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Unsteady Wave Generation by an Oscillating Cylinder Advancing under the Free Surface

by

D.C. Hong*

Abstract

The radiation problem for an oscillating cylinder advancing under the free surface with a constant horizontal velocity is studied using the Green integral equation in the frequency domain. The Green function expressed in terms of the complex exponential integral, is derived using the damped free surface condition.

Special attention is given to the behavior of the numerical solution in the vicinity of the critical Brard number $\gamma_c=\omega\cdot u/g=0.25$, where ω is the circular frequency of encounter, u the advancing speed and g the gravitational acceleration.It is shown that the solution is finite in the vicinity of γ_c although the Green function becomes singular at γ_c. It is also shown that the computed hydrodynamic coefficients agree well with those obtained from the solution of the same problem formulated in the time domain.

I. Introduction

The Green function associated with a three dimensional pulsating source advancing with a constant horizontal velocity has been presented by Brard in 1948[1]. He has shown that the behavior of the far field waves changes according to a non-dimensional number $\gamma=\omega\cdot u/g$. From his theory and experimental results, he has concluded that there are two different regimes of far-field waves separated by a transition zone located in the vicinity of the critical Brard number γ_c. Haskind has also studied the same problem but in two dimension[2]. It is well known that the expressions of Green functions in both memoirs are singular at γ_c.

In this paper, the Green function associated with a two dimensional pulsating source is derived using the damped free surface condition. The present expression of the Green funciton is also singular at γ_c. Since the most probabel cause of the singularity is the linearization of the free surface condition, the non-linear free surface effects must be taken into account to remove the singularity. But the aim of the present work is to show the behavior of the linearized solution in the vicinity of γ_c to provide correct reference to existing or future non-linear solutions. It can also be compared with the results computed in the time domain to confirm mutual agreement[3].

II. Formulation of the problem

The fluid is assumed to occupy a space D bounded by the wetted surface S of an immersed cylindrical body and by the free sunface F of deep water under

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* Member, Ship Research Station, KIMM

gravity. The body performs simple harmonic oscillations of small amplitude with circular frequency ω about its mean position which is assumed to advance with uniform horizontal velocity u. Cartesian coordinates (x, y) attached to the mean position of the body, are employed with the origin O in the undisturbed free surface above the body and the y axis vertically upwards. The plane xoy is perpendicular to the generatrices of the cylindrical body in order that the problem may be treated in two dimensions.

With the usual assumptions of an incompressible fluid and irrotational flow without capillarity, the fluid velocity $\vec{v}$ is given by the gradient of a velocity potential $\Phi_T(x, y, t)$. To find the potential Φ_T, the boundary conditions on S, on F and at infinity must be known. Brard has avoided the difficulty due to the ignorance of the condition at infinity by introducing the notion of the almost perfect fluid characterized by a friction force per unit volume equal to $-\varepsilon\vec{v}$ where ε is an infinitesimal positive real number. Following Brard, the governing equations for Φ_T can be given as follows:

$$\nabla^2\Phi_T=O \qquad \text{in } D \qquad (2.1)$$

$$\frac{\partial\Phi_T}{\partial n}=\vec{v}_e\cdot\vec{n} \qquad \text{on } S \qquad (2.2)$$

$$\frac{\partial'^2\Phi_T}{\partial t^2}+g\frac{\partial\Phi_T}{\partial y}+\varepsilon\frac{\partial'\Phi_T}{\partial t}=O \qquad \text{on } F \qquad (2.3)$$

Here $\vec{n}$ denotes the normal vector directed into the fluid region D. It should be noted that the problem is linearized assuming that the square of $\vec{v}$ is small enough to be neglected compared with other quantities. In(2.2), $\vec{v}_e$ denotes the entrainment velocity.

$$\vec{v}_e(M)=u\vec{e}_1+\dot{a}_1\vec{e}_1+\dot{a}_2\vec{e}_2+\dot{a}_3\vec{e}_3\times O_1M, \quad M\in S \qquad (2.4)$$

Here $a_j(j=1,2,3)$ denote the displacements due to surge, heave and pitch and O_1 the center of rotation of the body.

Since $\vec{v}_e$ is composed of the steady velocity u and oscillating velocities $\dot{a}_j(j=1,2,3)$, the potential Φ_T can be decomposed as follows:

$$\Phi_T=\Phi_0(x,y)+\Phi(x,y,t) \qquad (2.5)$$

Here Φ_0 is the steady potential of the stationary lee waves.

Assuming that $\nabla\Phi_0$, $\nabla\Phi$ and a_j are all small and comparable to each other in magnitude, the coupling effect between Φ_0 and Φ can be disregarded in this linear problem. Then each of the problems for Φ_0 and Φ can be solved separately. In this paper the problem for Φ only is treated.

Since the cartesian coordinates employed here are attached to S_0, the body surface at its mean position, the normal vector $\vec{n}$ on S and the time derivative $\frac{\partial'}{\partial t}$ can be expressed as follows:

$$\vec{n}|_S=\vec{n}|_{S_0}+a_3\vec{e}_3\times\vec{n}|_{S_0} \qquad (2.6)$$

$$\frac{\partial'}{\partial t}=\frac{\partial}{\partial t}-u\vec{e}_1\cdot\nabla \qquad (2.7)$$

Substuting (2.6) and (2.7) into (2.2) and (2.3) respectively, the boundary conditions for Φ can be found:

$$\frac{\partial\Phi}{\partial n}=[\dot{a}_1\vec{e}_1+(\dot{a}_2-ua_3)\vec{e}_2+\dot{a}_3\vec{e}_3\times\overrightarrow{O_1M}]\cdot\vec{n} \quad \text{on } S_0 \qquad (2.8)$$

$$\frac{\partial^2\Phi}{\partial t^2}-2u\frac{\partial^2\Phi}{\partial x\partial t}+u^2\frac{\partial^2\Phi}{\partial x^2}+g\frac{\partial\Phi}{\partial y}+\varepsilon\frac{\partial\Phi}{\partial t}-\varepsilon u\frac{\partial\Phi}{\partial x}=O \qquad \text{on } F \qquad (2.9)$$

Considering the condition (2.8), it can be found that Φ takes the following form:

$$\Phi=R_e\left\{-i\omega\sum_{j=1}^{3}[(a_j^*+ia_j^{**})\phi_j\, e^{-i\omega t}]-u(a_3^*+ia_3^{**})\phi^2e^{-i\omega t}\right\} \qquad (2.10)$$

with

$$\phi_j=\phi_j^*+i\phi_j^{**} \qquad (2.11)$$

$$a_j=R_e\,\{(a_j^*+ia_j^{**})e^{-i\omega t}\} \qquad (2.12)$$

Here $a_j^*+ia_j^{**}$ represent the magnitude and phase of the displacement due to the oscillation in the jth direction and ϕ_j the complex valued elementary potential.

Taking into account of formulas (2.8) and (2.10), the body boundary conditions for $\phi_j(j=1,2,3)$ can be found:

$$\frac{\partial\phi_j}{\partial n}=\vec{e}_j\cdot\vec{n} \qquad \text{on } S_0 \text{ for } j=1,2 \qquad (2.13)$$

$$\frac{\partial\phi_3}{\partial n}=(\vec{e}_3\times\overrightarrow{O_1M})\cdot\vec{n} \qquad \text{on } S_0 \qquad (2.14)$$

III. Construction of the Green function

One can solve the present boundary value problem by making use of the Green integral equation. The following expression of the Green function has already been presented by Hong[4]. Here, a brief description of the process to derive the Green function in the complex plane $z=x+iy$ will be given.

In order to simplify the calculation, the following expression of a complex potential is adopted:

$$F(z,z';t)=\frac{q^*}{2\pi}\cos\omega t\,\log\frac{z-z'}{z-\bar{z}'}+f(z,z';t) \tag{3.1}$$

Here, the function f is a complex potential which will enable the complex potential $F(z,z';t)$ to satisfy the free surface condition issued from (2.9):

$$\boldsymbol{R}_e\left\{\left[\frac{\partial^2}{\partial t^2}-2u\frac{\partial^2}{\partial z\partial t}+u^2\frac{\partial^2}{\partial z^2}+ig\frac{\partial}{\partial z}+\varepsilon\frac{\partial}{\partial t}-\varepsilon u\frac{\partial}{\partial z}\right]F(z,z';t)\right\}_{|y=0}=0 \tag{3.2}$$

Expressing the function f in the form of a Fourier integral

$$f(z,z';t)=\frac{1}{\pi}\int_0^\infty\hat{\phi}(k,t;z')e^{-ikz}dk \tag{3.3}$$

the equation (3.2) becomes

$$\left[\frac{\partial^2}{\partial t^2}+(2iuk+\varepsilon)\frac{\partial}{\partial t}+gk-u^2k^2+i\varepsilon uk\right]\hat{\phi}=-gq^*e^{ik\bar{z}'}\cos\omega t \tag{3.4}$$

The above ordinary differential equation for the Fourier transform $\hat{\phi}$ admits the following particular solution:

$$\hat{\phi}=\frac{gq^*}{2}e^{ik\bar{z}'}(e^{-i\omega t}/D_1+e^{i\omega t}/D_2) \tag{3.5}$$

with,

$$D_1=D_1'-i\varepsilon(uk-\omega),\quad D_1'=(uk-\omega)^2-gk \tag{3.5a}$$

$$D_2=D_2'-i\varepsilon(uk+\omega),\quad D_2'=(uk+\omega)^2-gk \tag{3.5b}$$

Substitution of (3.5) to (3.3) yields

$$f(z,z';t)=f_1(z,z';t)+f_2(z,z';t) \tag{3.6}$$

$$f_1(z,z';t)=\frac{gq^*}{2}e^{-i\omega t}\int_0^\infty e^{-ik(z-\bar{z}')}/D_1\cdot dk \tag{3.6a}$$

$$f_2(z,z';t)=\frac{gq^*}{2\pi}e^{i\omega t}\int_0^\infty e^{-ik(z-\bar{z}')}/D_2\cdot dk \tag{3.6b}$$

Decomposing $1/D$ in two fractional equations, the integrals in the equations (3.6a) and (3.6b) become,

$$I_1=\frac{1}{\sqrt{1+4\gamma}}\int_0^\infty\left\{\frac{\exp[-iK(z-\bar{z}')]}{K-(K_2+i\delta)}-\frac{\exp[-iK(z-\bar{z}')]}{K-(K_1+i\delta)}\right\}dK \tag{3.7a}$$

$$I_2=\frac{1}{\sqrt{1-4\gamma}}\int_0^\infty\left\{\frac{\exp[-iK(z-\bar{z}')]}{K-(K_4+i\delta)}-\frac{\exp[-iK(z-\bar{z}')]}{K-(K_3-i\delta)}\right\}dK,\qquad \gamma\neq\frac{1}{4} \tag{3.7b}$$

Here, the following non-dimensionalyzation is introduced:

$Z=z/L,\qquad K=k\cdot L,\qquad \tilde{u}=u/\sqrt{gL}$

$\tilde{\omega}=\omega\sqrt{L/g},\qquad \gamma=\tilde{u}\tilde{\omega}=u\omega,\qquad L=$characteristic length

The non-dimensional parameter $\pm\delta,\delta$ being positive, in the denominators of the integrands of I_1 and I_2 is the product of ε and the sign of $(uk\pm\omega)$ in the vicinity of the roots K_1, K_2, K_3 and K_4 of D_1' and D_2'.

$$K_i=[1+2\gamma+(-1)^i\sqrt{1+4\gamma}]/2\tilde{u}^2,\qquad i=1,2 \tag{3.8a}$$

$$K_i=[1-2\gamma+(-1)^i\sqrt{1-4\gamma}]/2\tilde{u}^2s\qquad i=3,4 \tag{3.8b}$$

It should be noted that D_1' posseses alway two distinct real roots K_1 and K_2. But D_2' can admit two distinst real roots, double root or two complex roots according to the value of γ greater, equal or less than 0.25, the critical Brard number γ_c. However, as far as the damping parameter is taken into account, D_1 and D_2 posses always two distinct complex roots whatever the value of γ is. So, when γ is equal to 0.25, the integral I_2 takes the following form:

$$I_2=\frac{1}{i\delta}\int_0^\infty\left\{\frac{\exp[-iK(z-\bar{z}')]}{K-(K_c+i\delta)}-\frac{\exp[-iK(z-\bar{z}')]}{K-(K_c-i\delta)}\right\}dK,\qquad \gamma=\frac{1}{4} \tag{3.9}$$

$$K_c=(1-2\gamma)/2\tilde{u}^2=1/4u^2 \tag{3.10}$$

Since the integral in (3.9) does not vanish, the value of I_2 at γ_c will diverge when δ approaches zero. However, the formula (3.9) shows that the value of I_2 at γ_c when γ approaches γ_c from the domain $\gamma>\gamma_c$ is identical with the one from the domain $\gamma<\gamma_c$ as far as the artificial damping parameter is retained.

According to the table in the appendix, I_1 and I_2 take finally the following form:

$$I_1=\frac{1}{\sqrt{1+4\gamma}}\{e^{\zeta_2}[\mathcal{E}_1(\zeta_2)+2i\pi]-e^{\zeta_1}[\mathcal{E}_1(\zeta_1)+2i\pi]\},\qquad \forall\gamma \tag{3.11}$$

$$I_2=\begin{cases}\dfrac{1}{\sqrt{1-4\gamma}}\{e^{\zeta_4}[\mathcal{E}_1(\zeta_4)+2i\pi]-e^{\zeta_3}\mathcal{E}_1(\zeta_3)\}, & \gamma<\dfrac{1}{2},\ \gamma\neq\dfrac{1}{4} \quad (3.12a)\\[2ex] \dfrac{1}{\sqrt{1-4\gamma}}[e^{\zeta_4}E_1(\zeta_4)-e^{\zeta_3}E_1(\zeta_3)], & \gamma\geq\dfrac{1}{2} \quad (3.12b)\end{cases}$$

with,

$$\zeta_j=-i\ K_j(Z-\bar{Z}'),\qquad j=1,2,3,4 \tag{3.13}$$

By using formulas (3.11) to (3.13), one has

$$F(Z,Z';t)=\frac{q^*}{2\pi}\ (\cos\omega t\log\frac{Z-Z'}{Z-\bar{Z}'}+I_1e^{-i\omega t}+I_2e^{i\omega t}) \tag{3.14}$$

Now, let us consider the farfield behavior of waves generated by a pulsating point source in uniform translation. By making use of the properties of the complex exponential integral

$$E_1(\zeta)\to 0 \qquad \text{for } |\zeta|\to\infty$$

$$\mathcal{E}_1(\zeta)=\begin{cases}\to 0 & \text{for } |\zeta|\to\infty \text{ and } I_m(\zeta)>0\\ \to -2i\pi & \text{for } |\zeta|\to\infty \text{ and } I_m(\zeta)<0\end{cases}$$

one can find the following expressions of the Green function at infinity:

$$F(Z,Z';\ t)|_{\gamma<\gamma_c}=\begin{cases}iq^*\left[\dfrac{e^{-i\omega t}}{\sqrt{1+4\gamma}}(e^{\zeta_2}-e^{\zeta_1})+\dfrac{e^{i\omega t}}{\sqrt{1-4\gamma}}e^{\zeta_4}\right] & \text{for } (X-X')\to-\infty\\ iq^*e^{i\omega t}e^{\zeta_3}/\sqrt{1-4\gamma} & \text{for } (X-X')\to\infty\end{cases} \tag{3.15}$$

$$F(Z,Z';\ t)|_{\gamma>\gamma_c}=\begin{cases}iq^*\dfrac{e^{-i\omega t}}{\sqrt{1+4\gamma}}(e^{\zeta_2}-e^{\zeta_1}) & \text{for } (X-X')\to-\infty\\ 0 & \text{for } (X-X')\to\infty\end{cases} \tag{3.16}$$

Denoting the celerity of a wave related with $K_j(j=1,2,3,4)$ by $C_j(j=1,2,3,4)$, one has

$$C_1=-u\frac{\sqrt{1+4\gamma}+1}{2\gamma}\left(=\frac{-g}{\omega}\frac{\sqrt{1+4\gamma}+1}{2}\right)\qquad \text{for } (X-X')\to-\infty,\ \forall\gamma \tag{3.17}$$

$$C_2=u\frac{\sqrt{1+4\gamma}-1}{2\gamma}\left(=\frac{g}{\omega}\frac{\sqrt{1+4\gamma}-1}{2}\right)\qquad \text{for } (X-X')\to-\infty,\ \forall\gamma \tag{3.18}$$

$$C_3=u\frac{\sqrt{1-4\gamma}+1}{2\gamma}\left(=\frac{g}{\omega}\frac{\sqrt{1-4\gamma}+1}{2}\right)\qquad \text{for } (X-X')\to\infty,\ \gamma<\gamma_c \tag{3.19}$$

$$C_4=u\frac{1-\sqrt{1-4\gamma}}{2\gamma}\left(=\frac{g}{\omega}\frac{1-\sqrt{1-4\gamma}}{2}\right)\qquad \text{for } (X-X')\to-\infty,\ \gamma<\gamma_c \tag{3.20}$$

The above formulas show that there are four waves for $\gamma<\gamma_c$ and two waves for $\gamma>\gamma_c$.

By using the expression of I_2 at γ_c given by the formula (3.9), it can be shown that

$$C_3=C_4=2u \qquad \text{for } \gamma=\gamma_c \tag{3.21}$$

It signifies that at γ_c, the total amount of energy influx by the K_4 wave is carried away by the K_3 wave. So the resonance will not occur in the fluid contained in the region bounded by the free surface and a space-fixed geometric surface. But the energy of K_3 and K_4 waves cannot be radiated from the source since their energy transmission velocity equals the steady advancing velocity of the source. In consequence, the resonance will occur locally around the source which is in steady translating motion accompanied by small oscillations. But it is not physically acceptable and the linearized solution fails at γ_c.

IV. Green Integral Equation

Using the expression of the complex potential given by the formula (3.14), the Green function $G(z, z')$ is defined as follows.

$$G(Z, Z')=G^*(Z, Z')+iG^{**}(Z, Z') \tag{4.1}$$

$$R_e[F(Z, Z';\ t)]=R_e[G(Z, Z')e^{-i\omega t}]\cdot q^* \tag{4.2}$$

The Green function $G(z, z')$ is complex valued in accordance with the elementary potentials.

Applying Green's theorem to one of the elementary potentials and the Green function in the fluid region D, a Fredholm integral equation of the second kind for the potential can be obtained:

$$\frac{\phi(Z)}{2}+\int_{S_0}\phi(Z')\frac{\partial G(Z, Z')}{\partial n_{Z'}}dS_{Z'}=\int_{S_0}\frac{\partial\phi(Z')}{\partial n}G(Z, Z')dS_{Z'} \qquad P\in S_0 \tag{4.3}$$

Substituting the boundary conditions (2.13) and (2.14) in turn into the above equation, it can be solved with the aid of the discretization. Then using the Bernoulli equation, the pressure on S_0 can be obtained:

$$p=-\rho\left(\frac{\partial\Phi}{\partial t}-u\frac{\partial\Phi}{\partial x}\right) \tag{4.4}$$

Then the hydrodynamic pressure forces and moment due to the unsteady potential can be obtained as

$$\vec{F}=-\int_{S_0}p\vec{n}dS \text{ on } S_0 \tag{4.5}$$

$$\vec{M}=-\int_{S_0}p(\overrightarrow{O_1M}\times\vec{n})dS, \qquad M\in S_0 \tag{4.6}$$

Using the expression of Φ given by (2.10), the following explicit form of F and M can be found:

$$F_i=-\rho L^2\sum_{j=1}^{3}[M_{ij}\ddot{a}_j+\omega D_{ij}\dot{a}_j], \qquad i=1, 2, 3 \tag{4.7}$$

or

$$F_i=-\rho gL\sum_{j=1}^{3}[\mu_{ij}\ddot{a}_j/\omega^2+\lambda_{ij}\dot{a}_j/\omega], \qquad i=1, 2, 3 \tag{4.8}$$

with

$$F_3=\vec{e}_3\cdot\vec{M}/L \text{ and } F_i=\vec{F}\cdot\vec{e}_i, \qquad i=1, 2 \tag{4.9}$$

Here, the non-dimensional coefficients M_{ij} or μ_{ij} are known as the added-mass coefficients and D_{ij} or λ_{ij} the wave-damping coefficients.

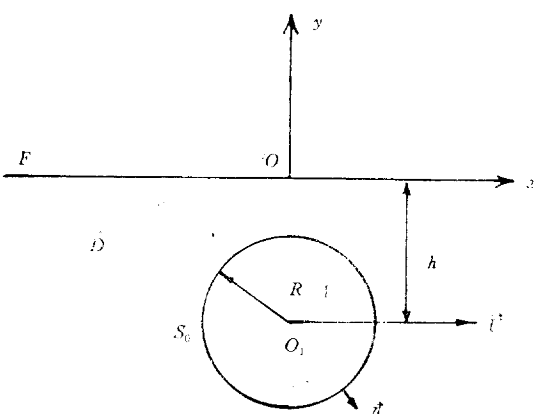

Fig. 1 Coordinate systems

V. Numerical Results and Discussion

The added-mass and wave-damping coefficients of a circular cylinder completely immersed in water of infinite depth are computed (see Fig. 1).

In Fig. 2, the surge (or heave) induced surge (or heave) added-mass coefficients are presented with other numerical results for $u/\sqrt{gR}=0.25$. The results of Kim are obtained from the wave-damping coefficients computed in the time domain using the Kramers-Kronig relationship. As shown in this figure, the present results agree well with the results of Kim and there is no singular behavior in the vicinity of γ_c. It seems that there are some numerical errors in the results of Park for $0.6<\omega\sqrt{R/g}\leq 1.0$.

In Fig. 3, μ_{11} and λ_{11} are presented and compared with the results of Hwang and Kim computed in the time domain for $u/\sqrt{gR}=0.4$. They agree well with each other in this case too. But the present results show that there is discontinuity in the slopes of μ and λ curves at γ_c. In Fig. 4, the μ and λ

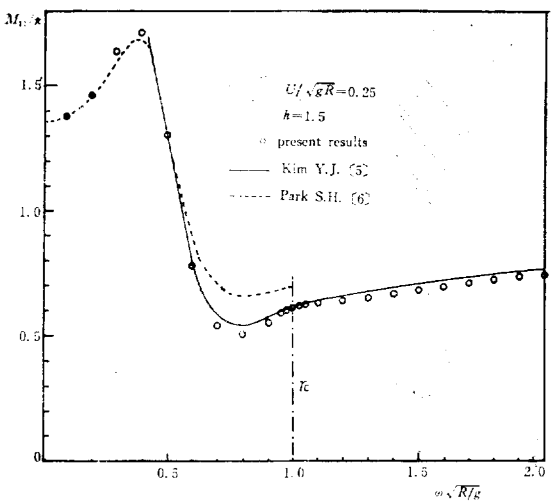

Fig. 2 Surge induced surge added-mass coefficients

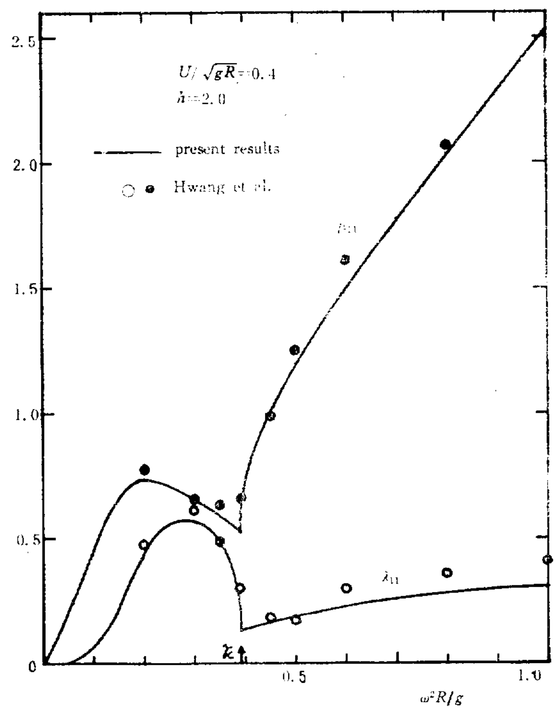

Fig. 3 Surge induced surge hydrodynamic coefficients

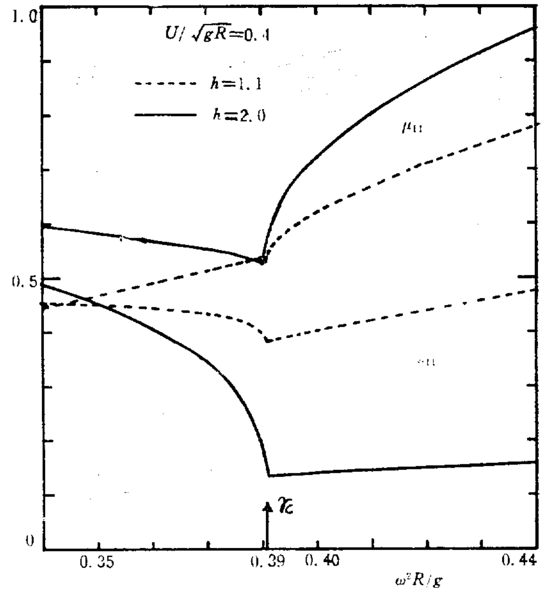

Fig. 4 Surge induced surge hydrodynamic coefficients

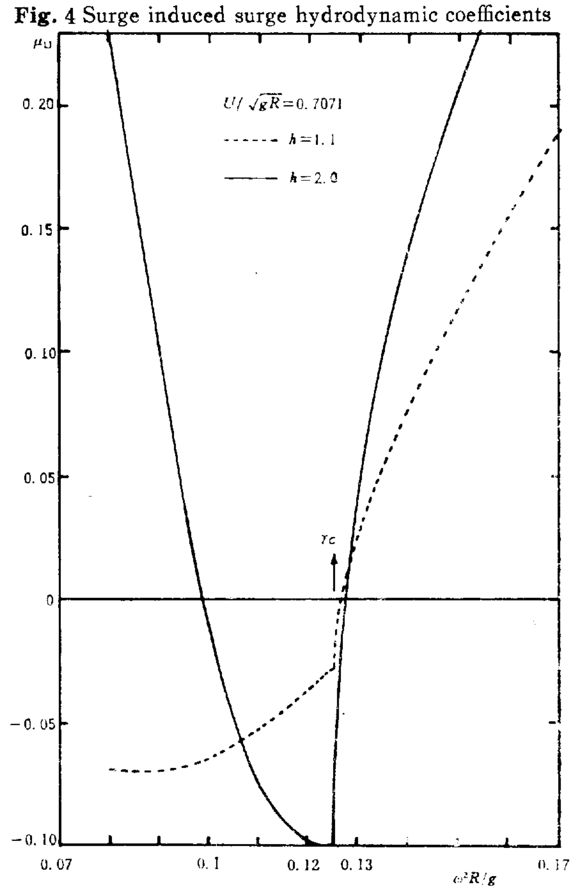

Fig. 5 Surge induced surge added-mass coefficients

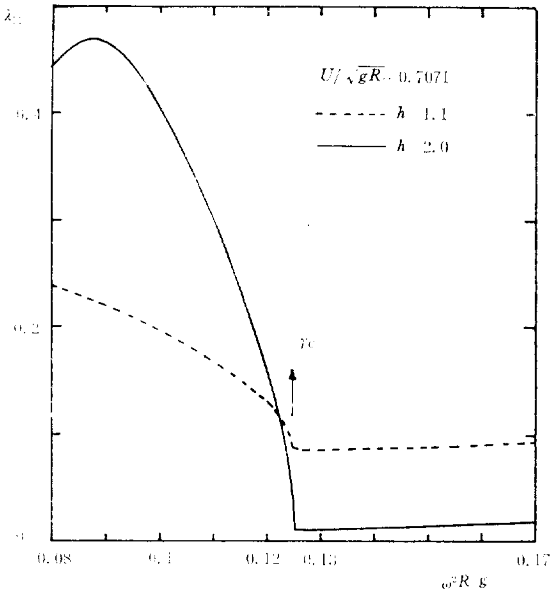

Fig. 6 Surge induced surge wave-damping coefficients

curves are presented in a greater scale to show that the values of μ and λ remain finite in the vicinity of γ_c. Here μ and λ curves for $h=1.1$ are also presented. In Figs. 5 and 6, μ and λ curves for $h=1.1$ and $h=2.0$ are presented for $u/\sqrt{gR}=0.7071$. The values of μ and λ are again finite in the vicinity of γ_c in spite of the discontinuity in their slopes. The curves in Figs. 4 to 6 show that the changes at γ_c for $h=2.0$ are more abrupt than those for $h=1.1$. Since the discontinuity is entirely due to the contribution from K_3 and K_4 waves, the changes across γ_c will be great or slight according as the contribution from K_3 and K_4 waves is greater or smaller than one from K_1 and K_2 waves.

In conclusion, the following facts are expected:

1. The proposition of Brard presented in the introduction is confirmed numerically by showing the discontinuity in the slopes of μ and λ curves at γ_c.

2. The numerical solutions remain finite in the vicinity of γ_c where $(\gamma-\gamma_c)$ is of order 10^{-4}.

3. The non-linear analysis is required to obtain the exact solution at γ_c where the linear solution fails.

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Appendix

The integrals in the expression of the Green Function can be represented by

$$I(K_0, W)=\int_0^\infty \frac{e^{-kW}}{k-K_0}\,dk,\quad Re(W)>0 \qquad (A.1)$$

with

$$W=u+iv=\rho e^{i\theta},\quad |\theta|<\pi/2 \qquad (A.2)$$

$$K_0=k_0+im_0,\quad m_0\neq 0 \qquad (A.3)$$

It is well known that this type of integral can be evaluated by using the residue theorem with a suitable contour Γ. A priori, Γ can be taken in the first or fourth quadrant of the complex plane K.

$$K=k-im=\gamma e^{i\alpha}, \quad |\alpha|<\pi/2 \qquad (A.4)$$

The contour consists of the positive real axis $\overline{OA}$, a circular arc of radius R tending to infinity $\widehat{AB}$ and a straight line returning to the origin $\overline{BO}$. The line BO makes an angle x with the positive real axis. Considering that

$$-kW=-\gamma\rho[\cos(\theta+\alpha)+i\sin(\theta+\alpha)] \qquad (A.5)$$

the value of x should be

$$x=-\theta, \quad \theta=\tan^{-1}v/u \qquad (A.6)$$

Therefore the contour Γ will be taken in the first or fourth quadrant according as the sign of θ is negative or positive in order that the integral may converge at infinity.

After some mathematical operations and making use of the following formula

$$\int_0^\infty \frac{e^{-t}}{t-\zeta}dt=e^{\zeta}\int_\zeta^\infty \frac{e^{-t}}{t}dt=e^{\zeta}E_1(\zeta), \quad |\arg\zeta|<\pi \qquad (A.7)$$

the integral can be evaluated using one of the following expressions according to the values of k_0 and m_0:

Table of I (K_0, W)

	$k_0<0$	$k_0=0$	$k_0>0$
$m_0<0$	$e^W E_1(W)$	$e^W E_1(W)$	$e^W \mathcal{E}_1(W)$
$m_0>0$	$e^W E_1(W)$	$e^W E_1(W)$	$e^W[\mathcal{E}_1(W)+2i\pi]$

Here $E_1(\zeta)$ is the complex exponential integral and $\mathcal{E}_1(\zeta)$ is the modified complex exponential integral defined as follows:

$$\mathcal{E}_1(\zeta)=\begin{cases} E_1(\zeta) & \text{for } I_m(\zeta)>0 \\ E_1(\zeta)-2i\pi & \text{for } I_m(\zeta)<0 \end{cases}$$

The numerical computing techniques for $E_1(\zeta)$ can be found in the reference [7]