\sim}{\phi}(k) \end{aligned}$$

for $k = 1,2,\cdots$. Then, $\underset{\sim}{\phi}(k)$ is the solution vector of the Yule-Walker equations (1.2) for $j = 1,\cdots,k$, which can be efficiently solved by the Levinson(1947) and Durbin(1960) algorithm;

$$\underset{\sim}{\phi}(k+1) = \begin{pmatrix} \underset{\sim}{\phi}(k) - \theta(k)\, \tilde{\underset{\sim}{\phi}}(k)/\lambda(k) \\ \theta(k)/\lambda(k) \end{pmatrix}$$

$$\lambda(k+1) = \lambda(k)\{1 - \theta(k)^2/\lambda(k)^2\}.$$

The initial values are $\theta(0) = \rho_1$ and $\lambda(0) = 1$. Hereafter we denote the l-th element of $\underset{\sim}{\phi}(k)$ by $\phi_{k,l}$ and let $\phi_{k,0} = -1$.

For a given realization $\{y_1,\cdots,y_T\}$ we estimate the ACVF and the ACRF by

$$\hat{\sigma}(i) = \frac{1}{T}\sum_{l=1}^{T-i}(y_{l+1} - \bar{y})(y_l - \bar{y}) \quad \text{and} \quad \hat{\rho}_i = \hat{\sigma}(i)/\hat{\sigma}(0)$$

where $\bar{y}=\Sigma y_l/T$. When the sample ACRF is used instead of the ACRF, the solutions of the Yule-Walker equations are called the Yule-Walker estimates of the AR parameters. Throughout this note we use the sample ACRF and the Yule-Walker estimates. Other parameters are estimated by putting the sample ACVF and the Yule-Walker estimates instead of the ACVF and the AR coefficients into their definitions.

It is well-known that the partial autocorrelation at lag k is equal to $\phi_{k,k}$. Thus we can redefine it as follows.

Definition 1.1 The partial autocorrelation at lag k is defined by

$$\phi_{k,k}=\theta(k-1)/\lambda(k-1), \quad k=1,2,\cdots.$$

It is known (Dixon [1944] and Quenouille[1949]) that if the underlying process is from an AR(p) model then $\phi_{k,k}$'s, $k>p$, are asymptotically independent and identical random variables. The purpose of this paper is to present a neat derivation of the asymptotic distribution using complex analysis.

2. The contour integral derivation

The nonsingularity of B(k) and the consistency of the sample ACRF imply the following property. (See, e.g., Tsay and Tiao[1984].)

Property 2.1 If the underlying process is from the AR(p) model in (1.1), then the following holds for $k(\geq p)$.

(a) $\underset{\sim}{\phi}(k)=(\phi_1,\cdots,\phi_p,0,\cdots,0)^t$.
(b) $\hat{\phi}_{k,l}=\phi_l+O_p(T^{-1/2})$, where ϕ_l is understood to be 0 for $l>p$.
(c) $\hat{\lambda}(k)$ is consistent to $\sigma^2/\sigma(0)$.

The following asymptotic property of the sample ACRF is due to Bartlett(1946).

Property 2.2 If $\{y_1,\cdots,y_T\}$ is a T-realization of the AR(p) process in (1.1), the random variables $T^{1/2}(\hat{\rho}_1-\rho_1),\cdots,T^{1/2}(\hat{\rho}_n-\rho_n)$ have asymptotic normal distributions with means 0 and covariances

$$\lim_{T\to\infty} T\ \mathrm{COV}(\hat{\rho}_g,\hat{\rho}_h)$$
$$=\frac{4\pi}{\sigma^2(0)}\int_{-\pi}^{\pi}(\mathrm{COS}\ \lambda h-\rho_h)(\mathrm{COS}\ \lambda g-\rho_g)\, S^2(\lambda)\, d\lambda$$

where $S(\lambda)$ is the spectral density of the AR process, i.e.,

$$S(\lambda)=\frac{\sigma^2}{2\pi}|\phi(e^{i\lambda})|^{-2}.$$

Theorem. If $\{y_1,\cdots,y_T\}$ is from the AR(p) model in (1.1), then, for $k>p$, $T^{1/2}\hat{\phi}_{k,k}$'s are asymptotically independent normal random variables with means 0 and variances 1.

Proof. If we let $\phi_{k-1,0}=-1$, then the sampled version of the Yule-Walker equations imply that

$$\hat{\theta}(k-1) = -\sum_{r=0}^{k-1}\sum_{s=0}^{k-1} \hat{\phi}_{k-1,r}\hat{\phi}_{k-1,s}\rho_{k-r-s}.$$

Then, Property 2.1(b) implies that $\hat{\theta}(k-1)$ equals

$$-\sum_{r=0}^{k-1}\sum_{s=0}^{k-1} \{\phi_r+O_p(T^{-1/2})\}(\phi_s+O_p(T^{-1/2})\}\hat{\rho}_{k-r-s}.$$

$$= -\sum_{r=0}^{p}\sum_{s=0}^{p} \phi_r\phi_s\hat{\rho}_{k-r-s} + O_p(\frac{1}{T}).$$

Thus, $T^{1/2}\hat{\theta}(k-1)$ has the same asymptotic distribution as

$$V_k = -T^{1/2}\sum_{r=0}^{p}\sum_{s=0}^{p} \phi_r\phi_s\hat{\rho}_{k-r-s}.$$

We are going to show that, for $k>p$, the random variables V_k's are asymptotically independently and normally distributed with means 0 and variances

$$\lim_{T\to\infty} \mathrm{Var}(V_k) = \{\sigma^2/\sigma(0)\}^2 .$$

It is known (see, e.g., Anderson[1971, p.217]) that the random variables V_k and V_j are asymptotically normally distributed. The consistency of the sample ACRF and the Yule-Walker equations imply that their means are zeroes. Property 2.2 implies that the asymptotic covariance is

$$\lim \mathrm{COV}(V_j,\ V_k)$$

$$= \lim_{T\to\infty} T\sum_{r=0}^{p}\sum_{s=0}^{p}\phi_r\phi_s\sum_{u=0}^{p}\sum_{v=0}^{p}\phi_u\phi_v\ \mathrm{Cov}(\hat{\rho}_{k-r-s},\ \hat{\rho}_{j-u-v})$$

$$= \frac{2\pi}{\sigma^2(0)}\int_{-\pi}^{\pi}\sum_{r=0}^{p}\sum_{s=0}^{p}\sum_{u=0}^{p}\sum_{v=0}^{p}\phi_r\phi_s\phi_u\phi_v\, e^{i\lambda(k+j-r-s-u-v)}\, S^2(\lambda)d\lambda$$

$$+ \frac{2\pi}{\sigma^2(0)}\int_{-\pi}^{\pi}\sum_{r=0}^{p}\sum_{s=0}^{p}\sum_{u=0}^{p}\sum_{v=0}^{p}\phi_r\phi_s\phi_u\phi_v\, e^{i\lambda(k-j-r-s+u+v)}\, S^2(\lambda)d\lambda$$

$$-\frac{8\pi}{\sigma^2(0)}\int_{-\pi}^{\pi}\sum_{r=0}^{p}\sum_{s=0}^{p}\sum_{u=0}^{p}\sum_{v=0}^{p}\phi_r\phi_s\phi_u\phi_v\rho_{j-u-v}\, e^{i\lambda(k-r-s)}\, S^2(\lambda)d\lambda$$

$$+\frac{4\pi}{\sigma^2(0)}\int_{-\pi}^{\pi}\sum_{r=0}^{p}\sum_{s=0}^{p}\sum_{u=0}^{p}\sum_{v=0}^{p}\phi_r\phi_s\phi_u\phi_v\,\rho_{k-r-s}\rho_{j-u-v}\, S^2(\lambda)d\lambda\ .$$

If we let $Z=\exp(i\lambda)$, then the first integral of the RHS equals

$$\left(\frac{\sigma^2}{2\pi}\right)^2 \oint_C Z^{k+j}\left\{\frac{\phi(Z^{-1})}{\phi(Z)}\right\}^2\frac{1}{iz}dz$$

where C is the unit circle with its center at the origin. Since $\phi(z)$ has all the zeroes outside the unit circle, and since $j+k$ is greater than 2p, the integrand has no poles inside or on the simple closed curve C. Cauchy's integral formula yields that the first integral is zero. The third integrand equals

$$\left\{\sum_{r=0}^{p}\sum_{s=0}^{p}\phi_r\phi_s\, e^{i\lambda(k-r-s)}\right\}\left\{\sum_{u=0}^{p}\sum_{v=0}^{p}\phi_u\phi_v\rho_{j-u-v}\right\}S^2(\lambda)$$

Its second factor is equal to zero by (1.2), because $j>0$. Similarly we can show that the fourth integrand is zero. The second integral equals

$$\int_{-\pi}^{\pi} e^{i\lambda(k-j)}\,\phi^2(e^{-i\lambda})\,\phi^2(e^{i\lambda})\left\{\frac{\sigma^2}{2\pi}\frac{1}{|\phi(e^{i\lambda})|^2}\right\}^2 d\lambda$$

$$=\left(\frac{\sigma^2}{2\pi}\right)^2\int_{-\pi}^{\pi} e^{i\lambda(k-j)}\,d\lambda$$

$$=\left(\frac{\sigma^4}{2\pi}\right)\delta_{k,j}\ ,$$

where $\delta_{k,j}$ is the Kronecker delta. Thus,

$$\lim_{T\to\infty}\mathrm{COV}(V_k,V_j)=\{\sigma^2/\sigma(0)\}^2\delta_{k,j}.$$

We know that $T^{1/2}\hat{\phi}_{k,k}=T^{1/2}\hat{\theta}(k-1)/\hat{\lambda}(k-1)$. Since Property 2.1(c) says that the denominator $\hat{\lambda}(k-1)$ is consistent to $\sigma^2/\sigma(0)$, $T^{1/2}\hat{\phi}_{k,k}$'s are asymptotically independently normally distributed with means 0 and variances 1. ■

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