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HOMEOMORPHISM THEOREMS FOR LOCALLY EXPANSIVE OPERATORS*

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1. Introduction

Let X and Y be Banach spaces and $T : X \to Y$ a local homeomorphism. In this paper, we investigate conditions on T under which T is actually a global homeomorphism. This paper is motivated from the following well-known theorem:

THEOREM. [5, Theorem 4.10] *Let X and Y be Banach spaces and $T : X \to Y$ be a local homeomorphism. If there is a continuous nonincreasing function $c : [0, \infty) \to (0, \infty)$ with $\int_0^\infty c(t)\,dt = \infty$ such that T is locally c-expansive, that is, every $x \in X$ has a neighborhood U such that*

$$c(max\{\|u\|, \|v\|\})\|u-v\| \leqq \|Tu-Tv\|, \quad u, v \in U,$$

then T is a homeomorphism of X onto Y.

The above theorem has wide applications to nonlinear functional analysis and many authors have studied and extended this theorem (See [1, 2, 3, 4, 5, 6, 8, 9]). Torrejon [9] obtained a surjectivity theorem for locally c-expansive mappings having closed graph without assuming that c is nonincreasing. And recently, the first author calculated the precise range of a locally m-expansive operator where $m : X \to (0, \infty)$ is a continuous function satisfying some conditions relating curve-integrals [2].

The aim in this paper is to obtain some homeomorphism theorems for local homeomorphisms and to extend the above theorem for locally m-expansive operators. Our theorems will be applied to Gateaux differentiable operators. For the proofs, we use the covering mapping theorem in [5]. For terminologies and notations, see [2].

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2. Main results

In this section, we prove some homeomorphism theorems.

For the definitions of a (*locally*) *rectifiable curve*, *rectifiably pathwise connected space* and a *rectifiably simply connected space*, we follow [2].

A mapping T from a metric space X into a metric space Y is said to be *locally m-expansive* where $m : X \to (0, \infty)$ is a continuous function, if each $x \in X$ has a neighborhood U such that

(1) $\quad \min\{m(u), m(v)\}\, d(u, v) \leqq d(Tu, Tv), \quad u, v \in U.$

Now we state and prove the following basic homeomorphism theorem:

THEOREM 2.1. *Let X be a complete metric space, Y a metric space and let T be a local homeomorphism of X into Y. Suppose that X is locally pathwise connected, Y is connected and that each point of Y has a neighborhood in Y which is rectifiably pathwise connected and rectifiably simply connected. Suppose that T is locally m-expansive, where $m : X \to (0, \infty)$ is a continuous function satisfying the condition*:

(2) *if $g : [0, 1) \to X$ is any locally rectifiable curve satisfying*

$$\int_0^1 m(g(t))\, ds_g(t) < \infty,$$

then g is rectifiable.

Then T is a covering mapping of X onto Y. Moreover, if X is connected and Y is simply connected, then T is a homeomorphism of X onto Y.

To prove Theorem 2.1, we need the following result of Browder [5, Theorem 4.8].

LEMMA 2.2. *Let X and Y be such as in Theorem 2.1 and T a local homeomorphism of X into Y. Suppose that if $h : [0, 1) \to Y$ is a rectifiable curve, any lifting g of h against T (that is, $g : [0, 1) \to X$ is a continuous curve with $h = T \circ g$) is also rectifiable. Then T is a covering mapping of X onto Y. In particular, if X is connected and Y is simply connected, then T is a homeomorphism of X onto Y.*

Using Lemma 2.2, the proof of Theorem 2.1 directly follows from the following lemma.

LEMMA 2.3. *Let X and Y be metric spaces and T a locally m-expansive* *mapping from X into Y, where* $m: X \to (0, \infty)$ *is a continuous function satisfying the condition* (2). *If* $g: [0, 1) \to X$ *is a continuous curve such that* $h = T \circ g : [0, 1) \to Y$ *is rectifiable, then g is also rectifiable.*

Proof. First we show that g is locally rectifiable. For $u \in [0, 1)$, let

$$0 = t_0 < t_1 < \cdots < t_n = u$$

be a subdivision of $[0, u]$. Since g is continuous and $[0, u]$ is compact, $m = \inf\{m(g(t)) \mid 0 \leqq t \leqq u\} > 0$. Then by using (1) it is easy to show that

$$\sum_{i=1}^{n} d(g(t_{i-1}), g(t_i)) \leqq \frac{s_h(t)}{m}.$$

Since h is rectifiable, the above inequality shows that g is locally rectifiable.

Now we claim that for any $r \in [0, 1)$,

$$\int_0^r m(g(t)) ds_g(t) \leqq s_h(r). \tag{3}$$

Therefore the left hand side of (3) is bounded from above by the length of the curve h, and by hypothesis (2) g is rectifiable as desired.

Let $\varepsilon > 0$ be given. Since $m(g(t))$ is uniformly continuous on the compact set $[0, r]$, there is a $\delta_1 > 0$ such that if $|t - t'| < \delta_1$ with $t, t' \in [0, r]$, then we have $|m(g(t)) - m(g(t'))| < \varepsilon$. Also since g is continuous and T is locally m-expansive, there is a $\delta_2 > 0$ such that if $|t - t'| < \delta_2$ with $t, t' \in [0, r]$, then (1) holds for $u = g(t)$ and $v = g(t')$. Now let

$$0 = t_0 < t_1 < \cdots < t_n = r$$

be a subdivision of $[0, r]$ such that $t_i - t_{i-1} < \min\{\delta_1, \delta_2\}$ for all $i = 0, 1, \cdots, n$. For any subinterval $[t_{i-1}, t_i]$, $1 \leqq i \leqq n$, we have a sequence of points

$$t_{i-1} = t_{i0} < t_{i1} < \cdots < t_{in_i} = t_i$$

satisfying

$$s_g(t_i) - s_g(t_{i-1}) < \sum_{j=1}^{n_i} d(g(t_{ij-1}), g(t_{ij})) + \frac{\varepsilon}{n}.$$

Then for any ξ_i with $t_{i-1} \leqq \xi_i \leqq t_i$, $1 \leqq i \leqq n$ we have

$$m(g(\xi_i))\{s_g(t_i) - s_g(t_{i-1})\}$$
$$\leqq m(g(\xi_i))\{\sum_{j=1}^{n_i} d(gt_{ij-1}), g(t_{ij})) + \frac{\varepsilon}{n}\}$$

$$\leqq \sum_{j=1}^{n_i} [\min\{g(t_{ij-1}), g(t_{ij})\} + \varepsilon]\ d(g(t_{ij-1}), g(t_{ij})) + \frac{M}{n}\varepsilon.$$

$$\leqq \varepsilon\{s_g(t_i) - s_g(t_{i-1})\} + s_h(t_i) - s_h(t_{i-1}) + \frac{M}{n}\varepsilon,$$

where $M = \sup\{m(g(t)) \mid 0 \leqq t \leqq r\} < \infty$. Therefore the Riemann-Stieltjes sum of $m \circ g$ with respect to s_g satisfies

$$\textstyle\sum_{i=1}^{n} m(g(\xi_i))\{s_g(t_i) - s_g(t_{i-1})\} < s_g(r) + s_h(r) + M\varepsilon.$$

Since ε is arbitrary, we complete the proof of (3).

Note that the condition (2) is equivalent to the following condition:

(4) if $h : [0, \tau) \to X$ is any locally rectifiable curve with

$$\int_0^\tau m(h(t))\, ds_h(t) < \infty,$$

then h is rectifiable.

Also note that every Banach space satisfies all the conditions imposed on X and Y in Theorem 2.1 (See [2]). Hence as a simple application of Theorem 2.1, we have the following extension of [5, Theorem 4.10]. It may be applied to a number of important classes of nonlinear mappings in Banach spaces (See [4], [5] and [8]).

COROLLARY 2.4. *Let X and Y be Banach spaces and T a local homeomorphism of X into Y. Suppose that T is locally m-expansive, where $m : X \to (0, \infty)$ is a continuous function satisfying* (2) (*or* (4)). *Then T is a homeomorphism of X onto Y.*

Even though the operator T is not continuous, analogous result as in Theorem 2.1 can be obtained for the class of mappings whose graphs are closed. Now we state this fact as follows.

THEOREM 2.5. *Let (X, d) and (Y, d) be complete metric spaces and T an open mapping of (X, d) into (Y, d) which is locally m-expansive having closed graph, where $m : X \to (0, \infty)$ is a continuous function satisfying the condition* (2). *Suppose that Y is connected and every neighborhood of each point of Y contains a subneighborhood which is rectifiably pathwise connected and rectifiably simply connected.*

Let ρ be another metric on X defined by

(5) $\rho(x, y) = max\{d(x, y), d(Tx, Ty)\},\quad x, y \in X.$

Then T is a covering mapping of (X, ρ) onto Y. Moreover, if (X, ρ)

is connected and Y is simply connected, then T is a homeomorphism of (X, ρ) onto Y.

Proof. Since T has closed graph, (X, ρ) is also a complete metric space by Lemma 2.4 of [2]. Note that the topology on X induced by the new metric ρ is finer than the original topology on X induced by d. As a result $m : (X, \rho) \to (0, \infty)$ is still continuous. Also since T is an open mapping and locally m-expansive, and since $T : (X, \rho) \to Y$ is continuous, we see that $T : (X, \rho) \to Y$ is a local homeomorphism. As matter as Y is locally pathwise connected, so is (X, ρ). Therefore in order to show that $T : (X, \rho) \to Y$ is an onto covering mapping, by applying Lemma 2.2, we only need to prove that if a continuous curve $g : [0, 1) \to (X, \rho)$ has its projection $h = T \circ g : [0, 1) \to Y$ which is rectifiable, then g is also rectifiable (with respect to the metric ρ). Since g is also a continuous curve in (X, d), we know that by Lemma 2.3 g is rectifiable with respect to the original metric d. Denote the length of g with respect to d by M, and the length of h by N. Then for any sequence of points

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

we have

$$\begin{aligned} &\textstyle\sum_{i=1}^{n} \rho(g(t_i), g(t_{i-1})) \\ &= \textstyle\sum_{i=1}^{n} \max\{d(g(t_i), g(t_{i-1})),\ d(h(t_i), h(t_{i-1}))\} \\ &\leqq \textstyle\sum_{i=1}^{n} \{d(g(t_i), d(g(t_{i-1})) + d(h(t_i), h(t_{i-1}))\} \\ &\leqq M + N. \end{aligned}$$

Therefore g is also rectifiable with respect to the new metric ρ, and we complete the proof.

Note that in particular, under some suitable conditions, Theorem 2.5 says that T is one-to-one. And it may be the first paper in this kind for the class of mappings having closed graph. The authors in [1, 2, 6, 9] proved only the surjectivity of the mapping T under the condition that Y is metrically convex. For Banach spaces we obtain the following as a simple application of Theorem 2.5.

COROLLARY 2.6. *Let X and Y be Banach spaces and T an open mapping of X into Y having closed graph. Suppose that T is locally m-expansive, where $m : X \to (0, \infty)$ is a continuous function satisfying* (2). *Then T is* *surjective. Moreover, if T is continuous on every straight line in X, then T is bijective.*

Proof. Surjectivity of T follows from Theorem 2.5. Suppose that T is continuous on every straight line in X. Then for any $x, y \in X$, the curve $h : [0,1] \to X$ defined by $h(t) = (1-t)x + ty$, $0 \leq t \leq 1$, is continuous with respect to the new metric ρ introduced in Theorem 2.5. Hence X is pathwise connected with respect to the metric ρ. Therefore by the last paragraph of the above theorem, T is actually a homeomorphism of (X, ρ) onto Y, which completes the proof.

In a certain case, although the domain of T is not the whole space X, we can obtain the same homeomorphism theorem. For example, consider $Tx = \tan x$ on $D = (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ in the real line R. One can easily see that the graph of T is closed in R^2.

THEOREM 2.7. *Let X and Y be such as in theorem 2.1, D a nonempty (open) subset of X and let T be a local homeomorphism of D into Y having closed graph in $X \times Y$. Suppose that T is locally m-expansive, where $m : D \to (0, \infty)$ is a continuous function which satisfies the condition (2) for any locally rectifiable curve $g : [0, 1) \to D$. If D is locally pathwise connected and Y is simply connected, then T is a homeomorphism of D onto Y.*

Proof. Let ρ be another metric introduced as in (5). Since T is continuous, the topologies induced by the original metric d and the new metric ρ are same. But since T has closed graph in $X \times Y$, (D, ρ) is complete by Lemma 2.4 of [2]. Therefore by the same method of the proof of Theorem 2.5, one can easily check that all the hypotheses of Lemma 2.2 are satisfied, and hence the proof is completed.

Let X and Y be Banach spaces and T a mapping from an open subset D of X into Y. We say that T is *Gateaux differentiable* if for each $x \in D$, there is a mapping $dT_x : X \to Y$ satisfying

$$\lim_{t \to 0^+} \frac{T(x+th) - T(x)}{t} = dT_x(h), \quad h \in X.$$

If dT_x is a bounded linear operator and the above limit is attained uniformly for all $h \in X$ with $\|h\| \leqq 1$, then T is said to be *Frechet differentiable.*

It is well-known that if T is a C^1-mapping of X into Y and dT_x is an isomorphism of X onto Y for each $x \in X$ with $\|[dT_x]^{-1}\| \leqq m$ for some constant m, then T is a C^1-diffeomorphism. In [2], the first author proved a surjectivity theorem for such mapping by assuming $\|[dT_x]^{-1}\| \leqq 1/m(x)$, where $m : D \to (0, \infty)$ is a continuous function satisfying some conditions about curve-integrals.

In the following, we give condition under which such mapping is a C^1-diffeomorphism.

THEOREM 2.8. *Let X and Y be Banach spaces, D a nonempty open subset of X and let $T : D \to Y$ be a C^1-mapping such that for each $x \in D$, dT_x is an isomorphism of X onto Y. Suppose that $\|[dT_x]^{-1}\| \leqq 1/m(x)$ for each $x \in D$, where $m : D \to (0, \infty)$ is a continuous function.*

(a) *If T has closed graph in $X \times Y$ and m satisfies the condition* (2) *for any locally rectifiable curve $g : [0, 1) \to D$, then T is a covering mapping. Moreover, if D is connected, then T is a C^1-diffeomorphism of D onto Y.*

(b) *If $D = X$ and m satisfies the condition* (2), *then T is a C^1-diffeomorphism of X onto Y.*

Proof. By [2, Lemma 5.3], for any $\varepsilon \in (0, 1)$, T is locally εm-expansive. Since the function εm still satisfies the condition (2) for any fixed $\varepsilon \in (0, 1)$, (a) is directly derived from Theorem 2.7. (b) follows from (a).

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