Journal of the Korean
OR/MS Society
Vol. 13 No. 1 June 1988

—Research Paper—

Algorithms and Planning Horizons for a One-Plant Multi-Retailer System*

Sang-Bum Lee**

ABSTRACT

This paper examines a deterministic, discrete-time, finite-horizon, production/distribution problem for a one-plant multi-retailer system. Production may occur at the plant in each time period. Customer demands at each retailer over a finite number of periods are known and must be met without backlogging. The plant as well as the retailers can serve as stocking points. The problem is to find a minimum-cost production/distribution schedule satisfying the known demands. We show that under a certain cost structure a nested policy is optimal, and present an efficient algorithm to find such an optimal policy. Planning horizon results and some computational saving schemes are also presented.

1. Introduction

This paper is concerned with a production/distribution system consisting of one plant and N demand locations (called "retailers"), shown in Figure 1. A single product may be produced at the plant in each time period. Customer demands at each retailer over a finite number T of periods are known, and must be met without backlogging. The plant as well as the retailers can serve as stocking points. The problem is to find a production/distribution schedule which minimizes the total cost of production, shipment and inventory over the T periods.

The problem structure is often described as a "one-warehouse multi-retailer system" or a "two-echelon arborescence system" in the literature. A similar structure has been treated assuming stationary, continuous demand over an infinite horizon (e.g., Roundy [5]). The problem can also be viewed as representing a two-stage production process, involving several products instead of several locations.

The two known algorithms for the discrete-time problem with concave costs are Veinott's [6] generalization of Zangwill's [10] facilities-in-series algorithm to general arborescence systems and an algorithm of Kalymon [2]. The running time of the Zangwill-Veinott algorithm for the one-plant N-retailer system is polynomial in the number T of time periods, but is exponential in the number N of retailers. On the other hand, the running time of Kalymon's algorithm is linear in N, but exponential in T. Thus neither is likely to be useful when both parameters

* 이 논문은 1987년도 문교부 자유공모과제 학술연구조성비에 의하여 연구되었음.

** Department of Management, Seoul City University.

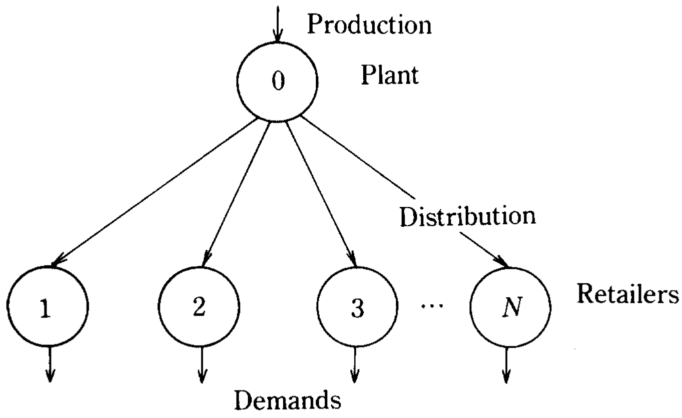

Figure 1. A One-Plant N-Retailer Production-Distribution System.

are large.

Under some restrictons on cost structures, it is often possible that optimal solutions for multi-facility production-inventory systems have the nested property, that is, each facility orders every time its immediate predecessor does. For example, Love [3] has shown that under reasonable conditions on cost functions, a nested policy is optimal for the facilities-in-series system. For arborescence systems, Veinott [6] considers a special cost structure under which an optimal policy has the nested property, and presents a polynomial-time algorithm to find such a policy.

For the one-plant N-retailer system, the cost structure of Veinott is described as follow. (i) Production and shipment costs (to each retailer) are fixed-plus-linear, and inventory holding costs (at the plant and all the retailers) are linear. (ii) In each period, the fixed cost for production is at least as large as the sum of the fixed shipment costs of all retailers. (iii) The fixed shipment cost at retailer i in period t is incurred if and only if a shipment occurs to retailer i and no production occurs at the plant in period t. (iv) The variable unit production and shipment costs are nonincreasing in time periods, and in each period the holding cost at the plant is less than or equal to the holding cost at any retailer. Of these, assumptions (ii) and (iii) seem to be somewhat restrictive, especially assumption(iii).

In our cost structure, we retain assumptions (i) (except for the shipment costs) and (iv), but do not need(ii) and (iii) by taking another shipment cost function which has the form of joint fixed-plus-linear. This cost structure also yields the nested property of an optimal solution. Furthermore, the cost structure leads to a very efficient algorithm including some computational saving schemes, and enables us to develop planning horizon results.

In Section 2, we describe the cost structure and present the mathematical formulation of the problem. Section 3 shows the nested property of an optimal solution. Exploiting the property, Section 4 presents an efficient dynamic programming algorithm including some computational saving schemes. Finally, Section 5 develops planning horizon results.

2. Cost structure and Formulation

The following notation is used :

t = index for time periods, $t=1, 2,\cdots,T$,

i = index for retailers, $i=1, 2,\cdots,N$,

x_t = production at the plant in period t,

y_{it} = shipment to retailer i in period t,

z_t = total shipment in period t, i.e., $z_t = \sum_{i=1}^{N} y_{it}$,

I_t = inventory at the plant at the end of period t,

J_{it} = inventory at retailer i at the end of period t,

d_{it} = demand at reatiler i in period t.

We assume the following cost structure :

Production costs : $K_t\delta(x_t)+a_tx_t$, where $\delta(x_t)=\begin{cases} 1 & \text{if } x_t>0 \\ 0 & \text{otherwise} \end{cases}$

Shipment costs : $S\,\delta(\sum_{i=1}^{N} y_{it})+\sum_{i=1}^{N} c_{it}\,y_{it}$

Holding costs at the plant : H_tI_t

Holding costs at each retailer : $h_{it}J_{it}$

In addition we make the following assumptions on the cost parameters :

$$K_t \geqq K_{t+1},\quad a_t \geqq a_{t+1},\quad c_{it} \geqq c_{i\ t+1}, \text{ and}$$
$$H_t \leqq h_{it},\quad \text{for } 1 \leqq t \leqq T,\ 1 \leqq i \leqq N.$$

Note that the shipment costs have no individual fixed costs. This cost structure may describe situations where transshipment functions at the plant are centralized in one department, so that individual preparing costs for shipments to each retailer are negligible in comparison with a large joint fixed cost. This shipment cost structure may also describe a situation where shipments to *all* retailers are made by a single tour of a large vehicle(e.g., boat or train).

The problem is now formulated as follows :

$$(P)\text{: Min } \sum_{t=1}^{T} \{K_t\delta(x_t)+a_tx_t+H_tI_t+S_t\delta(z_t)+\sum_{i=1}^{N} c_{it}y_{it}+\sum_{i=1}^{N} h_{it}J_{it}\} \tag{1}$$

subject to

$$I_{t-1}+x_t-z_t-I_t=0, \qquad t=1,\cdots,T, \tag{2}$$

$$z_t=\sum_{i=1}^{N} y_{it} \qquad t=1,\cdots T, \tag{3}$$

$$J_{i\ t-1}+y_{it}-J_{it}=d_{it}, \qquad i=1,\cdots,N,\ t=1,\cdots,T, \tag{4}$$

$$I_0=I_T=J_{i0}=J_{it}=0 \qquad i=1,\cdots,N, \tag{5}$$

$$x_t \geqq 0,\ y_{it} \geqq 0,\ I_t \geqq 0,\ J_{it} \geqq 0. \tag{6}$$

This model can be interpreted as a minimum-cost network flow problem. The network for $N=3$ and $T=4$ is shown in Figure 2, where the nodes $(0, t)$, (t) and (i, t) correspond to the balance equations (2), (3) and (4), respectively, and the single source (node 0) is a dummy node.

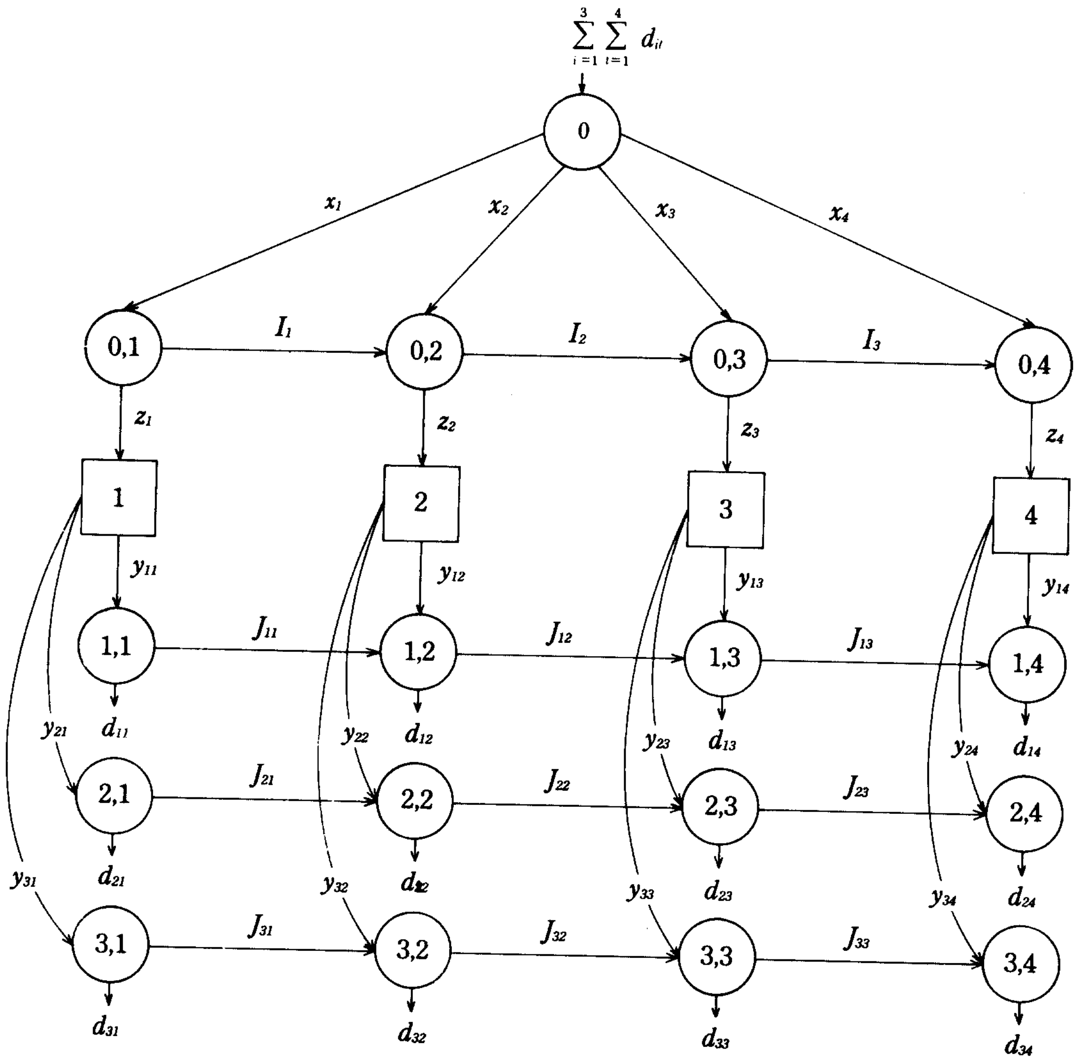

Figure 2. A Network Representation of (P) for $N=3$, $T=4$.

It should be noted that the results in this paper can be easily generalized to a situation where lead times are nonzero (cf. Zangwill [8]).

3. Optimality Properties

Since the objective function (1) is concave, there exists an extreme-point solution which minimizes (1). Hence we first characterize the extreme points of the feasible set of (P).

Proposition 1. (Extreme-Point Property)

In problem (P), *a feasible solution is an extreme-point solution if and only if*

(i) $I_{t-1}x_t=0, \qquad t=1,\cdots,T$ (7a)

(ii) $J_{i,t-1}y_{it}=0, \qquad i=1,\cdots,N,\ t=1,\cdots,T.$ (7b)

Proof. Follows directly from the results of Zangwill [9], using the network interpretation of (P). □

Proposition 2. (Nested Property)

There exists an optimal extreme-point solution such that

(i) *if* $x_t>0$, *then* $z_t>0$, $t=1\cdots,T$, *and* (8a)

(ii) $z_t(\sum_{i=1}^{N} J_{i,t-1})=0$, $t=1,\cdots T.$ (8b)

Proof. (i) Suppose there is an optimal extreme-point solution in which (8a) does not hold, so $x_k>0$ and $z_k=0$ for some k. We show this leads to a contradiction. By (7a), we have $I_{k-1}=0$, and $I_k=x_k>0$ from(2). Again by (7a), we have $x_{k+1}=0$. Consider the following alternative solution : $x_k'=0$, $x'_{k+1}=x_k$, $I'_k=0$ and the values of all other variables are the same as before. Obviously this alternative solution is feasible, and it saves at least the holding cost H_kx_k, since $K_K\geq K_{k+1}$ and $a_k\geq a_{k+1}$.

(ii) Suppose that $z_k(\sum_{i=1}^{N} J_{i,k-1})>0$ for some k in an optimal extreme-point solution, that is, $z_k>0$ and $\sum_{i=1}^{N} J_{i,k-1}>0$. Let $A=\{i : J_{i,k-1}>0,\ i=1,\cdots,N\}$. Then we have $0<|A|<N$ and $y_{ik}=0$ for $i\in A$ by (3) and (7b). Since $I_{k-1}+x_k=0$ from (2) and $I_{k-1}x_k=0$ from (7a), we consider the following two cases of the current solution :

Case 1 : $x_k>0$ and $I_{k-1}=0$.

Construct an alternative solution in which

$$x'_k=x_k+\sum_{i\in A} J_{i,k-1},$$

$$z'_k=z_k+\sum_{i\in A} J_{i,k-1},$$

$$y'_{ik}=J_{i,k-1} \text{ for } i\in A,$$

$$J'_{i,k-1}=0 \text{ for } i\in A,$$

and all other variables before period k have values reflecting the above changes. (For example, reduce the earlier production and shipment that led to the inventories $J_{i,k-1}$, $i\in A$.) It is easily seen that this solution saves at least the holding cost $\sum_{i\in A} h_{i,k-1} J_{i,k-1}$.

Case 2 : $x_k=0$ and $I_{k-1}>0$.

Construct an alternative solution in which

$$I'_{k-1}=I_{k-1}+\Sigma J_{i,k-1},$$

$$z'_k=z_k+\sum_{i\in A} J_{i,k-1},$$

$$y'_{ik}=J_{i,k-1} \text{ for } i\in A.$$

$$J'_{i,k-1}=0 \text{ for } i\in A,$$

and all other variables before period k have values reflecting the above changes. This solution saves at least the cost $\sum_{i\in A}(h_{i,k-1}-H_{k-1})J_{i,k-1}$. □

Proposition 2 implies that if production occurs, then shipment must occur, and whenever shipments do occur, the amount shipped must satisfy demands for the same integral number of periods for all retailers.

From Propositions 1 and 2, we have the following corollary :

Corollary 1. *There exists an optimal solution such that*

$$x_t(I_{t-1}+\sum_{i=1}^{N}J_{i,t-1})=0,\quad t=1,\cdots,T.$$

4. A Nested Algorithm

Using the nested property, this section presents an efficient Wagner-Whitin [7] type forward algorithm.

Definition 1.

(i) A period t is a *production point* if $x_t>0$.

(ii) A period t is a *shipment point* if $z_t>0$.

(iii) A period t is a *regeneration point* if $I_t=0$ and $J_{it}=0$, $i=1,\cdots,N$.

(iv) A period t is a *subregeneration point* if $J_{it}=0$, $i=1,\cdots,N$.

From Proposition 2 and Corollary 1, it follows that if period t is a production point, then that period must be a shipment point and period t- 1 must be a regeneration point. Further, if period t is a shipment point, then period t-1 must be a subregeneration point.

Denote the t-period problem by $P(t)$, $t=1, 2,\cdots\cdots$ Let

$F(t)$ = the minimum cost for problem $P(t)$,
$F_j(t)$ = the minimum cost for $P(t)$ when the last production point is restricted to period j,
$j^*(t)$ = the last production point in the optimal solution to $P(t)$,
$C(j, t)$ = the minimum cost for subproblem $Q(j, t)$ which is defined below.

Subproblem $Q(j, t)$ is the problem for finding the minimum-cost inventory and shipment schedule during periods $j, j+1,\cdots,t$ when j is the last production point in $P(t)$ (i. e., when total demand from period j to t is produced in period j). Specifically $Q(j, t)$ is defined as follows :

$$Q(j,t):\ Min\ \sum_{k=j}^{t}\{S_k\delta(z_k)+\sum_{i=1}^{N}c_{ik}y_{ik}+H_kI_k+\sum_{i=1}^{N}h_{ik}J_{ik}\}$$

s. t.

$$\sum_{i=1}^{N}y_{ik}=z_k \qquad k=j, j+1,\cdots,t,$$

$$z_j+I_j=\sum_{i=1}^{N}\sum_{k=j}^{t}d_{ik}$$

$$I_{k-1}-z_k-I_k=0, \qquad k=j+1,\cdots,t,$$

$$J_{i,k-1}+y_{ik}-J_{ik}=d_{ik}, \qquad i=1,\cdots,N,\ k=j,\cdots,t,$$

$$I_t=J_{i,j-1}=J_{it}=0, \qquad i=1,\cdots,N,$$

$$y_{ik}\geq 0,\ I_k\geq 0,\ J_{ik}\geq 0.$$

(Observe $Q(j, t)$ as a subnetwork in Figure 2.)

Define $\langle r, s\rangle=\{r, r+1,\cdots,s\}$ for convenience. We now have the following forward recursion :

$$F(t)=F_{j^*(t)}(t) = \min_{j\in\langle 1,t\rangle} F_j(t), \qquad t=1, 2, \cdots, \tag{9}$$

where $\quad F_j(t)=F(j-1)+K_j+a_j\sum_{i=1}^{N}\sum_{k=j}^{t}d_{ik}+C(j,t)$

with $\quad F(0)=0.$

Recursion (9) is implemented in the usual manner given all the values of $C(j, t)$ $j=1,2\cdots,t$, $t=1,2\cdots$

We now show how to solve subproblem $Q(j, t)$. In fact we need again a Wager-Whitin type algorithm. Let

$C_\ell(j, t)$ = the minimum cost for $Q(j, t)$ with ℓ as the last shipment point,
$\ell^*(j, t)$ = the last shipment point in the optimal solution to $Q(j, t)$.

Since $Q(j, t)$ is decomposed at a subregeneration point, we have

$$C(j,t) = C_{\ell^*(j,t)}(j,t) = \min_{\ell\in\langle j,t\rangle} C_\ell(j,t), \quad j=1,2\cdots,t, \ t=1,2\cdots, \tag{10}$$

where $$C_\ell(j,t)=C(j,\ell-1)+\Big(\sum_{k=j}^{\ell-1}H_k\Big)\Big(\sum_{i=1}^{N}\sum_{k=\ell}^{t}d_{ik}\Big)+S_\ell + \sum_{i=1}^{N}c_{i\ell}\Big(\sum_{k=\ell}^{t}d_{ik}\Big)+\sum_{i=1}^{N}\sum_{k=\ell}^{t-1}h_{ik}\Big(\sum_{r=k+1}^{t}d_{ir}\Big), \tag{11}$$

with $C(j, j-1)=0$.

We now discuss the computational effort required by the algorithm in terms of number of periods T and number of retailers N. For a T-period problem, we must compute $C(j, t)$ for $j=1, 2\cdots,t$ and $t=1, 2\cdots, T$. Note that solving $Q(1, T)$ with forward recursion (10) yields the values of $C(1, 1)$, $C(1, 2),\cdots$, $C(1, T)$; thus, in general, solving $Q(j, T)$ yields the values of $C(j, j)$, $C(j, j+1)$, $\cdots$, $C(j, T)$. After computing each cost sums appearing in (11), $Q(j, T)$ is solved in $O[(T-j+1)^2]$ time for $j=1,2,\cdots T$. Hence we need at most $O(T^3)+O(NT^2)$ operations to obtain all $C(j, t)$ values for $j=1,2,\cdots, t$, $t=1,2\cdots,T$. Since $F(T)$ can be computed in $O(T^2)+O(NT)$ time given all $C(j, t)$ values, the overall computational effort of the algorithm is at most $O(T^3)+O(NT^2)$.

Since most of the computational effort is spent on computing $C(j, t)$ values, the following proposition is important as a computational saving scheme.

Proposition 3. $\ell^*(j, t+1)\geq \ell^*(j, t)$, $j=1,2\cdots,t$, $t=1,2\cdots$

Proof. It is enough to show that

$C_\ell(j, t+1)\geq C_{\ell^*(j,t)}(j, t+1)$ for all $\ell\in\langle j, \ell^*(j, t)\rangle$

$$C_\ell(j,t+1)=C(j,\ell-1)+(\sum_{k=j}^{\ell-1}H_k)(\sum_{i=1}^{N}\sum_{k=\ell}^{t+1}d_{ik})$$

$$+S_\ell+\sum_{i=1}^{N}C_{i\ell}(\sum_{k=\ell}^{t+1}d_{ik})+\sum_{i=1}^{N}\sum_{k=\ell}^{t}h_{ik}(\sum_{r=k+1}^{t+1}d_{ir})$$

$$=C(j,\ell-1)+(\sum_{k=j}^{\ell-1}H_k)(\sum_{i=1}^{N}\sum_{k=\ell}^{t}d_{ik})+(\sum_{k=j}^{\ell-1}H_k)(\sum_{i=1}^{N}d_{i,t+1})+S_\ell$$

$$+\sum_{i=1}^{N}c_{i\ell}(\sum_{k=\ell}^{t}d_{ik})+\sum_{i=1}^{N}c_{i\ell}d_{i,t+1}+\sum_{i=1}^{N}\sum_{k=\ell}^{t-1}h_{ik}(\sum_{r=k+1}^{t}d_{ir})+\sum_{i=1}^{N}\sum_{k=\ell}^{t}h_{ik}d_{i,t+1}$$

$$=C_\ell(j,t)+(\sum_{k=j}^{\ell-1}H_k)(\sum_{i=1}^{N}d_{i,t+1})+\sum_{i=1}^{N}c_{i\ell}d_{i,t+1}+\sum_{i=1}^{N}\sum_{k=\ell}^{t}h_{ik}d_{i,t+1}.$$

Similarly,

$$C_{\ell^*(j,t)}(j,t+1)=C_{\ell^*(j,t)}(j,t)+(\sum_{k=j}^{\ell^*(j,t)-1}H_k)(\sum_{i=1}^{N}d_{i,t+1})$$

$$+\sum_{i=1}^{N}c_{i,\ell^*(j,t)}d_{i,t+1}+\sum_{i=1}^{N}\sum_{k=\ell^*(j,t)}^{t}h_{ik}d_{i,t+1}.$$

Therefore,

$$C_\ell(j,t+1)-C_{\ell^*(j,t)}(j,t+1)$$

$$=C_\ell(j,t)-C_{\ell^*(j,t)}(j,t)+\sum_{i=1}^{N}(c_{i\ell}-c_{i,\ell^*(j,t)})d_{i,t+1}$$

$$+\sum_{i=1}^{N}\sum_{k=\ell}^{\ell^*(j,t)-1}h_{ik}d_{i,t+1}-(\sum_{k=\ell}^{\ell^*(j,t)-1}H_k)(\sum_{i=1}^{N}d_{i,t+1})$$

$\geqq 0$ (since $C_\ell(j,t)\geqq C_{\ell^*(j,t)}(j,t)$, $c_{i\ell}\geqq c_{i,\ell^*(j,t)}$ and $h_{ik}\geqq H_k,\ \forall i,k$). □

Using Proposition 3, recursion (10) can now be restated by

$$C(j,t)=\min_{\ell\in\langle\ell^*(j,t-1),t\rangle} C_\ell(j,t). \qquad (10')$$

We shall show another computational saving scheme for recursion (9) in the next section.

5. Planning Horizon Results

The planning horizon issue addresses how much future demand information is really needed for an optimal initial decision to an infinite horizon problem. This section develops planning horizon results from the forward algorithm.

The two related terms of planning and forecast horizons are defined as follows :

Definition 2. A period $\bar{t}$ is called a *planning horizon* if the decisions for the first $t(\geqq 1)$ periods in the optimal solution to the $\hat{t}$-period problem $P(t)$ remain optimal for any longer horizon problem, irrespective of demand information beyond period $\hat{t}$; such a period $\hat{t}$ is called a *forecast horizon.*

Our purpose is to find such periods $\bar{t}$ and $\hat{t}$. For this, we use the planning horizon method developed by Lundin and Morton [4].

Definition 3. A *regeneration set* $R(t)$ at the end of period t is a finite set of periods such that for any $t' > t$, problem $P(t')$ has at least one optimal solution with a regeneration point in this set.

Proposition 4. *Let $f^*(t)$ denote the first regeneration point (strictly greater than zero) in the optimal solution to $P(t)$. If $f^*(j)$ is a positive constant π for all $j \in R(k)$ for some k, then π is a planning horizon and k is a forecast horizon.*

Proof. See Lundin and Morton [4].□

Observe that the key for finding planning and forecast horizons by the Lundin and Morton method is how regeneration sets can be found during the implementation of the forward algorithm. For developing regeneration sets, we first prove the following lemma :

Lemma 1. *For any $t' > t$, we have*
$C(j, t') - C(j, t) \geqq C(t, t') - C(t, t)$ *for* $j < t$, $t = 1, 2 \cdots$

Proof. Consider an optimal solution to $Q(j, t')$ (one such solution is shown in Figure 3). First note that the total amount of inventory at the end of period t must be $\sum_{i=1}^{N} \sum_{k=t+1}^{t'} d_{ik}$, and the distribution of the inventory among the plant and retailers at the end of period t takes on the following form by the optimality properties :

$$I_t = \sum_{i=1}^{N} \sum_{k=\alpha+1}^{t'} d_{ik} \text{ and } J_{it} = \sum_{k=t+1}^{\alpha} d_{ik}, i = 1, \cdots, N, \text{ for some } \alpha,\ t \leqq \alpha \leqq t'$$

(where $I_t = 0$ for $\alpha = t'$, and $J_{it} = 0$, $i = 1, \cdots, N$, for $\alpha = t$).

Without loss of generality, we assume that the last shipment before period $t+1$ occurs in some period m, $j \leqq m \leqq t$, so that the inventory J_{it}, $i = 1, \cdots N$, is due to that shipment.

In the solution, if we reduce the flow on chains (i.e., directed paths) between the single source node$(0, j)$ to the destination nodes (i, k), $i = 1, \cdots N$, $k = t+1, \cdots, t'$, by d_{ik}, respectively, then the remaining solution must be a feasible solution to $Q(j, t)$ with a cost not less than $C(j, t)$.

Now observe that the reduction in cost from the previous redution of flow is greater than or equal to

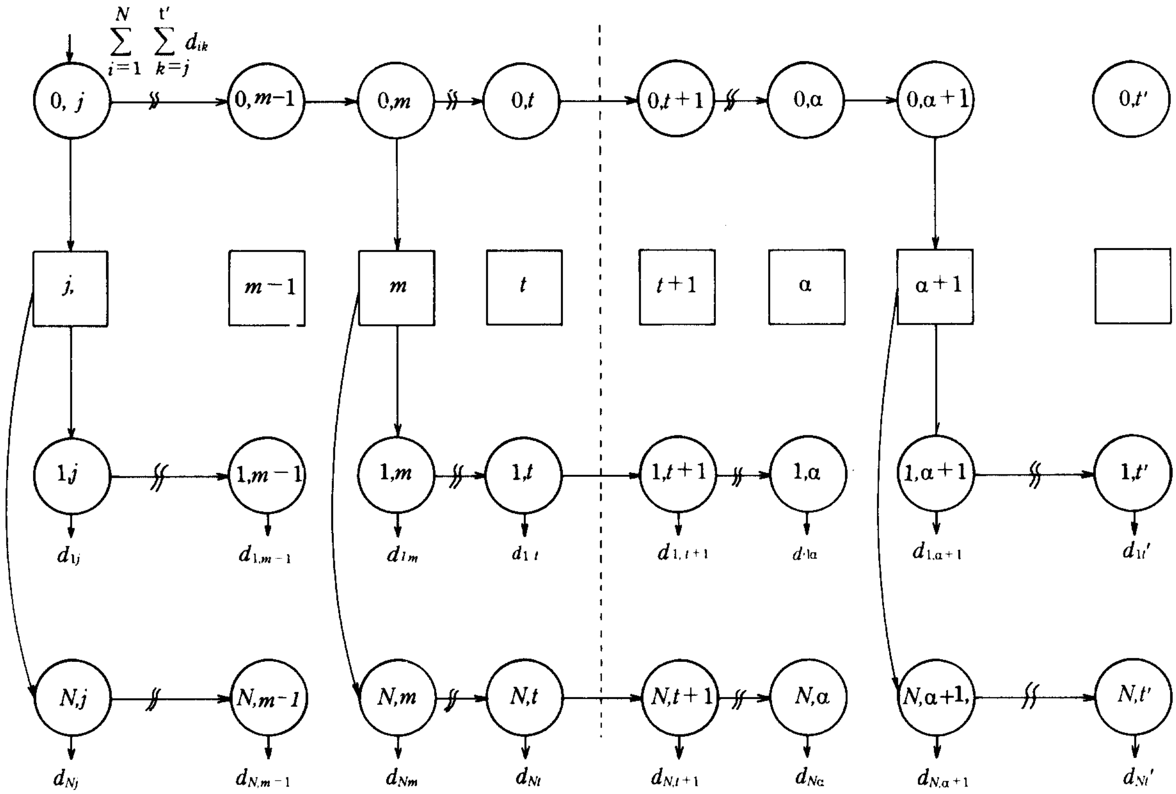

Figure 3.

$$\sum_{i=1}^{N} c_{im}\Big(\sum_{k=t+1}^{\alpha} d_{ik}\Big) + \Big(\sum_{k=t}^{\alpha} H_k\Big)\Big(\sum_{i=1}^{N}\sum_{k=\alpha+1}^{t'} d_{ik}\Big) + \sum_{i=1}^{N}\sum_{k=t}^{\alpha-1} h_{ik}\Big(\sum_{r=k+1}^{\alpha} d_{ir}\Big) + C(\alpha+1, t').$$

We denote this total cost reduction by TCR for convenience. Then

$$\begin{aligned} &C(j, t') - C(j, t) \\ &\geq TCR \\ &\geq \sum_{i=1}^{N} c_{it}\Big(\sum_{k=t+1}^{\alpha} d_{ik}\Big) + \Big(\sum_{k=t}^{\alpha} H_k\Big)\Big(\sum_{i=1}^{N}\sum_{k=\alpha+1}^{t} d_{ik}\Big) + \sum_{i=1}^{N}\sum_{k=t}^{\alpha-1} h_{ik}\Big(\sum_{r=k+1}^{\alpha} d_{ir}\Big) + C(\alpha+1, t'), \qquad (12) \end{aligned}$$

(since $c_{im} \geq c_{it}$, $i=1, \cdots, N$) for some α, $t \leq \alpha \leq t'$.

We now have

$$C(t,t') \leqq S_t + \sum_{i=1}^{N} c_{it}\left(\sum_{k=t}^{\alpha} d_{ik}\right) + \left(\sum_{k=t}^{\alpha} H_k\right)\left(\sum_{i=1}^{N}\sum_{k=\alpha+1}^{t'} d_{ik}\right)$$

$$+ \sum_{i=1}^{N}\sum_{k=t}^{\alpha-1} h_{ik}\left(\sum_{r=k+1}^{\alpha} d_{ir}\right) + C(\alpha+1, t'), \tag{13}$$

for any α, $t \leqq \alpha \leqq t'$,

since the right hand side of the inequality is the total cost of a feasible solution to $Q(t, t')$.

Finally note that

$$C(t,t) = S_t + \sum_{i=1}^{N} c_{it} d_{it}. \tag{14}$$

The proof now follows from (12), (13) and (14).□

We can now prove the following important proposition.

Proposition 5. *If $F_j(t) \geqq F_t(t)$ for $j < t$, then $F_j(t') \geqq F_t(t')$ for any $t' > t$.*

Proof. $F_j(t') = F(j-1) + K_j + a_j \sum_{i=1}^{N}\sum_{k=j}^{t'} d_{ik} + C(j, t')$

$$= F(j-1) + K_j + a_j \sum_{i=1}^{N}\sum_{k=j}^{t} d_{ik} + a_j \sum_{i=1}^{N}\sum_{k=t+1}^{t'} d_{ik}$$

$$+ C(j,t) + C(j,t') - C(j,t)$$

$$= F_j(t) + a_j \sum_{i=1}^{N}\sum_{k=t+1}^{t'} d_{ik} + C(j,t') - C(j,t).$$

Similarly, $F_t(t') = F_t(t) + a_t \sum_{i=1}^{N}\sum_{k=t+1}^{t'} d_{ik} + C(t,t') - C(t,t)$.

Therefore,

$$F_j(t') - F_t(t') = F_j(t) - F_t(t) + (a_j - a_t)\sum_{i=1}^{N}\sum_{k=t+1}^{t'} d_{ik}$$

$$+ C(j,t') - C(j,t) - C(t,t') + C(t,t)$$

$$C(j,t') - C(j,t) - C(t,t') + C(t,t)$$

(since $F_j(t) \geqq F_t(t)$ and $a_j \geqq a_t$)

$\geqq 0$ (from Lemma 1). □

Proposition 5 implies that if $F_j(t) \geqq F_t(t)$, then we can eliminate period j from consideration as the optimal last production point in any problem with horizon longer than t. (A similar property to Proposition 5 has been obtained by Chand [1] for the facilities-in-series problem with stationary fixed-plus-linear production costs and stationary linear holding costs.)

Using Proposition 5, we can now develop a regeneration set. Let

$$
\begin{aligned}
&E(0)=\phi, \text{ and} \\
&E(t)=E(t-1) \cup \{t\}-L(t),\ t=1, 2\cdots, \qquad (15)\\
\text{where}\quad &L(t)=\{j : F_j(t) \geqq F_t(t),\ j \in E(\mathrm{T}-1)\}.
\end{aligned}
$$

Now observe that $E(t)$ is the set of periods (among periods $1,\cdots,t$) that must be considered as the possible optimal last production point for any problem with horizon longer than t. Hence given $E(t-1)$, $E(t-1) \cup \{t\}$ is the set of possible last production points for the optimal solution to problem $P(t)$. After solving $P(t)$, we can generate the set $L(t)$ which is the subset of periods in $E(t-1)$ that can be eliminated from consideration as the optimal last production point for any problem $P(t')$. $t' > t$. Note that $j^*(t)$ and t always belong to $E(t)$.

Proposition 6. *Given $E(t)$, the optimal solution to $P(t')$ for any $t' > t$ has a production point in some period $k \in E(t)$.*

Proof. From Proposition 5 and (15), we know $j^*(t') \in E(t) \cup < t+1, t'>$ If $j^*(t') \in E(t)$, then the proposition holds trivially. If $j^*(t') \in \langle t+1, t'\rangle$, then the optimal solution to $P(t')$ has a first production point after t in some period $r \in \langle t+1, t'\rangle$. It follows that $j^*(r) \in E(t)$.□

Since a production point is immediately preceded by a regeneration point from Corollary 1, a regeneration set can be given by

$$R(t)=\{k-1 : k \in E(t)\},\ t=1,2\cdots.$$

Using Proposition 4, we can now find planning and forecast horizons.

Observe that recursion (9) is now restated by

$$F(t)=\min_{j \in E(t-1) \cup \{t\}} F_j(t),\ t=1, 2,\cdots \qquad (9')$$

Therefore, using (9′) and (10′) together can save considerable computational effort during the implementation of the forward algorithm.

A Numerical Example

We illustrate the forward algorithm and the planning horizon results with the following 2-retailer 4-period problem. We assume all the costs are stationary, so we let $a_t = c_{it} = 0$, $i = 1,2$, $t=1, 2, 3, 4$, without loss of generality. Set

$(d_{11}, d_{12}, d_{13}, d_{14})=(1, 4, 3, 4)$, $(d_{21}, d_{22}, d_{23}, d_{24})=(2, 5, 3, 4)$,
$K_t=10$, $S_t=5$, $H_t=1$, and $h_{1t}=h_{2t}=2, \forall t$.

Calculations for $F(t)$ and $C(j, t)$ are given in Tables 1 and 2, respectively. In Table 2, observe that we need not calculate $C_1(1, 3)$ and $C_2(2, 4)$, since $\ell^*(1, 2)=2$ and $\ell^*(2, 3)=3$. Also observe that we need not calculate $F_1(4)$ in Table 1 and $C(1, 4)$ in Table 2, since $F_1(3) \geqq F_3(3)$, so

$L(3)=\{1\}$.

Since $j^*(4)=4$ so $R(4)=\{3\}$, period 4 is a forecast horizon and period 1 (i. e., the first regeneration point in the optimal solution to $P(3)$) is a planning horizon.

From the two Tables, one can easily find the optimal solution :

$(x_1, x_2, x_3, x_4)=(3, 15, 0, 8)$,
$(z_1, z_2, z_3, z_4)=(3, 9, 6, 8)$,
$(I_1, I_2, I_3, I_4)=(0, 6, 0, 0)$,
$(y_{11}, y_{12}, y_{13}, y_{14})=(1, 4, 3, 4)$,
$(y_{21}, y_{22}, y_{23}, y_{24})=(2, 5, 3, 4)$,
$(J_{11}, J_{12}, J_{13}, J_{14})=(0, 0, 0, 0)$,
$(J_{21}, J_{22}, J_{23}, J_{24})=(0, 0, 0, 0)$, and total cost is 56.

Table 1. Calculation for $F(t)$

j \ t	1	2	3	4
1	$F_1(1)=F(0)+K+C(1,1)$ $=10+5$ $=⑮$	$F_1(2)=F(0)+K+C(1,2)$ $=10+19$ $=㉙$	$F_1(3)=F(0)+K+C(1,3)$ $=10+36$ $=46$	–
2		$F_2(2)=F(1)+K+C(2,2)$ $=15+10+5$ $=30$	$F_2(3)=F(1)+K+C(2,3)$ $=15+10+16$ $=㊶$	$F_2(4)=F(1)+K+C(2, 4)$ $=15+10+37$ $=62$
3			$F_3(3)=F(2)+K+C(3,3)$ $=29+10+5$ $=44$	$F_3(4)=F(2)+K+C(3, 4)$ $=29+10+18$ $=57$
4				$F_4(4)=F(3)+K+C(4, 4)$ $=41+10+5$ $=$ (56)
$F(t)$	15	29	41	56
$j^*(t)$	1	1	2	4
$f^*(t)$	1	2	1	1
$L(t)$	–	–	{1}	{2,3}
$E(t)$	{1}	{1,2}	{2,3}	{4}
$R(t)$	{0}	{0,1}	{1,2}	{3}

Table 2. Calculation for $C(j,\ t)$

j \ t	1	2	3	4
1	$C(1,1)=5,\ \ell^{*}(1,1)=1$	$C(1,2)=19,\ \ell^{*}(1,2)=2$	$C(1,3)=36,\ \ell^{*}(1,3)=3$	$C(1,4)$ −
		$C_1(1,2)=23$ $C_2(1,2)=$ ⑲	$C_1(1,3)$ − $C_2(1,3)=37$ $C_3(1,3)=$ ㊱	
2		$C(2,2)=5,\ \ell^{*}(2,2)=2$	$C(2,3)=16,\ \ell^{*}(2,3)=3$	$C(2,4)=37, 1^{*}(2,4)=4$
			$C_2(2,3)=17$ $C_3(2,3)=$ ⑯	$C_2(2,4)$ − $C_3(2,4)=40$ $C_4(2,4)=$ ㊲
3			$C(3,3)=5,\ \ell^{*}(3,3)=3$	$C(3,4)=18,\ \ell^{*}(3,4)=4$
				$C_3(3,4)=21$ $C_4(3,4)=$ ⑱
4				$C(4,4)=5,\ \ell^{*}(4,4)=4$

References

1. Chand, S., "Rolling Horizon Procedures for the Facilities in Series Inventory Model with Nested Schedules," *Management Science,* Vol. 29, No. 2, pp. 237-249, 1983.
2. Kalymon, B. A., "A Decomposition Algorithm for Arborescence Inventory Systems," *Operations Research,* Vol. 20, No. 4, pp. 860-874, 1972.
3. Love, S. F., "A Facilities in Series Inventory Model with Nested Schedules," *Management Science,* Vol. 18, No. 5, pp. 327-338, 1972.
4. Lundin, R. A. and Morton, T. E., "Planning Horizons for the Dynamic Lot Size Model : Zabel vs. Protective Procedures and Computational Results," *Operations Research,* Vol. 23, No. 4, pp. 711-734, 1975.
5. Roundy, R., "98%-Effective Integer-Ratio Lot-Sizing for One-Warehouse Multi-Retailer Systems," *Management Science,* Vol. 31, No. 11, pp. 1416-1430, 1985.
6. Veinott, A. F., Jr., "Minimum Concave Cost Solution of Leontief Substitution Models of Multi-Facility Inventory Systems," *Operations Research,* Vol. 17, No, pp. 262-291, 1969.
7. Wagner, H. M. and Whitin, T. M., "Dynamic Version of the Economic Lot Size Model," *Management Science,* Vol. 5, No. 1, pp. 89-96, 1958.
8. Zangwill, W. I., "A Deterministic Multiproduct, Multifacility Production and Inventory Model," *Operations Research,* Vol. 14, No. 3, pp. 486-507, 1966.
9. Zangwill, W. I., "Minimum Concave Cost Flows in Certain Networks," *Management Science,* Vol. 14, No. 7, pp. 429-450, 1968.
10. Zangwill, W. I., "A Backlogging Model and a Multi-Echelon Model of a Dynamic Economic Lot Size Production System - A Network Approach," *Management Science,* Vol. 15, No. 9, pp. 506-527, 1969.