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TUBE FORMULAS FROM CHERN'S KINEMATIC FORMULA

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1. Introduction

In 1966 Chern [1] proved the following remarkable kinematic formula. For any compact p-dimensional submanifold P of Euclidean n-space $\boldsymbol{R}^n$, Chern considered invariants I_e of $P, 0\leq e$ even$\leq p$, which are given by

$$(1)\qquad I_e=\frac{(p-e)!}{2^{e/2}p!}\sum\delta\binom{\alpha}{\beta}R_{\alpha_1\alpha_2\beta_1\beta_2}\cdots R_{\alpha_{e-1}\alpha_e\beta_{e-1}\beta_e},$$

R_{ijkl} being the curvature tensor of P, while $\delta\binom{\alpha}{\beta}$ is $+1$ or -1 according as $\alpha_1, \cdots, \alpha_e$ are distinct and an even or odd permutation of $\beta_1, \cdots, \beta_e$, and otherwise $\delta\binom{\alpha}{\beta}$ is zero. The summation in I_e is taken over all α's and β's running from 1 to p. When P is oriented, the integral of I_e over P is denoted by $\mu_e(P)$. Let P and Q be compact manifolds of dimensions p and q imbedded in $\boldsymbol{R}^n$, and let g be an element of the group $E(n)$ of proper motions of $\boldsymbol{R}^n$. For almost all $g\in E(n)$, $P\cap gQ$ is again a submanifold of dimension $p+q-n$, and $\mu_e(P\cap gQ)$ are meaningful quantities. Chern proved that, if $0\leq e$ even$\leq p+q-n$, then

$$(2)\qquad \int\mu_e(P\cap gQ)\,dg=\sum_{0\leq i \text{ even}\leq e}c_i\mu_i(P)\mu_{e-i}(Q)$$

for constants c_i depending on p, q, n and e, while the integration extends over $E(n)$, and dg is the Haar measure on $E(n)$.

In this paper, by employing this kinematic formula and the generalized Gauss-Bonnet formula, we derive formulas related to the volume of a tube about a submanifold in $\boldsymbol{R}^n$. Specifically let $V_P{}^n(r)$ be the n-dimensional volume of a tube of radius r about P in $\boldsymbol{R}^n$. Throughout the paper we assume that $r>0$ is less than or equal to the distance from P to its nearest focal point. We will derive Weyl's tube formula

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[5] for odd dimensional P

(3) $$V_P{}^n(r)=\sum_{c=0}^{[p/2]}\frac{\pi^{(n-p)/2}k_{2c}(P)}{2^c\Gamma\left(\frac{n-p}{2}+c+1\right)}r^{n-p+2c},$$

where the invariants

(4) $$k_{2c}(P)=\frac{p!}{2^c(p-2c)!c!}\mu_{2c}(P).$$

In this derivation we need the generalized Gauss-Bonnet formula as follows:

(5) $$\mu_p(P)=\frac{(2\pi)^{p-2}}{(p-1)(p-3)\cdots3\cdot1}\chi(P),$$

where $p=\dim P$ is even and $\chi(P)$ is the Euler characteristic of P. We also need the product expression [4] for c_i in (2) as

(6) $$c_i=O_{n+1}\cdots O_2\cdot\frac{\dfrac{O_{p+q-n+1}O_{p+q-n+2}\left(\frac{e}{2}\right)!}{O_{p+q-n-e+2}}}{\left(\dfrac{O_{p+1}O_{p+2}\left(\frac{i}{2}\right)!}{O_{p-i+2}}\right)\left(\dfrac{O_{q+1}O_{q+2}\left(\frac{e-i}{2}\right)!}{O_{q-e+i+2}}\right)}$$

where $O_m=\dfrac{2\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)}$ is the volume of the unit sphere of dimension $m-1$ in $\boldsymbol{R}^m$. Moreover with the formula due to Nijenhuis [4]

(7) $$k_{2c}(P\times Q)=\sum_{i=0}^{c}k_{2i}(P)k_{2c-2i}(Q)$$

it is not difficult to obtain the following product formula

(8) $$V_{P\times Q}{}^n(r)=\sum_{a=0}^{[p/2]}\sum_{b=0}^{[q/2]}\frac{\pi^{(n-p-q)/2}k_{2a}(P)k_{2b}(Q)}{2^{a+b}\Gamma(a+b+1+\frac{n-p-q}{2})}r^{n-p-q+2a+2b}$$

when $p+q$ is odd. Here $P\times Q$ is the Riemannian product of P and Q.

Similarly we can derive the expression for $k_{2c}(P_r)$, where P_r is the tubular hypersurface of radius r about P, in terms of $k_{2a}(P)$ and r. If p is odd and n is even, then we have

(9) $$k_{2c}(P_r)=\sum_{a=0}^{[p/2]}\frac{2^{c-a+1}\pi^{(n-p-1)/2}\Gamma\left(c+\frac{1}{2}\right)}{\Gamma\left(\frac{n-p}{2}+a\right)}\binom{n-p+2a-1}{2c}$$

$$\times k_{2a}(P)r^{n-p+2a-2c-1}.$$

REMARKS.

(1) We essentially follow Wolf [6] for the derivation of (3). But computations are simplified greatly with the expression (6) of c_i.

(2) In general the formulas (3), (8) and (9) hold without assumptions on parity (see for example [2, 3, 5]). But we derive them as applications of Chern's kinematic formula under the parity assumptions.

(3) The Haar measure dg on $E(n)=\boldsymbol{R}^n \dot{\times} SO(n)$ is normalized so that $dg=dx\wedge dg_0$, where dx is the volume element on $\boldsymbol{R}^n$ and dg_0 is the Haar measure on $SO(n)$ such that

$$\int dg_0=O_n O_{n-1}\cdots O_2. \tag{10}$$

(4) A direct proof of (7) is given in [2].

2. Derivations from Chern's kinematic formula

Proof of (3). Let P be a compact imbedded submanifold of $\boldsymbol{R}^n$ and let $p=\dim P$ be odd. We will apply Chern's kinematic formula (2) with P as the stationary submanifold and with $S^{n-1}(r)$ as the moving submanifold of $\boldsymbol{R}^n$. Here $S^{n-1}(r)$ is the $(n-1)$-sphere of radius r, and $r>0$ is less than or equal to the distance from P to its nearest focal point. Let x be the center of $gS^{n-1}(r)$, $g\in E(n)$. Since $E(n)$ is the semidirect product $\boldsymbol{R}^n \dot{\times} SO(n)$ we can write $gS^{n-1}(r)=g_0 S_x{}^{n-1}(r)$, where $g_0\in SO(n)$ and $S_x{}^{n-1}(r)$ denotes the $(n-1)$-sphere of radius r with the center x. If $d(x,P)>r$, then $P\cap gS^{n-1}(r)$ is empty. Hence we can say that

$$\begin{aligned}\int \mu_{p-1}(P\cap gS^{n-1}(r))\,dg &= \int_{\boldsymbol{R}^n}\left(\int_{SO(n)}\mu_{p-1}(P\cap g_0 S_x{}^{n-1}(r))\,dg_0\right)dx \\ &= \int_{T(P,r)}\left(\int_{SO(n)}\mu_{p-1}(P\cap g_0 S_x{}^{n-1}(r))\,dg_0\right)dx,\end{aligned} \tag{11}$$

where dg_0 is the Haar measure on $SO(n)$ normalized so that $\int_{SO(n)} dg_0 =O_n O_{n-1}\cdots O_2$, and $T(P,r)=\{x\in\boldsymbol{R}^n \mid d(x,P)\leq r\}$. To evaluate the integral (11) we may assume $d(x, P)<r$ since the measure of the boundary of $T(P,r)$ is equal to 0. Then $P\cap g_0S_x{}^{n-1}(r)$ is homeomorphic to a $(p-1)$-sphere. Now by the Gauss-Bonnet formula (5)

$$\text{(12)}\qquad \mu_{p-1}(P\cap g_0S_x{}^{n-1}(r))=\frac{(2\pi)^{(p-1)/2}}{(p-2)(p-4)\cdots3\cdot1}\cdot2$$

since the Euler characteristic of an even-dimensional sphere is 2. Furthermore

and
$$\int_{T(P,r)}dx=V_P{}^n(r)$$
$$\mu_e(S^{n-1}(r))=O_nr^{n-e-1}.$$

Putting the kinematic formula (2) and the Gauss-Bonnet formula (12) together we obtain from (11)

$$\text{(13)}\qquad V_P{}^n(r)=\sum_{0\le i\ \text{even}\le p-1}\frac{(p-2)(p-4)\cdots3\cdot1}{2^{(p+1)/2}\pi^{(p-1)/2}O_{n-1}\cdots O_2}c_i\mu_i(P)r^{n-p+i}.$$

According to (6), c_i in (13) is given by

$$c_i=\frac{\left(\frac{p-1}{2}\right)!O_{n-1}\cdots O_3O_{n-p+i+2}}{\left(\frac{i}{2}\right)!\left(\frac{p-i-1}{2}\right)!O_{p-i+1}}\cdot\frac{O_pO_{p+1}O_{p-i+1}O_{p-i+2}}{O_{p+1}O_{p+2}}.$$

Since Legendre duplication formula implies $O_{k+1}O_k=2^{k+2}\pi^{k+1}/k!$, we have

$$c_i=\frac{2^{p-i+1}\pi^{(n+1)/2}p\left(\frac{p-1}{2}\right)!\,O_{n-1}\cdots O_3}{\left(\frac{i}{2}\right)!(p-i)!\Gamma\left(\frac{n-p+i+2}{2}\right)}$$
$$=\frac{p!\pi^{(n+1)/2}2^{(p-2i+3)/2}O_{n-1}\cdots O_3}{\left(\frac{i}{2}\right)!(p-i)!(p-2)(p-4)\cdots3\cdot1\cdot\Gamma\left(\frac{n-p+i+2}{2}\right)}.$$

Hence c_i can be written as

$$\text{(14)}\qquad c_i=\frac{p!}{2^{i/2}\left(\frac{i}{2}\right)!(p-i)!}\cdot\frac{2^{(p+1)/2}\pi^{(p-1)/2}O_{n-1}\cdots O_2}{(p-2)(p-4)\cdots3\cdot1}$$
$$\times\frac{\pi^{(n-p)/2}}{2^{i/2}\Gamma\left(\frac{n-p+i+2}{2}\right)}.$$

Substituting (4) and (14) to (13) we obtain the result

(15) $$V_P{}^n(r)=\sum_{0\leq i\ \mathrm{even}\leq p}\frac{\pi^{(n-p)/2}k_i(P)}{2^{i/2}\Gamma\left(\frac{n-p}{2}+\frac{i}{2}+1\right)}r^{n-p+i},$$

where p is odd.

Proof of (8). Let the Riemannian product $P\times Q$ be an imbedded submanifold of $\boldsymbol{R}^n$ and let $p+q$ be odd. Then from (3) and (7) we have (8).

Proof of (9). Let $p=\dim P$ be odd and let n be even. We will apply Chern's kinematic formula with the tubular hypersurface P_r as the stationary submanifold and with $S^{n-1}(s), 0<s<r$, as the moving submanifold of $\boldsymbol{R}^n$. Since $V_{P_r}{}^n(s)=V_P{}^n(r+s)-V_P{}^n(r-s)$, we have from (15)

(16) $$V_P{}^n(r+s)-V_P{}^n(r-s)=\sum_{c=0}^{(n-2)/2}\frac{\pi^{1/2}k_{2c}(P_r)}{2^c\Gamma\left(c+1+\frac{1}{2}\right)}s^{2c+1}.$$

By (3) we can express the left-hand side as

(17) $$V_P{}^n(r+s)-V_P{}^n(r-s)$$
$$=\sum_{a=0}^{[p/2]}\sum_{c=0}^{(n-2)/2}\frac{\pi^{(n-p)/2}k_{2a}(P)r^{n-p+2a-2c-1}}{2^{a-1}\Gamma\left(\frac{n-p}{2}+a+1\right)}\binom{n-p+2a}{2c+1}s^{2c+1}.$$

Finally we obtain (9) by comparing the coefficients of powers of s in (16) and (17).

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