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ω-LIMIT SETS FOR MAPS OF THE CIRCLE*

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1. Introduction

Let I be a closed interval, S^1 a circle and X a topological space, and let $C^0(X,X)$ denote the set of continuous maps of X into itself. For any $f\in C^0(X,X)$, let $f^0: X\to X$ be the identity, and define, inductively, $f^n=f\circ f^{n-1}$ for any positive integer n. Let $P(f)$, $\Lambda(f)$ and $\Omega(f)$ denote the sets of periodic, ω-limit and nonwandering points of f, respectively. Let $\omega(x,f)$ denote the set of ω-limit points of $x\in X$ and denote $\Lambda(A,f)=\bigcup_{x\in A}\omega(x,f)$ for any $A\subset X$. (see §2 for definitions).

Let $\Lambda^0(f)=X$, and define, inductively, $\Lambda^n(f)=\Lambda(\Lambda^{n-1}(f),f)$ for any positive integer n.

Recently, there have been a number of excellent studies ([2], [4], [8], [9], [10]) for ω-limit sets of maps of the interval. For any $f\in C^0(I,I)$, A. N. Sarkovskii [6] has shown that $\Lambda(f)$ is closed, and hence that $\overline{P(f)}\subset\Lambda(f)$; and Z. Nitecki [5] has shown that if f is piecewise monotone, then $\Lambda(f)\subset\overline{P(f)}$, so that in this case $\Lambda(f)=\overline{P(f)}$. Especially Xiong [10] has proved that $\Lambda^2(f)=\Lambda(\overline{P(f)},f)=\Lambda(\Omega(f),f)$ for any $f\in C^0(I,I)$. Hence $\Lambda^n(f)$ is closed for some $1<n\leq\infty$ if and only if $\Lambda^n(f)=\overline{P(f)}$ for any $1<n\leq\infty$. However there is a continuous map $\chi_\infty\in C^0(I,I)$ which satisfies the condition $\Lambda^2(\chi_\infty)\neq\overline{P(f)}$ even if periods of its periodic points are powers of 2 (see [4]).

In this paper we obtain similar results for maps of the circle. We will prove the following theorems in the section 3.

THEOREM A. *Let $f\in C^0(S^1,S^1)$. Then we have $\overline{P(f)}\subset\Lambda(f)\subset\Omega(f)$.*

THEOREM B. *Let $f\in C^0(S^1,S^1)$ and $P(f)$ be nonempty. Then we have*

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$$\Lambda^2(f)=\Lambda(\overline{P(f)},f)=\Lambda(\Omega(f),f).$$

Here we have added to the corresponding result ([8], [10]) for maps of the interval the obvious requirement that the set of periodic points is nonempty–consider an irrational rotation.

2. Definitions, Notations and Preliminaries

Let X be a compact metric space and let $f\in C^0(X,X)$. A point $x\in X$ is called a *periodic point* of f if $f^n(x)=x$ for some positive integer n. The *period* of x is the least such integer n. The *orbit* of x is the set $\{f^k(x)\mid k=0,1,2,\dots\}$, and denoted by orb$(x)$. If x is a periodic point of period n, orb(x) contains exactly n points, each of which is a periodic point of period n. A point $x\in X$ is called a *nonwandering point* of f if for every neighbourhood U of x, there exists an integer $n\geq 1$ such that $f^n(U)\cap U\neq\phi$. A point $y\in X$ is called an *ω-limit point* of x if there is a sequence $n_i\to\infty$ of positive integers such that $f^{n_i}(x)\to y$. For any $f\in C^0(X,X)$, it is easy to see that $\omega(x,f)$ is nonempty, closed and invariant relative to f, and that there is a minimal set of f in it. Additional results concerning ω-limit sets for maps of the interval have been obtained by Block and Coven [2], and Xiong ([8], [9], [10]).

The following two technical lemmas are proved using essentially the same arguments used to prove the corresponding results [7] for maps of the interval.

LEMMA 1. *Let $f\in C^0(S^1,S^1)$ and let K be a subset of S^1 satisfying the condition $f(K)\subset K$. If $x\in\Omega(f)-K$, then $f^n(x)\notin K^0$ for any $n\geq 0$. (K^0 being the interior of K).*

Proof. Suppose that $x\in\Omega(f)-K$. Then by Lemma 2 of [3], we can choose a sequence $x_i\to x$ of points of S^1 and a sequence $k_i\to\infty$ of positive integers such that $f^{k_i}(x_i)=x$ for each $i>0$. Assume that $f^n(x)\in K^0$ for some $n\geq 0$. By the continuity of f, we have $f^n(x_i)\to f^n(x)$. Hence there exists some $N>0$ such that $f^n(x_i)\in K$, whenever $i>N$. Choose $k_j>n$ such that $j>N$. Then $f^{k_j-n}(f^n(x_j))=x\notin K$. This is a contradiction to the assumption of the Lemma, because $f(K)\subset K$ implies $f^k(K)\subset K$ for each $k\geq 0$.

LEMMA 2. *Suppose that $f \in C^0(S^1, S^1)$, $K \subset S^1$ has only finitely many connected components, and $f(K) \subset K$. Then the orbit of each point of $\Omega(f) \cap (\bar{K} - K)$ is finite.*

Proof. Let $x \in \Omega(f) \cap (\bar{K} - K)$, and let $n \geq 0$. By Lemma 1, $f^n(x) \notin K^0$ for all $n \geq 0$. On the other hand, $f(K) \subset K$ implies $f^n(K) \subset K$. Hence $f^n(\bar{K}) \subset \bar{K}$. Therefore, $f^n(x) \in \bar{K} - K^0$. $\bar{K} - K^0$ is finite because K has only finitely many connected components. Thus the orbit of x is finite.

We will use the following notations throughout this paper. Let $a, b \in S^1$ with $a \neq b$. We write $[a, b]$, (a, b) and $(a, b]$ or $[a, b)$ to denote the closed, open and half-open intervals from a counterclockwise to b, respectively. Then we can give an order on the above intervals as the usual one on $\boldsymbol{R}$.

LEMMA 3. *Let $f \in C^0(S^1, S^1)$. If $J \cap P(f) = \phi$ and $f(\bar{J}) \subset \bar{J}$, where J is an open interval with $\bar{J} \neq S^1$, then $J \cap \Omega(f) = \phi$.*

Proof. Let $J = (a, b)$. Then either $x < f(x)$ or $x > f(x)$ for all $x \in J$. We may assume that $x < f(x)$ for all $x \in J$. Let $x \in J$. Then there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $(x - \varepsilon_1, x + \varepsilon_1) \cap (f(x) - \varepsilon_2, \ b] = \phi$ and $f((x - \varepsilon_1, x + \varepsilon_1)) \subset (f(x) - \varepsilon_2, b]$. Hence $f^n((x - \varepsilon_1, \ x + \varepsilon_1)) \subset (f(x) - \varepsilon_2, b]$ for all integers $n \geq 1$, that is, $f^n((x - \varepsilon_1, \ x + \varepsilon_1)) \cap (x - \varepsilon_1, \ x + \varepsilon_1) = \phi$ for all integers $n \geq 1$. Therefore $x \notin \Omega(f)$.

Formally we will think of the circle as $\boldsymbol{R}/\boldsymbol{Z}$ and use π to denote the canonical projection. Thus every continuous map f of the circle has countably many lifts, i. e., continuous maps $F : \boldsymbol{R} \to \boldsymbol{R}$ satisfying $f \circ \pi = \pi \circ F$. Let $f \in C^0(S^1, S^1)$ and let $F \in C^0(\boldsymbol{R}, \boldsymbol{R})$ be a lifting of f to the covering space. If F and F' are liftings of the same map f, then $F = F' + k$ for some integer k. There exists a unique integer N such that $F(X+1) = F(X) + N$ for all lifts F and all $X \in R$, which is called the *degree* of f, denote by $\deg(f)$.

A technical concept we will use is that of f-covering. We say of two closed intervals J and K that J *f-covers* K if $f(J') = K$ for some subinterval J' of J. For maps of the interval this is equivalent to $f(J) \supset K$; for maps of the circle, it is stronger. The importance of

f-coverings lies in the fact that if J f^n-covers itself for some n, then f has a periodic point in J.

LEMMA 4. *Let $f\in C^0(S^1, S^1)$ with $deg(f)\neq 0$, and let J be a component of $S^1 - \overline{P(f)}$ with $\bar{J}\neq S^1$. If $J\cap\Omega(f)\neq\phi$, then exactly one of the followings holds:*

(1) *If $x\in J$ and $f(x)\in J$, then $x<f(x)$.*

(2) *If $x\in J$ and $f(x)\in J$, then $x>f(x)$.*

Proof. Let $J=(a, b)$ be a component of $S^1-\overline{P(f)}$ with $a\neq b$ and suppose that there exist $x, y\in J$ such that $f(x), f(y)\in J$, $x<f(x)$ and $y>f(y)$.

Case 1. First assume that $\bar{J}$ f-covers $\bar{J}$. Then there is a closed subinterval $I_1\subset\bar{J}$ with $f(I_1)=\bar{J}$. We may assume that $I_1\neq\bar{J}$ by Lemma 3. Since $J\cap\overline{P(f)}=\phi$, I_1 contains a fixed point of f, which is an end point of J, say a, without loss of generality. Then we may also assume $I_1=[a, c]$ for some $a<c<b$ with $f(c)=b$. Note that $f(z)>z$ for all $z\in[a, c]$. Since $f(y)<y$, we may see that $y\in(c, b)$. Suppose that $[c, y]$ does not f-cover $[b, f(y)]$. Then since $f(c)>c$ and $f(y)<y$, f has a fixed point in $[c, y]$. Hence $[c, y]$ f-covers $[b, f(y)]$. Therefore there is a closed subinterval I_2 of $[c, y]$ with $f(I_2)=[a, f(y)]$. Also note that there are a closed subinterval I_3 of $[a, f(y)]$ and an integer $k\geq 1$ such that $f^k(I_3)=I_2$. For this, first note that $f([a, c])=[a, b]$ with $f(c)=b$. Put $c=x_0$ and by induction we can find a sequence (x_i) with $x_{i+1}\in(a, x_i), f(x_{i+1})=x_i$ and $f([a, x_{i+1}])=[a, x_i]$ for $i=0, 1, 2, \ldots$, so that $f^{i+1}(x_i)=b$ and $f^{i+1}([a, x_i])=[a, b]$. Hence since $x_0>x_1>\ldots>x_i>\ldots>a$, we may assume that $x_i\to z$ for some $z\in[a, b)$. Then we have $f(z)=z$ and hence we know that $z=a$. Now $x_i\to a$ implies that there is some integer $k\geq 1$ such that $x_{k-1}\in(a, f(y))$, and we can find I_3 in (a, x_{k-1}). In this case $f^{k+1}(I_3)=f(I_2)=[a, f(y)]\supset I_3$, and hence f has a periodic point in I_3, which leads a contradiction.

Case 2. Assume that $\bar{J}$ does not f-cover $\bar{J}$. Then we may see that $[x, y]$ (if $x<y$) or $[y, x]$ (if $y<x$) f-covers $[b, a]$, otherwise f has a fixed point in J. Hence $[b, a]$ does not f-cover $\bar{J}=[a, b]$. In this case, since $P(f)$ is invariant, $f(a)\notin J$ and $f(b)\notin J$, and hence $\deg(f)$

is zero, which completes the proof.

LEMMA 5. *Let* $f \in C^0(S^1, S^1)$ *with* $deg(f) \neq 0$, *and let* $J=(a, b), a \neq b$ *be a component of* $S^1-\overline{P(f)}$ *with* $\bar{J} \neq S^1$. *If* $z \in J \cap \Omega(f)$ *has an infinite orbit, then there exists an integer* $N \geq 1$ *such that*

$$z \notin \bigcup_{k=0}^{\infty} f^{kN}([a, z)) \text{ or } z \notin \bigcup_{k=0}^{\infty} f^{kN}((z, b]).$$

Proof. Note that $z \in \Omega(f^n)$ for all integers $n \geq 1$ by Lemma 3 of [3]. Also note that there is an integer $n \geq 1$ such that $f^n(J) \cap J \neq \phi$ and $f^n(J) \cap (S^1-\bar{J}) \neq \phi$ otherwise if $f^n(\bar{J}) \subset \bar{J}$ for some integer $n \geq 1$, then $J \cap \Omega(f^n)=\phi$ by Lemma 3. Hence $f^n(J)$ contains a periodic point of f, so that there are a point $x \in J$ and an integer $m \geq n$ such that $f^{jm}(x)=f^m(x)$ for all $j \geq 1$. Clearly $f^m(x) \notin J$. Without loss of generality, we may assume that $z<x$. Note that $z \in \Omega(f^m)$.

Suppose that $z \in \bigcup_{k=0}^{\infty} f^{km}([a, z))$, that is, there are a point $y \in (a, z)$ and an integer $k_1 \geq 1$ such that $f^{k_1 m}(y)=z$. Put $k_1 m=N$. Then we claim that $z \notin \bigcup_{k=0}^{\infty} f^{kN}((z, b])$ and hence we complete the proof. To prove the claim, put $g=f^N$. Then $g(x)$ is a fixed point of $g=f^N$ and $z \in \Omega(g)$, $g(y)=z$ and $\deg(g) \neq 0$. Now put $I_1=[y, x], I_2=[x, g(x)], I_3=[g(x), y]$. Then by Lemma 4, I_1 does not g-cover I_3, hence I_1 g-covers I_2. If I_3 does not g-cover I_3, then I_2 g-covers I_1, I_2 and I_3 or $\deg(g)=0$. But $\deg(g) \neq 0$ implies $I_1 \cap P(f) \neq \phi$, which is a contradiction. Hence I_3 g-covers I_3.

Case 1. Assume that I_3 does not g-cover I_1. Then $\deg(g)=1$, since $\deg(g) \neq 0$, and I_1 and I_2 do not g-cover I_1. Also by Lemma 4, we can see that $g(a)=a$. Now suppose that there are an integer $k \geq 1$ and a point $u \in (z, b)$ such that $g^k(u)=z$. Let G be a lifting of g with $\pi(0)=g(x), G(0)=0$. Then there exist $A, Y, Z, X, B \in (0,1)$ such that $\pi(A)=a, \pi(Y)=y, \pi(Z)=z, \pi(X)=x$ and $\pi(B)=b$. Also, we know that $G(0)=0, G(X)=G(1)=1$. Let $\pi(U)=u$, where $U \in (Z, B)$. Then, $G^k(0)=0$. $G^k(A)=A, G^k(1)=1$ and $G^k(X)=1$. Note that $G^k(U)=Z+i$ for some integer i. If $G^k(U) \leq Z<U$, then G^k has a periodic point in $[U, X]$ (if $U<X$) or $[X, U]$ (if $X<U$), which is a contradiction. Hence $G^k(U) \geq Z+1$, so that $[A, Z+1] \subset G^k([A, U])$. Therefore there exists a point $V \in [A, U]$ such that $G^k(V)=U$. Then

$v=\pi(V)<\pi(U)=u=g^k(v)$. But $z=g^k(u)<u$, which is a contradiction by Lemma 4.

Case 2. Assume that I_3 g-covers I_1. Let G be a lift of g such that $\pi(0)=g(x)$, $\pi(A)=a$, $\pi(Y)=y$, $\pi(Z)=z$, $\pi(X)=x$, $\pi(B)=b$ and $G(Y)=Z$. Then $G(X)=1$ by Lemma 4. Since I_2 does not g-cover $I_1, G(1)=1$.

(1) Suppose that $G(A)\geq B$. If $G(A)\geq 1+B$, then there is a closed subinterval $[C, D]\subset[A, Y]$ with $G([C, D])=[1+A, 1+B]$ and $G(D)=1+A$. But by Lemma 4, $G(U)\geq 1+U$ for all $U\in[C, D]$ and $1+A=G(D)\geq 1+D$, which is a contradiction. Hence if $G(A)\geq B$, then $B\leq G(A)\leq 1+A$.

Note that $G(0)\leq 0$ or $G(0)\geq 2$, since $\deg(f)\neq 0$. In each case, we obtain

(*) there exists a closed subinterval $I_4\subset[g(x), a]$ such that $g(I_4)=\bar{J}$.

(2) Suppose that $G(A)\leq A$. Then $G(A)=A$. If $G(0)=0$, then $\deg(g)=1$ and the above case shows that $z\notin\bigcup_{k=0}^{\infty} g^k((z, b])$. If $G(0)\leq -1$ or $G(0)\geq 2$, then we also have (*).

Hence, we may assume that (*) holds. Suppose that there are a point $u\in(z, b)$ and an integer $k\geq 1$ such that $g^k(u)=z$. Let $\pi(U)=u$ with $Z<U<B$. Then $G^k(X)=1$.

If $G^k(U)\leq Z$, then G^k has a periodic point in (A, B), so that g has a periodic point in J. Therefore $G^k(U)\geq 1+Z$. Hence there exists a closed subinterval $I_5\subset[u, x]$ (if $u<x$) or $[x, u]$ (if $x<u$) in J such that $g^k(I_5)=I_4$. Then $g^{k+1}(I_5)=g(I_4)=\bar{J}\supset I_5$, so that, g has a periodic in J, which is also a contradiction. Hence this completes the proof.

3. Proofs of Theorems

In this section, we will prove theorems A and B. We need the following lemma:

LEMMA 6. *Let $f\in C^0(S^1, S^1)$ with $\deg(f)=0$ and let F be a lifting of f. Then for any $X\in R$, we have*

$$\pi(\omega(X, F))=\omega(\pi(X), f).$$

Proof. It is easy to see that

$$\pi(\omega(X, F))\subset\omega(\pi(X), f).$$

Let $y\in\omega(\pi(X), f)$. Then there is a sequence (n_i) of integers with $n_i\to\infty$ such that $f^{n_i}(\pi(X))\to y$. Since $\deg(f)=0$, the image of F is compact, and hence there is a subsequence of $(F^{n_i}(X))$ which converges to some element Y of $\boldsymbol{R}$. Then it follows that $\pi(Y)=y$ and $y\in\pi(\omega(X, F))$, which completes the proof.

THEOREM A. *Let* $f\in C^0(S^1, S^1)$. *Then we have* $\overline{P(f)}\subset\Lambda(f)\subset\Omega(f)$.

Proof. By definition, it is clear that $P(f)\subset\Lambda(f)\subset\Omega(f)$. Suppose that $\deg(f)=0$ and let F be a lift of f. Then by [6], we have $\overline{P(F)}\subset\Lambda(F)$, and hence by Lemma 2 of [1] and Lemma 6

$$\overline{P(f)}=\pi(\overline{P(F)})\subset\pi(\Lambda(F))=\Lambda(f).$$

Therefore we may assume that $\deg(f)\neq 0$. Let $x\in\overline{P(f)}-P(f)$, and let (x_n) be a sequence of periodic points of f such that $x_n\to x$. Let the period of x_n be p_n with respect to f. Without loss of generality, we may assume that $x_n\in[x_1, x]$ and $x_1<x_2<\dots<x$ with order in $[x_1, x]$. For a fixed $i\geq 1$, let $g=f^{p_i}$ and n_j be the period of x_j with respect to g. Then

$$K=g(L)\cup g^2(L)\cup\cdots$$

is connected, where $L=[x_i, x)$. Now for each $k=1, 2, 3$, consider the sequence $(g^{4n_j-k}(x_j))$ in K, which has a subsequence converging to some $u_k\in\overline{K}$. Then we know that $g^k(u_k)=x$. Since $x\notin P(f)$, u_i, $i=1, 2, 3$, are distinct, so that, one of these points has to lie in K, say u_k. Then there are $v\in[x_i, x)$ and some $t\geq 0$ such that $g^t(v)=u_k$. Therefore

$$g^{t+k}(v)=f^{(t+k)p_i}(v)=x.$$

Now let $q_i=(t+k)p_i$ for each $i=1, 2, \dots$. Then $[x_i, x]$ f^{q_i}-covers $[x_i, x]$ or $[x, x_i]$. Suppose that $f^{q_i}([x_i, x])\not\supset[x_i, x]$, and $f^{q_i}([x, x_i])\not\supset[x_i, x]$.
Then since $f^{q_i}(x_i)=x_i$ and $\deg(f^{q_i})\neq 0$, we know that $f^{q_i}(x)\in[x_i, x]$. Therefore we conclude that $f^{2q_i}([x_i, x])\supset[x_i, x]$ or $f^{q_i}(x)\in[x_i, x]$. If for infinitely many $i, f^{q_i}(x)\in[x_i, x]$ holds, then $x\in\omega(x, f)$, so that $x\in\Lambda(f)$. Therefore we may assume that $f^{2q_i}([x_i, x])\supset[x_i, x]$ for all

$i \geq 1$. Let $t_i = \sum_{j=1}^{i} 2q_j$ and define $F_1 = [x_1, x]$, $F_2 = F_1 \cap f^{-t_1}([x_2, x])$, and inductively $F_{i+1} = F_i \cap f^{-t_i}([x_{i+1}, x])$. Then we know that each F_i is nonempty closed and $F_1 \supset F_2 \supset \ldots$. Therefore $\bigcap_{i=1}^{\infty} F_i \neq \phi$. Choose an arbitrary point $y \in \bigcap_{i=1}^{\infty} F_i$. Then $f^{t_i}(y) \in [x_{i+1}, x]$, $i = 1, 2, 3, \ldots$, shows that $x \in \omega(y, f) \subset \Lambda(f)$, which completes the proof.

LEMMA 7. *Let* $f \in C^0(S^1, S^1)$ *and* $x, y \in S^1$. *If* $\omega(x, f^n) = \omega(y, f^n)$ *for some integer* $n \geq 1$, *then* $\omega(x, f) = \omega(y, f)$.

Proof. By Lemma 6 of [9], $\omega(x, f) = \bigcup_{k=0}^{n-1} \omega(f^k(x), f^n)$ and $f^i(\omega(f^k(x), f^n)) = \omega(f^{k+i}(x), f^n)$. Since $\omega(f^k(x), f^n) = f^k(\omega(x, f^n)) = f^k(\omega(y, f^n)) = \omega(f^k(y), f^n)$, we have $\omega(x, f) = \omega(y, f)$.

LEMMA 8. *Let* $f \in C^0(S^1, S^1)$ *with* $deg(f) = 0$. *Then we have*

$$\Lambda^2(f) = \Lambda(\overline{P(f)}, f) = \Lambda(\Omega(f), f).$$

Proof. It suffices to show that $\Lambda(\overline{P(f)}, f) = \Lambda(\Omega(f), f)$ by Theorem A. Let F be a lift of f. Then note that $\pi(\Omega(F)) = \Omega(f)$ and $\pi(P(F)) = P(f)$ by Lemma 2 of [1] and $\Lambda(\overline{P(F)}, F) = \Lambda(\Omega(F), F)$ by Theorem 1 of [8]. Using the above facts and Lemma 7, we can easily obtain the results.

THEOREM B. *Let* $f \in C^0(S^1, S^1)$ *with* $P(f) \neq \phi$. *Then we have*

$$\Lambda^2(f) = \Lambda((\overline{P(f)}, f) = \Lambda(\Omega(f), f).$$

Proof. By Lemma 8, we may assume that the degree of f is not zero. Let $z \in \Omega(f) - \overline{P(f)}$. Then we claim that there is some point $x \in \overline{P(f)}$ such that $\omega(z, f) = \omega(x, f)$. If z has a finite orbit, that is, $f^m(z)$ is a fixed point of f^m for some integer m, then we have

$$\omega(z, f^m) = \omega(f^m(z), f^m) = \{f^m(z)\}.$$

By Lemma 7, since $\omega(z, f) = \omega(f^m(z), f)$, we have done.

Hence we may assume that z has an infinite orbit. If $P(f)$ is a singleton, then, since $P(f) = \Omega(f)$ by Theorem C of [3], we have done. Therefore we may also assume that $P(f)$ has more than one point. Let $J=(a,b), a\neq b$, be a component of $S^1-\overline{P(f)}$ containing z. Then there exists an integer $N\geq 1$ such that either $z\notin \bigcup_{k=0}^{\infty} f^{kN}([a,z))$ or $z\notin \bigcup_{k=0}^{\infty} f^{kN}((z,b])$ by Lemma 5. Suppose that $z\notin \bigcup_{k=0}^{\infty} f^{kN}([a,z))=K$. Then $f^N(K)\subset K$ and $z\in(\bar{K}-K)\cap\Omega(f^N)$ since $z\in\Omega(f^k)$ for every positive integer k by Lemma 3 of [3]. By Lemma 2, K has infinite components. Therefore $f^{kN}([a,z))$, $k=0,1,2,\ldots$, are pairwise disjoint. Hence $\operatorname{diam}(f^{kN}([a,z]))=\operatorname{diam}(f^{kN}([a,z)))\to 0$ as $k\to\infty$, that is, for each point $y\in[a,z]$, $\omega(y,f^N)=\omega(z,f^N)$. But, since $a\in \overline{P(f)}$, $\omega(a,f^N)=\omega(z,f^N)$, so that, $\omega(a,f)=\omega(z,f)=\omega(y,f)$ for every point $y\in[a,z]$. Therefore $\Lambda(\overline{P(f)},f)=\Lambda(\Omega(f),f)$, which completes the proof by Theorem A.

Corollary. *Let $f\in C^0(S^1,S^1)$ and $P(f)$ be nonempty. Then $\Lambda^n(f)$ is closed for some $1<n\leq\infty$ if and only if $\Lambda^n(f)=\overline{P(f)}$ for any $1<n\leq\infty$.*

Proof. By Theorem B, we have

$$\Lambda^2(f)=\Lambda(\overline{P(f)},f)\subset\overline{P(f)}.$$

By combining Theorem A, we obtain

$$\Lambda(f)\supset\overline{P(f)}\supset\Lambda^2(f)\supset\Lambda^3(f)\supset\ldots\supset\Lambda^\infty(f)\supset P(f),$$

and these relations give the proof.

References

1. J.S. Bae and S.K. Yang, *$\bar{P}=\bar{R}$ for maps of the circle*, Bull. Korean Math. Soc. **24**(1987), No. 2, 151–157.
2. L. Block and E.M. Coven, *ω-limit sets for maps of the interval*, Ergod. Th. & Dynam. Sys. **6**(1986), 335–344.
3. L. Block, E.M. Coven, I. Mulvey and Z. Nitecki, *Homoclinic and nonwandering points for maps of the circle*, Ergod. Th. and Dynam. Sys. **3**(1983), 521–532.
4. H. Chu and J.C. Xiong, *A counterexample in dynamical systems of the interval*, Proc. Amer. Math. Soc. **97**(1987), 361–366.
5. Z. Nitecki, *Periodic and limit orbits and the depth of the center for piecewise monotone interval maps*, Proc. Amer. Math. Soc. **80**(1980), 511–514.

6. A. N. Sarkovskii, ***On a theorem of G. D. Birkhoff*** (Ukrainian, Russian and English summaries), Dopovidi Akad. Nauk Ukrain. RSR Ser. A, (1967), 429-432.
7. J. C. Xiong, *Nonwandering sets of continuous interval self-maps*, Kexue Tongbao (English Ed), **29**(1984), 1431-1433.
8. J. C. Xiong, *The attracting centre of a continuous self-map of the interval*, ICTP, Preprint, IC/86/143.
9. J. C. Xiong, *A note on minimal sets of interval maps*, ICTP, Preprint, IC/86/207.
10. J. C. Xiong, *A note on ω-limit sets of continuous self-maps of the interval*, Preprint.

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