〈Research Paper〉

Journal of the Korean
Statistical Society
Vol. 17, No.1, 1988

Large-Sample Comparisons of Statistical Calibration Procedures When the Standard Measurement is Also Subject to Error: The Replicated Case

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ABSTRACT

The classical theory of statistical calibration assumes that the standard measurement is exact. From a realistic point of view, however, this assumption needs to be relaxed so that more meaningful calibration procedures may be developed. This paper presents a model which explicitly considers errors in both standard and nonstandard measurements. Under the assumption that replicated observations are available in the calibration experiment, three estimation techniques (ordinary least squares, grouping least squares, and maximum likelihood estimation) combined with two prediction methods (direct and inverse prediction) are compared in terms of the asymptotic mean square error of prediction.

1. Introduction

The classical linear calibration model for two measurement methods can be represented as

$$y=\alpha+\beta X+e$$

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where the standard measurement X is assumed to be exact (i.e., free of measurement error) while the nonstandard measurement y is contaminated by random error e. In a calibration experiment estimates of α and β are obtained based upon pairs of (X, y), and in the future the exact characteristic value(X) of a measurand is predicted based upon nonstandard measurement(s) of that measurand.

As discussed above, the fundamental assumption in the classical theory of calibration is that the standard method measures a certain characteristic without error. In many real world applications, however, one may face the problem of uncertainty not only in the non-standard but also in the standard measurement. For instance, a routine measurement instrument in a plant is frequently calibrated to a plant working standard which has better accuracy and precision, but is still subject to measurement error.

Several authors investigated the statistical calibration problem when both standard and nonstandard measurements are subject to error (e.g., see Mandel(1984), Carroll and Spiegelman (1986), Lwin and Spiegelman (1986)). However, they are largely concerned with the applicability or modification of the ordinary least squares procedure, and comparative studies on various estimation and prediction methods have received little attention.

This paper presents a predictive functional relationship model which explicitly deals with errors in both measurements. Assuming replicated observations in the calibration experiment, ordinary least squares(OLS), grouping least squares($GRLS$), and maximum likelihood(ML) estimation techniques are considered for the estimation of the relationship, and the direct and inverse approaches are compared in terms of the asymptotic mean square error($AMSE$) of prediction.

2. The Model and Estimators

In the calibration experiment, let X_i and Y_i be respectively the (unknown) true standard and nonstandard measurements related as

$$Y_i = \alpha + \beta X_i, \quad i = 1, 2, \cdots, n \tag{1}$$

where n is the number of measurands, a certain characteristic of which is of interest, and α and β are unknown constants. It is assumed that for each measurand, m replicated measurements are made by the nonstandard and standard instrument. That is,

$$\left.\begin{aligned} x_{ij} &= X_i + u_{ij} \\ y_{ij} &= Y_i + v_{ij} \end{aligned}\right\} \quad \begin{aligned} i &= 1, 2, \cdots, n \\ j &= 1, 2, \cdots, m \end{aligned} \tag{2}$$

In (2) random measurement errors u_{ij} and v_{ij} are assumed to be distributed as

$$\begin{pmatrix} u_{ij} \\ v_{ij} \end{pmatrix} \sim BVN\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{bmatrix} \sigma_u^2 & \rho\sigma_u\sigma_v \\ \rho\sigma_u\sigma_v & \sigma_v^2 \end{bmatrix}\right\} \tag{3}$$

where 'BVN' reads 'bivariate normal'. We further assume that σ_u, σ_v, and ρ are unknown, and error vectors $\{(u_{ij}\ v_{ij})\}$ are independent of each other. Besides, we require $n \geq 2$ and $m \geq 2$ for the identifiability of the unknown parameters(Kendall and Stuart, 1979).

The above formulation of the present calibration problem is obviously more realistic than the classical one. Another important feature is that the following inverse relationship is also meaningful due to the symmetric role of X and Y.

$$X_i = \gamma + \delta Y_i \tag{4}$$

where $\gamma = -\alpha / \beta$ and $\delta = 1 / \beta$.

The model in Eqs. (1) and (2) or (2) and (4) is commonly called an errors-in-variables model (EVM) in the literature. If the true (unobservable) $X_i (i = 1, \ldots\ldots, n)$ are assumed non stochastic, then the EVM is called a functional model. Otherwise, it is called structural. In the present investigation a functional model is assumed.

Statistical problems in EVM have been investigated by many authors with their major concerns being the behavior of the estimated coefficients of the relationship (1) or (4). For a comprehensive review of EVM one may refer to Madansky (1959), Sprent (1969), Moran (1971), and Kendall and Stuart(1979). However, their results are not directly applicable to the present calibration problem since it involves one further step, namely, prediction.

Define

$$S_{pq} = \sum_{i=1}^{n} \sum_{j=1}^{m} (p_{ij} - \bar{\bar{p}})(q_{ij} - \bar{\bar{q}})$$

$$S_{\bar{p}\bar{q}} = \sum_{i=1}^{n} (\bar{p}_i - \bar{\bar{p}})(\bar{q}_i - \bar{\bar{q}})$$

$$s_{pq} = \sum_{i=1}^{n} \sum_{j=1}^{m} (p_{ij} - \bar{p}_i)(q_{ij} - \bar{q}_i) / [n(m-1)]$$

where $\bar{\bar{p}}$ and $\bar{\bar{q}}$ represent the grand means of all p_{ij} and q_{ij}, respectively. Similarly, $\bar{p}_i$ and $\bar{q}_i$ are respectively the i-th group means of p_{ij} and q_{ij}, $j = 1, \ldots\ldots, m$. Then, the OLS estimators of β, α, δ, γ are respectively given by

$$b_{OLS} = S_{xy} / S_{xx}$$
$$a_{OLS} = \bar{\bar{y}} - b_{OLS} \bar{\bar{x}}$$
$$d_{OLS} = S_{xy} / S_{yy}$$
$$c_{OLS} = \bar{\bar{x}} - d_{OLS} \bar{\bar{y}}.$$

Similarly, the *GRLS* estimators of β, α, δ, γ are respectively given by

$$b_G = S_{\bar{x}\bar{y}} / S_{\bar{x}\bar{x}}$$
$$a_G = \bar{\bar{y}} - b_G \bar{\bar{x}}$$
$$d_G = S_{\bar{x}\bar{y}} / S_{\bar{y}\bar{y}}$$
$$c_G = \bar{\bar{x}} - d_G \bar{\bar{y}}.$$

Finally, the *ML* estimators of β, α, δ, γ are respectively given by (e.g., see Villegas(1961))

$$b_M = (S_{\bar{x}\bar{y}} - \lambda^* s_{xy}) / (S_{\bar{x}\bar{y}} - \lambda^* s_{xx})$$
$$a_M = \bar{\bar{y}} - b_M \bar{\bar{x}}$$
$$d_M = (S_{\bar{x}\bar{y}} - \lambda' s_{xy}) / (S_{\bar{y}\bar{y}} - \lambda' s_{xy})$$
$$c_M = \bar{\bar{x}} - d_M \bar{\bar{y}}$$

where

λ^*=the minimum root of $|V_1 - \lambda U_1| = 0$
λ'=the minimum root of $|V_2 - \lambda U_2| = 0$

$$V_1 = \begin{bmatrix} S_{\bar{x}\bar{x}} & S_{\bar{x}\bar{y}} \\ S_{\bar{x}\bar{y}} & S_{\bar{y}\bar{y}} \end{bmatrix} \qquad V_2 = \begin{bmatrix} S_{\bar{y}\bar{y}} & S_{\bar{x}\bar{y}} \\ S_{\bar{x}\bar{y}} & S_{\bar{x}\bar{x}} \end{bmatrix}$$

$$U_1 = \begin{bmatrix} s_{xx} & s_{xy} \\ s_{xy} & s_{yy} \end{bmatrix} \qquad U_2 = \begin{bmatrix} s_{yy} & s_{xy} \\ s_{xy} & s_{xx} \end{bmatrix}.$$

In the future experiment, suppose that nonstandard measurements y_k^0 are obtained for a measurand where

$$y_k^0 = Y^0 + v_k^0, \qquad k = 1, 2, \cdots, r$$
$$v_k^0 \sim NID(0, \sigma_v^2), \quad k = 1, 2, \cdots, r$$
$$Y^0 = \alpha + \beta X^0.$$

Then, the corresponding X^0 is estimated as follows using the above three estimation methods for the relationships (1) and (4).

$$\begin{aligned}
\hat{X}^0_{D,OLS} &= (\bar{y}^0 - a_{OLS})/b_{OLS} \\
\hat{X}^0_{D,G} &= (\bar{y}^0 - a_G)/b_G \\
\hat{X}^0_{D,M} &= (\bar{y}^0 - a_M)/b_M \\
\hat{X}^0_{I,OLS} &= c_{OLS} + d_{OLS}\,\bar{y}^0 \\
\hat{X}^0_{I,G} &= c_G + d_G \bar{y}^0 \\
\hat{X}^0_{I,M} &= c_M + d_M \bar{y}^0
\end{aligned} \tag{5}$$

where $y^0 = r^{-1}\Sigma^r_{k=1}\, y^0_k$. X^0's with subscripts D and I are called direct and inverse estimators of X^0, respectively.

In this paper the above six estimators of X^0 are compared in terms of the asymptotic mean square error.

3. Asymptotic Mean Square Errors

For simplicity, $AMSE$'s for estimators in (5) are determined when the true values for ρ in (3) is 0.

First consider the $AMSE$ of $\hat{X}^0_{D,OLS}$. It is well (e.g., see Yum (1985)) that when n is fixed and m becomes indefinitely large, b_{OLS} is asymptotically normally distributed with asymptotic bias($ABIAS$) and variance($AVAR$) as

$$ABIAS\ (b_{OLS}) = -\beta/(1+\tau_x)$$
$$AVAR\ (b_{OLS}) = N^{-1}\{(\sigma_v^2/\sigma_u^2)/(1+\tau_x) + \beta^2\tau_x(1+\tau_x^2)/(1+\tau_x)^4\}$$

where

$$\tau_x = \sum_{i=1}^{n} (X_i - \bar{X})^2/(n\sigma_u^2)$$
$$N = nm$$
$$\bar{X} = n^{-1}\sum_{i=1}^{n} X_i .$$

As shown in Appendix, the $AMSE$ of $\hat{X}^0_{D,OLS}$ is given by

$$AMSE(\hat{X}^0_{D,OLS}) = N^{-1}\{(X^0-\bar{X})^2+\frac{\theta\sigma_u^2}{r}\}\{\frac{\theta(1+\tau_x)^3}{\tau_x^4}+\frac{(1+\tau_x^2)}{\tau_x^3}\}$$
$$+\frac{\sigma_u^2}{N}+\frac{(1+\tau_x)^2\theta\sigma_u^2}{\tau_x^2}(\frac{1}{N}+\frac{1}{r})+\frac{(X^0-\bar{X})^2}{\tau_x^2}$$

where

$$\theta=\frac{\sigma_v^2}{\beta^2\sigma_u^2}\quad . \tag{6}$$

Similary, the $AMSE$ of $\hat{X}^0_{I,OLS}$ is given by

$$AMSE(\hat{X}^0_{I,OLS}) = N^{-1}\{(X^0-X)^2+\frac{\theta\sigma_u^2}{r}\}\{\frac{1}{\theta(1+\tau_Y)}+\frac{\tau_Y(1+\tau_Y^2)}{(1+\tau_Y)^4}\}$$
$$+\frac{\sigma_u^2}{N}+\frac{\tau_Y^2\theta\sigma_u^2}{(1+\tau_Y)^2}(\frac{1}{N}+\frac{1}{r})+\frac{(X^0-\bar{X})^2}{(1+\tau_Y)^2}$$

where

$$\tau_Y=\sum_{i=1}^{n}(Y_i-Y)^2/(n\sigma_v^2)$$
$$=\tau_x/\theta.$$

When n is fixed and m becomes indefinitely large, b_M consistently estimates β (Bradley and Gart, 1962), and Dolby and Lipton (1972) showed that b_M is asymptotically normal with variance

$$N^{-1}\{(\sigma_v^2/\sigma_u^2)/\tau_x+\beta^2/\tau_x\}.$$

Following a similar procedure in Appendix we obtain the $AMSE$ of $\hat{X}^0_{D,M}$ as follows.

$$AMSE(X^0_{D,M})=N^{-1}\{(X^0-\bar{X})^2+\frac{\theta\sigma_u^2}{r}\}\{\frac{1}{\tau_x}+\frac{\theta}{\tau_x}\}+\frac{\sigma_u^2}{N}+\theta\sigma_u^2(\frac{1}{N}+\frac{1}{r}).$$

Similary, the $AMSE$ of $\hat{X}^0_{I,M}$ is given by

$$AMSE(X^0_{I,M})=N^{-1}\{(X^0-\bar{X})^2+\frac{\theta\sigma_u^2}{r}\}\{\frac{1}{\tau_Y}+\frac{1}{\theta\tau_Y}\}+\frac{\sigma_u^2}{N}+\theta\sigma_u^2(\frac{1}{N}+\frac{1}{r}).$$

Yum(1985) showed that b_G and b_M are asymptotically equivalent in the sense that the asymptotic distributions, biases, and variances of the two estimators are the same when n is fixed and $m \to \infty$. Therefore, the $AMSE$'s of $\hat{X}^0_{D,G}$ and $\hat{X}^0_{I,G}$ are indentical to those of $\hat{X}^0_{D,M}$ and $\hat{X}^0_{I,M}$ respectively.

In passing, it is worth noting that τ and θ have some physical meanings. That is, $\tau_X(\tau_Y)$ is the ratio of the dispersion of $X(Y)$ to that of the error in $x(y)$. It is frequently called the "signal-to-noise ratio". Further, $1/\sqrt{\theta} = \sigma_u / (\sigma_u / |\beta|)$ represents the ratio of the standard deviation of x to that error of y, converted to the units of x. Mandel (1984) called $1/\sqrt{\theta}$ the "sensitivity of y with respect to x."

4. AMSE Comparisons

It can be easily shown that $AMSE(\hat{X}^0_{D,OLS}) > AMSE(\hat{X}^0_{D,M})$. Since $\tau_X = \theta\tau_Y$, $AMSE(\hat{X}^0_{D,G}) = AMSE(\hat{X}^0_{I,M})$ and consequently, $AMSE(\hat{X}^0_{D,G}) = AMSE(\hat{X}^0_{D,M}) = AMSE(\hat{X}^0_{I,G}) = AMSE(\hat{X}^0_{I,M})$.

To compare $AMSE(\hat{X}^0_{D,M})$ and $AMSE(\hat{X}^0_{I,OLS})$ we solve $AMSE(\hat{X}^0_{D,M}) < AMSE(\hat{X}^0_{I,OLS})$ to obtain

$$(P_1 + P_2/r)P_3/N + P_4(1/N + 1/r) < P_1 \tag{7}$$

where

$$P_1 = (X^0 - \bar{X})^2$$
$$P_2 = \theta\sigma_u^2$$
$$P_3 = \theta/\tau_x + 1/\tau_x + 1/\theta + 2\tau_x^2/(\theta+\tau_x)^2 + 2$$
$$P_4 = (\theta + 2\tau_x)\sigma_u^2$$

Note that P_1, P_2, P_3, and P_4 are all nonnegative.

To obtain the numerical results, a set of parameter values are selected. Without loss of generality, the range of X in the calibration experiment is taken to be [0, 1], and hence, τ_X is restricted in $[0, 0.25/\sigma_u^2]$. The values of σ_u are chosen to be 0.1%, 0.5 %, 1%, 2%, 5%, and 10% of the range of X, and θ are set to be 0.01, 0.02, 0.03, 0.05, 0.1, 0.2, 0.3, 0.5, 1, 2, 3, 5, 10, and 100. Then, a set of dominance curves as in Figure 1 may be constructed to determine which estimator is preferred when $AMSE$ is the criterion. In constructing Figure 1 the number of future measurements (r) is set to 1, and its effects will be studied separately. For given θ, σ_u, and τ_X, if the point for N and $\sqrt{P_1}$ is on the right-hand side of the cor-

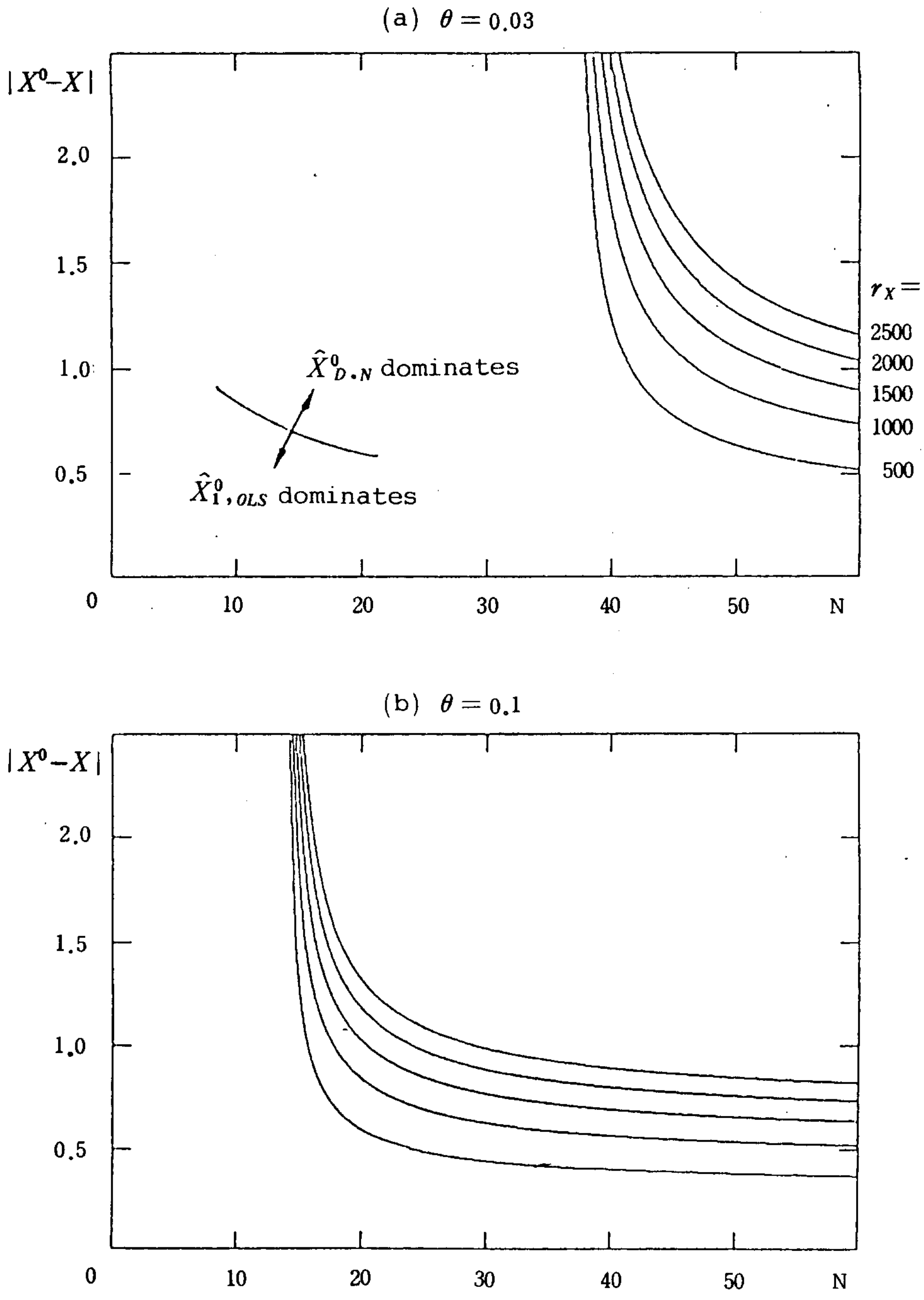


Figure 1. Contours of dominance between $\hat{X}^0_{D,M}$ and $\hat{X}^0_{1,OLS}$ with $|X^0 - \bar{X}|$ and N as coordinates (σ_u=0.01, r=1).

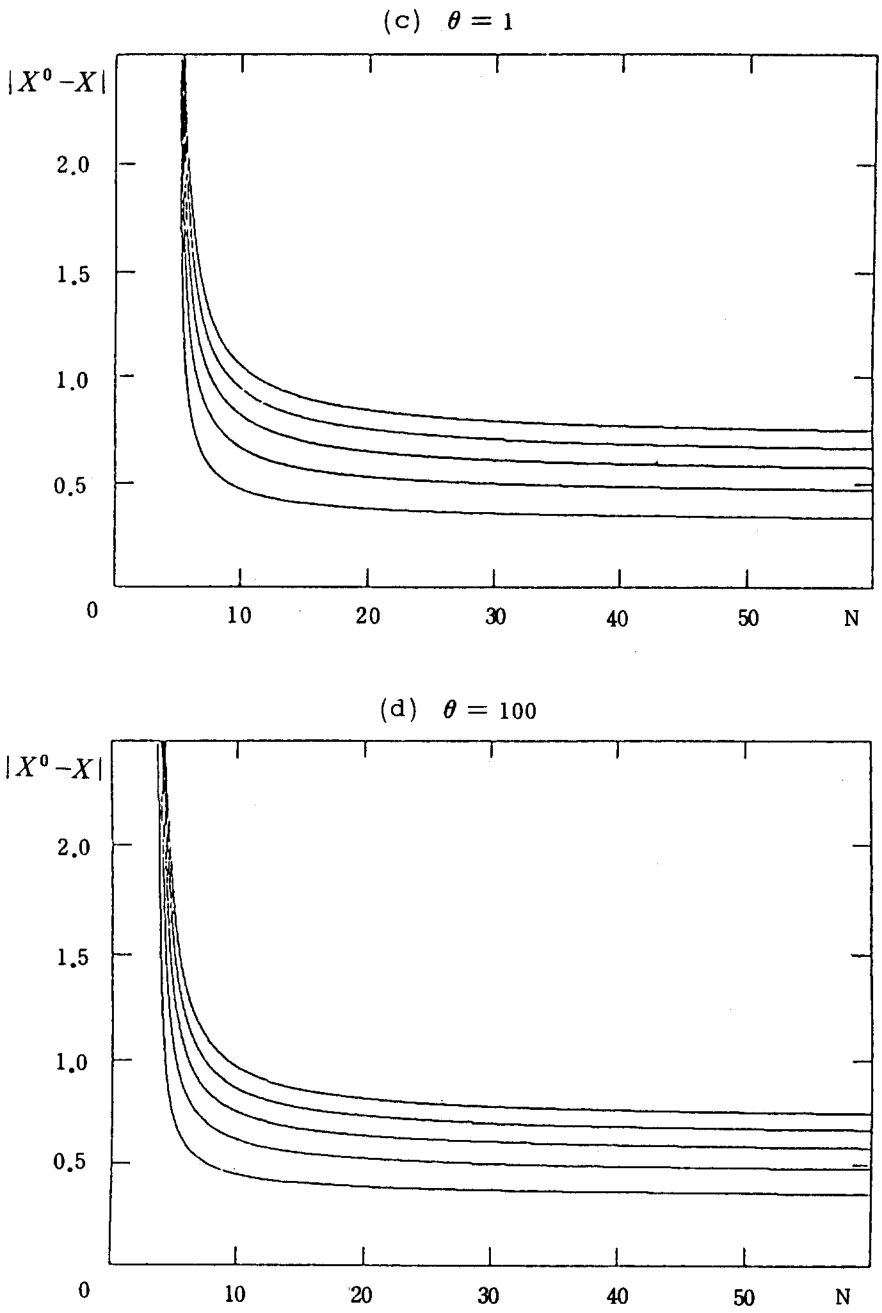


Figure 1. (continued)

responding curve, $\hat{X}^0_{D,M}$ dominates $\hat{X}^0_{I,OLS}$.

Based upon a series of figures generated, we first found that $\hat{X}^0_{D,M}$ is generally preferred to $\hat{X}^0_{I,OLS}$ when N is large. This can be also verified algebraically using (7), which can be rewritten as

$$P > (rP_1P_3 + P_2P_3 + rP_4)/(rP_1 - P_4) \quad \text{for} \quad rP_1 - P_4 > 0 \ .$$

If $rP_1 - P_4 < 0$, then there does not exist positive N which satisfies (7), and therefore, $\hat{X}^0_{I,OLS}$ is preferred. Secondly, $\hat{X}^0_{D,M}$ is preferred when $\sqrt{P_1} = |X^0 - \overline{X}|$ is large. This is clearly illustrated in Figure 1, and can be also verified using (7). That is, (7) can be rearranged to

$$P_1 > [P_2P_3 + P_4(N + r)] / [r(N - P_3)] \quad \text{for} \quad N - P_3 > 0.$$

If $N < P_3$, $\hat{X}^0_{I,OLS}$ are preferred for all values of P_1. This implies that in general $\hat{X}^0_{D,M}$ and $\hat{X}^0_{I,OLS}$ are preferred in extrapolation and interpolation, respectively. Another finding is that as θ increases, the dominance curves get more and more shifted towards the left, and consequently, $\hat{X}^0_{D,M}$ becomes better unless N and / or $\sqrt{P_1}$ is very small. Besides, we also observed from a series of figures generated that $\hat{X}^0_{D,M}$ becomes preferred as τ_X decreases. One exception is that the reverse is true when $\theta = 100$ and $\sigma_u = 0.1$.

In order to study the effect of r, a series of figures like Figure 2 were constructed with N and r as coordinates. In general, $\hat{X}^0_{D,M}$ is preferred for large r when N and $\sqrt{P_1}$ are small and τ_X is large. However, if N and $\sqrt{P_1}$ are large, then a relatively small r may still ensure the dominance of $\hat{X}^0_{D,M}$ over $\hat{X}^0_{I,OLS}$.

5. Conclusion

For the calibration problem when both measurements are subject to error, a predictive functional relationship model was proposed. Assuming replicated observations in the calibration experiment, various estimators for the unknown future standard measurement are compared in terms of the asymptotic mean square error. The major findings are as follows:

1. AMSE of $\hat{X}^0_{D,OLS}$ is always larger than that of $\hat{X}^0_{D,M}$.
2. AMSE$(\hat{X}^0_{D,G})$=AMSE$(\hat{X}^0_{D,M})$=AMSE$(\hat{X}^0_{I,G})$=AMSE$(\hat{X}^0_{I,M})$.
3. $\hat{X}^0_{D,M}$ is preferred for large N with n fixed.
4. $\hat{X}^0_{D,M}$ generally performs better in extrapolation while $\hat{X}^0_{I,OLS}$ is preferred in interpolation.
5. As θ increases and / or τ_X decreases, $\hat{X}^0_{D,M}$ becomes better.

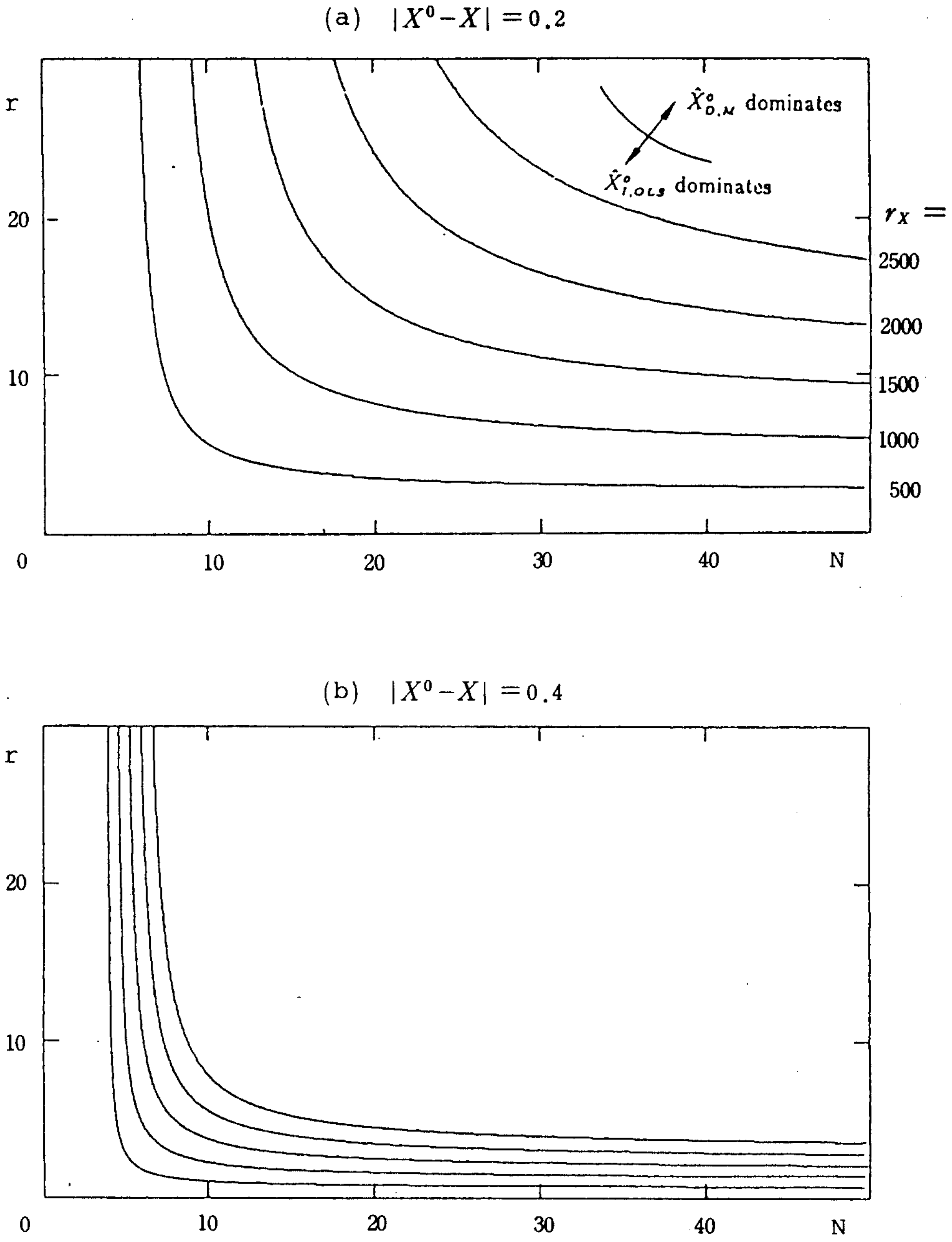
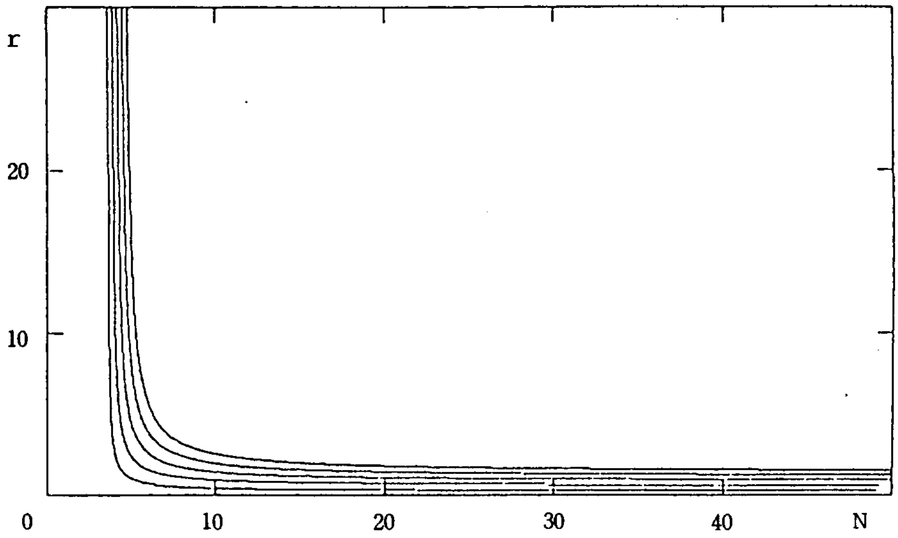


Figure 2. Contours of dominance between $\hat{X}^0_{0,M}$ and $\hat{X}^0_{1,OLS}$ with r and N as coordinates ($\theta=10$, $\sigma_u=0.01$).

(c) $|X^0 - X| = 0.6$



(d) $|X^0 - X| = 1.0$

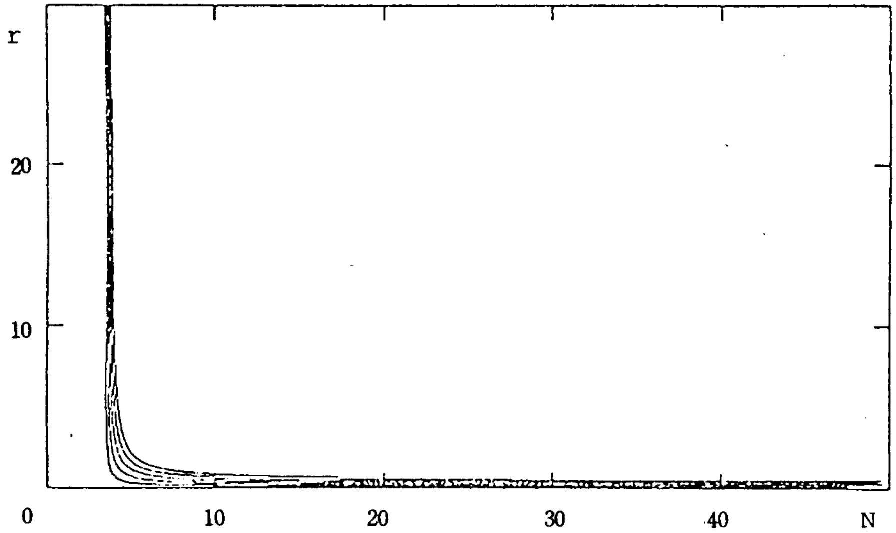


Figure 2. (Continued)

6. When N and $\sqrt{P_1}$ are small and τ_X is large a relatively large r is needed for the dominance of $\hat{X}_{\beta,M}$ over $\hat{X}_{\beta,OLS}$. Otherwise, a relatively small r suffices.

For AMSE comparisons the following estimates are suggested for the unknown parameters

$$\hat{\beta} = b_M$$
$$\hat{\sigma}_u^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_{ij} - \bar{x}_i)^2 / (N-n)$$
$$\hat{\sigma}_v^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} (y_{ij} - \bar{y}_i)^2 / (N-n)$$
$$\hat{\tau}_X = \sum_{i=1}^{n} (\bar{x}_i - \bar{\bar{x}})^2 / (n\sigma_u^2)$$
$$\hat{\bar{\bar{X}}} = N^{-1} \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} \, .$$

The estimate $\hat{\tau}_X$ was suggested by Richardson and Wu(1970) as a consistent estimate of τ_X. For β, b_M may be used since it consistently estimates β. Further, $\hat{\sigma}_u^2$ and $\hat{\sigma}_v^2$ are pooled sample variances which are unbiased.

Appendix

Derivation of the AMSE of $\hat{X}_{\beta,OLS}$

After some algebraic manipulation, direct and inverse estimators in (5) can be respectively reduced to

$$\hat{X}_D^0 - X^0 = (X^0 - \bar{X})(\beta/b - 1) + \bar{\bar{u}} - \bar{\bar{v}}/b - \bar{v}^0/b$$
$$\hat{X}_I - X^0 = (X^0 - \bar{X})(d/\delta - 1) + \bar{\bar{u}} - d\bar{\bar{v}} + d\bar{v}^0$$

where b and d are either the OLS, GRLS, or ML estimators of β and δ, respectively.

In this appendix, the AMSE of $\hat{X}_{\beta,OLS}$ is determined. AMSE's for the other estimators can be found in a similar manner.

First, difine a random vector

$$\omega = (b_{OLS}, \bar{\bar{u}}, \bar{\bar{v}})'.$$

Then, it can be shown that the probability limit of w when n is fixed and $m \to \infty$ is

$$plim\, \omega = \omega_0 = (\beta - \beta/(1+\tau_X), 0, 0)'.$$

Further, $\sqrt{m}\,(w-w_0)$ is asymptotically normally distributed with zero mean vector and covariance matrix

$$T=\text{diag}\{[(\sigma_v^2/\sigma_u^2)/(1+\tau_x)+\beta^2\tau_x(1+\tau_x^2)/(1+\tau_x)^4]/n,\ \sigma_u^2/n,\sigma_v^2/n\}.$$

Define

$$f(\omega)=\hat{X}^0_{D,OLS}-X^0.$$

Then,

$$f(\omega_0)=(X^0-\bar{X})/\tau_x+(1+\tau_x)\bar{v}^0/(\beta\tau_x).$$

Let ϕ be the vector of partial derivatives of f(w) with respect to the elements of w. Evaluating ϕ at w_0 yields

$$\phi_0=\begin{bmatrix}\dfrac{\partial f(\omega)}{\partial b_{OLS}}\\ \dfrac{\partial f(\omega)}{\partial\bar{\bar{u}}}\\ \dfrac{\partial f(\omega)}{\partial\bar{\bar{v}}}\end{bmatrix}=\begin{bmatrix}-(1+\tau_x)^2\{\dfrac{(X^0-\bar{X})}{(\beta\tau_x^2)}+\dfrac{\bar{v}^0}{(\beta\tau_x)^2}\}\\ 1\\ \dfrac{-(1+\tau_x)}{\beta\tau_x}\end{bmatrix}.$$

Then, using the Anderson theorem (1984, p.121) for given $\bar{v}^0$, we can show that $\sqrt{m}\,\{f(w)-f(w_0)\}$ is asymptotically normally distributed with mean zero and variance $\phi_0'T\phi_0$.

Therefore, the AMSE of $\hat{X}^0_{D,OLS}$ conditional on $\bar{v}^0$ is

$$AMSE(\hat{X}^0_{D,OLS}\mid\bar{v}^0)=\frac{(X^0-X)^2}{\tau_x^2}+\frac{2(X^0-\bar{X})\bar{v}^0(1+\tau_x)}{\beta\tau_x^2}+\frac{(\bar{v}^0)^2(1+\tau_x)^2}{(\beta\tau_x)^2}$$

$$+N^{-1}\{\frac{(X^0-\bar{X})^2(1+\tau_x)^4}{\beta^2\tau_x^4}+\frac{2(X^0-\bar{X})\bar{v}^0(1+\tau_x)^4}{\beta^3\tau_x^4}+\frac{(\bar{v}^0)^2(1+\tau_x)^4}{(\beta\tau_x)^4}\}$$

$$\{\frac{\sigma_v^2/\sigma_u^2}{1+\tau_x}+\frac{\beta^2\tau_x(1+\tau_x^2)}{(1+\tau_x)^4}\}+\frac{\sigma_u^2}{N}+\frac{(1+\tau_x)^2}{(\beta\tau_x)^2}\frac{\sigma_v^2}{N}.$$

Then, taking expectation of AMSE($\hat{X}^0_{D,OLS}|\bar{v}^0$) with respect to $\bar{v}^0$ yields the unconditional AMSE as

$$AMSE(\hat{X}^0_{D,OLS}) = N^{-1}\{(X^0-\bar{X})^2+\frac{\theta\sigma_u^2}{r}\}\{\frac{\theta(1+\tau_x)^3}{\tau_x^4}+\frac{(1+\tau_x^2)}{\tau_x^3}\}$$

$$+\frac{\sigma_u^2}{N}+\frac{(1+\tau_x)^2\theta\sigma_u^2}{\tau_x^2}(\frac{1}{N}+\frac{1}{r})+\frac{(X^0-\bar{X})^2}{\tau_x^2}$$

where θ is defined in Eq. (6).

References

(1) Anderson, T.W.(1984). *An Introduction to Multivariate Statistical Analysis,* 2nd Ed., Wiley, New York

(2) Bradley, V.D. and Gart, J.J. (1962). The Asymptotic Properties of ML Estimators When Sampling from Associated Populations, *Biometrika,* Vol. 49, 205~213.

(3) Carroll, R.J. and Spiegelman, C.H.(1986). The Effect of Ignoring Small Measurement Errors in Precision Instrument Calibration, *Journal of Quality Technology,* Vol. 18, 170~173.

(4) Dolby, G.R. and Lipton, S.(1972). Maximum Likelihood Estimation of the General Non-linear Functional Relationship with Replicated Observations and Correlated Errors, *Biometrika*, Vol. 59, 121~129.

(5) Kendall, M.G. and Stuart, A.(1979). *The Advanced Theory of Statistics,* Vol.2, 4th Ed., Griffin, London.

(6) Lwin, T. and Spiegelman, C.H.(1986). Calibration with Working Standards, *Applied Statistics*, Vol. 35, 256~261.

(7) Madansky, A.(1959). The Fitting Straight Lines When Both Variables are Subject to Error, *Journal of the American Statistical Association* Vol.54, 173~205.

(8) Mandel, J.(1984). Fitting Straight Lines When Both Variables are Subject to Error, *Journal of Quality Technology,* Vol.16, 1~14.

(9) Moran, P.A.P.(1971). Estimating Structural and Functional Relationships, *Journal of Multivariate Analysis*, Vol.1, 232~255.

(10) Richardson, D.H. and Wu, D.(1970). Least Squares and Grouping Method Estimators in the Errors in Variables Model, *Journal of the American Statistical Association,* Vol. 65, 724~748

(11) Sprent, P.(1963). A Generalized Least Squares Approach to Linear Functional Relationships, *Journal of the Royal Statistical Society, Series* B, Vol. 28, 278~297.

(12) Villegas, C.(1961). Maximum Likelihood Estimation of a Linear Functional Relationship, *The Annals of Mathematical Statistics*, Vol.32, 1048~1062.

(13) Yum, B.J.(1985). Asymptotic Properties of the OLS and GRLS Estimator for the Replicated Functional Relationship Model, *Communications in Statistics, Series* A, Vol.14, 1981~1996.