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A GENERALIZATION OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS II

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1. Introduction

Let S_p denote the class of functions of the form $f(z)=z^p+\sum_{n=1}^{\infty}a_{n+p}z^{n+p}$ which are analytic and p-valent in the unit disc $U=\{z : |z|<1\}$.

For $-1\leqslant A<B\leqslant 1$, $0<B\leqslant 1$ and $0\leqslant\alpha<p$, let $P(p, A, B, \alpha)$ be the class of those functions $f(z)$ of S_p for which $\frac{f'(z)}{z^{p-1}}$ is subordinate to $\frac{p+[pB+(A-B)(p-\alpha)]z}{1+Bz}$. In other words $f(z)\in P(p, A, B, \alpha)$ if and only if there exists a function $w(z)$ regular in U and satisfying $w(0)=0$, $|w(z)|<1$ for $z\in U$, such that

$$\frac{f'(z)}{z^{p-1}}=\frac{p+[pB+(A-B)(p-\alpha)]w(z)}{1+Bw(z)}.$$

Above condition is equivalent to

$$\left|\frac{\frac{f'(z)}{z^{p-1}}-p}{B\frac{f'(z)}{z^{p-1}}-[pB+(A-B)(p-\alpha)]}\right|<1, \quad z\in U. \tag{1.1}$$

Let T_p denote the subclass of S_p consisting of functions analytic and p-valent which can be expressed in the form

$$f(z)=z^p-\sum_{n=1}^{\infty}|a_{n+p}|z^{n+p}.$$

We denote by $P^*(p, A, B, \alpha)$ the class obtained by taking intersection of the class $P(p, A, B, \alpha)$ with T_p.

The subclasses $T_p^*(A, B, \alpha)$ and $C_p(A, B, \alpha)$ of T_p obtained by replacing $\frac{f'(z)}{z^{p-1}}$ with $\frac{zf'(z)}{f(z)}$ and $\left\{1+\frac{zf''(z)}{f'(z)}\right\}$ respectively in (1.1) have been studied by the author [1].

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In the present paper, we obtain sharp results concerning coefficient estimates, distortion theorem and radius of convexity for the class $P^*(p, A, B, \alpha)$. It is further shown that the class $P^*(p, A, B, \alpha)$ is closed under "arithmetic mean" and "convex linear combinations".

We also obtain class preserving integral operators of the form

$$F(z)=\frac{p+c}{z^c}\int_0^z t^{c-1} f(t)\,dt, \quad c>-p$$

for the class $P^*(p, A, B, \alpha)$. Conversely when $F(z)\in P^*(p, A, B, \alpha)$, radius of p-valence of $f(z)$ has been determined. Also we obtain distortion theorem for the fractional integral.

2. Coefficient estimates

THEOREM 1. *A function* $f(z)=z^p-\sum_{n=1}^{\infty}|a_{n+p}|z^{n+p}$ *is in* $P^*(p, A, B, \alpha)$ *if and only if*

$$\sum_{n=1}^{\infty}(n+p)(1+B)|a_{n+p}|\leqslant(B-A)(p-\alpha).$$

The result is sharp.

Proof. Let $|z|=1$. Then

$$\begin{aligned}&|f'(z)-pz^{p-1}|-|Bf'(z)-[pB+(A-B)(p-\alpha)]z^{p-1}|\\ &=\left|-\sum_{n=1}^{\infty}(n+p)|a_{n+p}|z^{n+p-1}\right|-\Bigg|(B-A)(p-\alpha)z^{p-1}-\\ &\qquad B\sum_{n=1}^{\infty}(n+p)|a_{n+p}|z^{n+p-1}\Bigg|\\ &\leqslant\sum_{n=1}^{\infty}(n+p)(1+B)|a_{n+p}|-(B-A)(p-\alpha)\leqslant 0, \text{ by hypothesis.}\end{aligned}$$

Hence by the maximum modulus theorem, $f(z)\in P^*(p, A, B, \alpha)$.

Conversely, suppose that

$$\begin{aligned}&\left|\frac{f'(z)-pz^{p-1}}{Bf'(z)-[pB+(A-B)(p-\alpha)]z^{p-1}}\right|\\ &=\left|\frac{-\sum_{n=1}^{\infty}(n+p)|a_{n+p}|z^{n+p-1}}{(B-A)(p-\alpha)z^{p-1}-B\sum_{n=1}^{\infty}(n+p)|a_{n+p}|z^{n+p-1}}\right|\leqslant 1, \quad z\in U.\end{aligned}$$

Since $|\mathrm{Re}(z)|\leqslant|z|$ for all z, we have

$$\mathrm{Re}\left\{\frac{\sum_{n=1}^{\infty}(n+p)\,|a_{n+p}|\,z^{n+p-1}}{(B-A)(p-\alpha)z^{p-1}-B\sum_{n=1}^{\infty}(n+p)\,|a_{n+p}|\,z^{n+p-1}}\right\}<1. \quad (2.1)$$

Choose values of z on the real axis so that $f'(z)/z^{p-1}$ is real. Upon clearing the denominator in (2.1) and letting $z \longrightarrow 1$ through real values, we have

$$\sum_{n=1}^{\infty}(n+p)(1+B)\,|a_{n+p}| \leqslant (B-A)(p-\alpha).$$

This completes the proof of the theorem.

The function

$$f(z)=z^p-\frac{(B-A)(p-\alpha)}{(n+p)(1+B)}z^{n+p} \quad (n\geqslant 1)$$

is an extremal function.

3. Distortion theorem

THEOREM 2. *If* $f(z)\in P^*(p, A, B, \alpha)$, *then*

$$r^p-\frac{(B-A)(p-\alpha)}{(1+p)(1+B)}r^{p+1}\leqslant |f(z)|\leqslant r^p+\frac{(B-A)(p-\alpha)}{(1+p)(1+B)}r^{p+1}(|z|=r), \quad (3.1)$$

and

$$pr^{p-1}-\frac{(B-A)(p-\alpha)}{(1+B)}r^p\leqslant |f'(z)|\leqslant pr^{p-1}+\frac{(B-A)(p-\alpha)}{(1+B)}r^p(|z|=r). \quad (3.2)$$

The estimates are sharp.

Proof. From Theorem 1, we have

$$(1+p)(1+B)\sum_{n=1}^{\infty}|a_{n+p}|\leqslant\sum_{n=1}^{\infty}(n+p)(1+B)\,|a_{n+p}|\leqslant(B-A)(p-\alpha).$$

This implies that

$$\sum_{n=1}^{\infty}|a_{n+p}|\leqslant\frac{(B-A)(p-\alpha)}{(1+p)(1+B)}.$$

Hence

$$|f(z)|\leqslant|z|^p+\sum_{n=1}^{\infty}|a_{n+p}|\,|z|^{n+p}\leqslant r^p\left(1+r\sum_{n=1}^{\infty}|a_{n+p}|\right)$$
$$\leqslant r^p+\frac{(B-A)(p-\alpha)}{(1+p)(1+B)}r^{p+1}$$

and

$$|f(z)|\geqslant|z|^p-\sum_{n=1}^{\infty}|a_{n+p}||z|^{n+p}\geqslant r^p(1-r\sum_{n=1}^{\infty}|a_{n+p}|)$$
$$\geqslant r^p-\frac{(B-A)(p-\alpha)}{(1+p)(1+B)}r^{p+1}.$$

Thus (3.1) follows.
Also

$$|f'(z)|\leqslant p|z|^{p-1}+\sum_{n=1}^{\infty}(n+p)|a_{n+p}||z|^{n+p-1}$$
$$\leqslant r^{p-1}(p+r\sum_{n=1}^{\infty}(n+p)|a_{n+p}|)$$
$$\leqslant pr^{p-1}+\frac{(B-A)(p-\alpha)}{(1+B)}r^p.$$

Similarly

$$|f'(z)|\geqslant p|z|^{p-1}-\sum_{n=1}^{\infty}(n+p)|a_{n+p}||z|^{n+p-1}$$
$$\geqslant r^{p-1}(p-r\sum_{n=1}^{\infty}(n+p)|a_{n+p}|)$$
$$\geqslant pr^{p-1}-\frac{(B-A)(p-\alpha)}{(1+B)}r^p.$$

This completes the proof of the theorem.

The bounds are sharp since the equalities are attained for the function

$$f(z)=z^p-\frac{(B-A)(p-\alpha)}{(1+p)(1+B)}z^{p+1} \qquad (z=\pm r).$$

COROLLARY 1. *Let* $f(z)\in P^*(p,A,B,\alpha)$. *Then the disc* $|z|<1$ *is mapped onto a domain that contains the disc*

$$|w|<\frac{1+B+p(1+A)+\alpha(B-A)}{(1+p)(1+B)}.$$

The result is sharp for the extremal function

$$f(z)=z^p-\frac{(B-A)(p-\alpha)}{(1+p)(1+B)}z^{p+1}.$$

Proof follows upon letting $r\longrightarrow 1$ in left hand side of (3.1).

4. Integral operators

THEOREM 3. *Let* c *be a real number such that* $c>-p$. *If* $f(z)\in P^*(p,A,B,\alpha)$, *then the function* $F(z)$ *defined by*

$$F(z)=\frac{p+c}{z^c}\int_0^z t^{c-1}f(t)\,dt \tag{4.1}$$

also belongs to $P^*(p, A, B, \alpha)$.

Proof. Let

$$f(z)=z^p-\sum_{n=1}^{\infty}|a_{n+p}|z^{n+p}.$$

Then from the representation of $F(z)$, it follows that

$$F(z)=z^p-\sum_{n=1}^{\infty}|b_{n+p}|z^{n+p},$$

where

$$|b_{n+p}|=\frac{(p+c)}{(n+p+c)}|a_{n+p}|.$$

Therefore

$$\begin{aligned}\sum_{n=1}^{\infty}(n+p)(1+B)|b_{n+p}|&=\sum_{n=1}^{\infty}(n+p)(1+B)\frac{p+c}{(n+p+c)}|a_{n+p}|\\&\leqslant\sum_{n=1}^{\infty}(n+p)(1+B)|a_{n+p}|\\&\leqslant(B-A)(p-\alpha),\end{aligned}$$

since $f(z)\in P^*(p, A, B, \alpha)$. Hence, by Theorem 1, $F(z)\in P^*(p, A, B, \alpha)$.

THEOREM 4. *Let c be a real number, $c>-p$. If $F(z)\in P^*(p, A, B, \alpha)$, then the function $f(z)$ defined in* (4.1) *is p-valent for $|z|<R_p^*$, where*

$$R_p^*=\inf_{n\geqslant 1}\left[\left(\frac{p+c}{n+p+c}\right)\cdot\frac{(1+B)p}{(B-A)(p-\alpha)}\right]^{\frac{1}{n}}.$$

The result is sharp.

Proof. Let

$$F(z)=z^p-\sum_{n=1}^{\infty}|a_{n+p}|z^{n+p}.$$

It follows then from (4.1) that

$$\begin{aligned}f(z)&=\frac{z^{1-c}}{p+c}\frac{d}{dz}(z^cF(z))\\&=z^p-\sum_{n=1}^{\infty}\left(\frac{n+p+c}{p+c}\right)|a_{n+p}|z^{n+p}.\end{aligned}$$

To prove the result it sufficies to show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|<p \quad \text{for} \quad |z|<R_p^*.$$

Now

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|=\left|-\sum_{n=1}^{\infty}(n+p)\left(\frac{n+p+c}{p+c}\right)|a_{n+p}|z^n\right|$$
$$\leqslant\sum_{n=1}^{\infty}(n+p)\left(\frac{n+p+c}{p+c}\right)|a_{n+p}|\,|z|^n.$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|<p \quad \text{if}$$

$$\sum_{n=1}^{\infty}(n+p)\left(\frac{n+p+c}{p+c}\right)|a_{n+p}|\,|z|^n<p. \tag{4.2}$$

But Theorem 1 confirms that

$$\sum_{n=1}^{\infty}\frac{(n+p)(1+B)p}{(B-A)(p-\alpha)}|a_{n+p}|\leqslant p.$$

Thus (4.2) will be satisfied if

$$(n+p)\left(\frac{n+p+c}{p+c}\right)|a_{n+p}|\,|z|^n\leqslant\frac{(n+p)(1+B)p}{(B-A)(p-\alpha)}|a_{n+p}|,\ n=1,2,\dots$$

or if

$$|z|\leqslant\left[\left(\frac{p+c}{n+p+c}\right)\frac{(1+B)p}{(B-A)(p-\alpha)}\right]^{\frac{1}{n}}. \tag{4.3}$$

The required result follows now from (4.3).

The result is sharp for the function

$$f(z)=z^p-\frac{(n+p+c)(B-A)(p-\alpha)}{(n+p)(p+c)(1+B)}z^{n+p}.$$

5. Radius of convexity for the class $P^*(p,A,B,\alpha)$

THEOREM 5. *If $f(z)\in P^*(p,A,B,\alpha)$, then $f(z)$ is p-valently convex in the disc*

$$|z|<R_p=\inf_n\left[\frac{(1+B)p^2}{(B-A)(p-\alpha)(n+p)}\right]^{\frac{1}{n}},\quad n=1,2,3,\dots.$$

The result is sharp.

Proof. To prove the theorem it is sufficient to show that

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right|\leqslant p \quad \text{for} \quad |z|<R_p.$$

We have

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right|=\left|\frac{-\sum_{n=1}^{\infty}n(n+p)\,|a_{n+p}|\,z^n}{p-\sum_{n=1}^{\infty}(n+p)\,|a_{n+p}|\,z^n}\right|$$

$$\leqslant\frac{\sum_{n=1}^{\infty}n(n+p)\,|a_{n+p}|\,|z|^n}{p-\sum_{n=1}^{\infty}(n+p)\,|a_{n+p}|\,|z|^n}\,.$$

Thus

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right|\leqslant p\quad\text{if}$$

$$\frac{\sum_{n=1}^{\infty}n(n+p)\,|a_{n+p}|\,|z|^n}{p-\sum_{n=1}^{\infty}(n+p)\,|a_{n+p}|\,|z|^n}\leqslant p$$

or

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)^2|a_{n+p}|\,|z|^n\leqslant 1.$$

But from Theorem 1, we obtain

$$\sum_{n=1}^{\infty}\frac{(n+p)(1+B)}{(B-A)(p-\alpha)}|a_{n+p}|\leqslant 1.$$

Hence $f(z)$ is p-valently convex if

$$\left(\frac{n+p}{p}\right)^2|z|^n\leqslant\frac{(n+p)(1+B)}{(B-A)(p-\alpha)}$$

or

$$|z|\leqslant\left[\frac{(1+B)p^2}{(B-A)(p-\alpha)(n+p)}\right]^{\frac{1}{n}},\quad n=1,2,3,\dots\,.$$

This completes the proof of the theorem.

The result is sharp for the function

$$f(z)=z^p-\frac{(B-A)(p-\alpha)}{(1+B)(n+p)}z^{n+p}.$$

6. Closure theorems

THEOREM 6. *If* $f(z)=z^p-\sum_{n=1}^{\infty}|a_{n+p}|z^{n+p}$ *and* $g(z)=z^p-\sum_{n=1}^{\infty}|b_{n+p}|z^{n+p}$ *are in* $P^*(p,A,B,\alpha)$, *then the function* $h(z)=z^p-\frac{1}{2}\sum_{n=1}^{\infty}|a_{n+p}+b_{n+p}|z^{n+p}$

is also in $P^*(p, A, B, \alpha)$.

Proof. Since $f(z)$ and $g(z)$ are in $P^*(p, A, B, \alpha)$. Therefore we have

$$\sum_{n=1}^{\infty}(n+p)(1+B)|a_{n+p}| \leqslant (B-A)(p-\alpha) \tag{6.1}$$

and

$$\sum_{n=1}^{\infty}(n+p)(1+B)|b_{n+p}| \leqslant (B-A)(p-\alpha). \tag{6.2}$$

From (6.1) and (6.2) we obtain

$$\frac{1}{2}\sum_{n=1}^{\infty}(n+p)(1+B)|a_{n+p}+b_{n+p}| \leqslant (B-A)(p-\alpha).$$

This completes the proof of the theorem.

THEOREM 7. *Let* $f_i(z)=z^p-\sum_{n=1}^{\infty}|a_{i,n+p}|z^{n+p}$ *be in the classes* $P^*(p, A, B, \alpha_i)$ *for each* $i=1, 2, 3\ldots, m$. *Then the function*

$$h(z)=z^p-\frac{1}{m}\sum_{n=1}^{\infty}\left(\sum_{i=1}^{m}|a_{i,n+p}|\right)z^{n+p}$$

is in the class $P^*(p, A, B, \alpha)$, *where* $\alpha=\underset{1\leqslant i\leqslant m}{\mathrm{Min}}\{\alpha_i\}$.

Proof. Since $f_i(z)\in P^*(p, A, B, \alpha_i)$ for each $i=1, 2, \ldots, m$, we have

$$\sum_{n=1}^{\infty}(n+p)(1+B)|a_{i,n+p}| \leqslant (B-A)(p-\alpha_i)$$

by Theorem 1. Hence we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}(n+p)\left(\frac{1}{m}\sum_{i=1}^{m}|a_{i,n+p}|\right)\\
&=\frac{1}{m}\sum_{n=1}^{\infty}\left\{\sum_{i=1}^{m}(n+p)|a_{i,n+p}|\right\}\\
&\leqslant\frac{1}{m}\sum_{i=1}^{m}\left\{\frac{(B-A)(p-\alpha_i)}{(1+B)}\right\}\\
&\leqslant\frac{(B-A)(p-\alpha)}{(1+B)},
\end{aligned}$$

because $\dfrac{(B-A)(p-\alpha')}{(1+B)} \geqslant \dfrac{(B-A)(p-\alpha'')}{(1+B)}$ for $\alpha'\leqslant\alpha''$.

Thus we get

$$\sum_{n=1}^{\infty}(n+p)(1+B)\left(\frac{1}{m}\sum_{i=1}^{\infty}|a_{i,n+p}|\right) \leqslant (B-A)(p-\alpha)$$

which shows that $h(z) \in P^*(p, A, B, \alpha)$.

THEOREM 8. *Let* $f_p(z) = z^p$, $f_{n+p}(z) = z^p - \frac{(B-A)(p-\alpha)}{(1+B)(n+p)} z^{n+p}$, $n=1, 2, 3, \ldots$. *Then* $f(z) \in P^*(p, A, B, \alpha)$ *if and only if it can be expressed in the form* $f(z) = \sum_{n=0}^{\infty} \lambda_{n+p} f_{n+p}(z)$ *where* $\lambda_{n+p} \geqq 0$ *and* $\sum_{n=0}^{\infty} \lambda_{n+p} = 1$.

Proof. Suppose
$$f(z) = \sum_{n=0}^{\infty} \lambda_{n+p} f_{n+p}(z) = z^p - \sum_{n=1}^{\infty} \frac{(B-A)(p-\alpha)}{(1+B)(n+p)} \lambda_{n+p} z^{n+p}.$$
Then
$$\sum_{n=1}^{\infty} \left[\frac{(1+B)(n+p)}{(B-A)(p-\alpha)} \lambda_{n+p} \frac{(B-A)(p-\alpha)}{(1+B)(n+p)} \right] = \sum_{n=1}^{\infty} \lambda_{n+p} = 1 - \lambda_p \leqq 1.$$
Hence by Theorem 1, $f(z) \in P^*(p, A, B, \alpha)$.

Conversely, suppose that $f(z) \in P^*(p, A, B, \alpha)$. Since
$$|a_{n+p}| \leqq \frac{(B-A)(p-\alpha)}{(1+B)(n+p)} \quad (n=1, 2, 3, \ldots),$$
we may set
$$\lambda_{n+p} = \frac{(1+B)(n+p)}{(B-A)(p-\alpha)} |a_{n+p}| \quad (n=1, 2, 3, \ldots)$$
and
$$\lambda_p = 1 - \sum_{n=1}^{\infty} \lambda_{n+p}.$$
Then
$$f(z) = \sum_{n=0}^{\infty} \lambda_{n+p} f_{n+p}(z).$$
This completes the proof of the theorem.

REMARKS.

(1) Putting $\alpha=0$ in the above theorems we get the results obtained by Shukla and Dashrath [4].

(2) Putting $p=1$ and taking $A=-\beta, B=\beta, 0<\beta\leqq 1$ in the above theorems we get the results obtained by Gupta and Jain [2].

7. Fractional integral

In 1978, Owa [3] gave the following definition for the fractional integral.

DEFINITION 1. The fractional integral of order k is defined by

$$D_z^{-k}f(z)=\frac{1}{\Gamma(k)}\int_0^z\frac{f(\zeta)d\zeta}{(z-\zeta)^{1-k}},$$

where $k>0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{k-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

THEOREM 9. *Let a function* $f(z)=z^p-\sum_{n=1}^{\infty}|a_{n+p}|z^{n+p}$ *be in the class* $P^*(p, A, B, \alpha)$. *Then we have*

$$|D_z^{-k}f(z)|\geqslant\frac{\Gamma(1+p)}{\Gamma(1+p+k)}|z|^{p+k}\left\{1-\frac{1}{1+p+k}\cdot\frac{(B-A)(p-\alpha)}{(1+B)}|z|\right\}$$

and

$$|D_z^{-k}f(z)|\leqslant\frac{\Gamma(1+p)}{\Gamma(1+p+k)}|z|^{p+k}\left\{1+\frac{1}{1+p+k}\cdot\frac{(B-A)(p-\alpha)}{(1+B)}|z|\right\}$$

for $0<k<1$ *and* $z\in U$. *The result is sharp.*

Proof. Let

$$\begin{aligned}F(z)&=\frac{\Gamma(1+p+k)}{\Gamma(1+p)}z^{-k}D_z^{-k}f(z)\\&=z^p-\sum_{n=1}^{\infty}\frac{\Gamma(n+p+1)\Gamma(1+p+k)}{\Gamma(n+p+1+k)\Gamma(1+p)}|a_{n+p}|z^{n+p}\\&=z^p-\sum_{n=1}^{\infty}A(n)|a_{n+p}|z^{n+p},\end{aligned}$$

where

$$A(n)=\frac{\Gamma(n+p+1)\Gamma(1+p+k)}{\Gamma(n+p+1+k)\Gamma(1+p)}\qquad(n\geqslant1).$$

Since

$$0<A(n)\leqslant A(1)=\frac{1+p}{1+p+k},$$

we have, with the help of Theorem 1,

$$|F(z)|\geqslant|z|^p-A(1)|z|^{1+p}\sum_{n=1}^{\infty}|a_{n+p}|$$

$$\geqslant |z|^p - \frac{1+p}{1+p+k} \cdot \frac{(B-A)(p-\alpha)}{(1+p)(1+B)} |z|^{1+p}$$
$$\geqslant |z|^p - \frac{1}{1+p+k} \cdot \frac{(B-A)(p-\alpha)}{(1+B)} |z|^{1+p}$$

and

$$|F(z)| \leqslant |z|^p + A(1)|z|^{1+p} \sum_{n=1}^{\infty} |a_{n+p}|$$
$$\leqslant |z|^p + \frac{1}{1+p+k} \cdot \frac{(B-A)(p-\alpha)}{(1+B)} |z|^{1+p}$$

which prove the inequalities of Theorem 9. Further, equalities are attained for the function

$$D_z^{-k} f(z) = \frac{\Gamma(1+p)}{\Gamma(1+p+k)} z^{p+k} \left\{ 1 - \frac{1}{1+p+k} \cdot \frac{(B-A)(p-\alpha)}{(1+B)} z \right\},$$

or

$$f(z) = z^p - \frac{(B-A)(p-\alpha)}{(1+p)(1+B)} z^{1+p}.$$

COROLLARY 2. *Under the hypotheses of Theorem* 9, $D_z^{-k} f(z)$ *is included in the disc with center at the origin and radius*

$$\frac{\Gamma(1+p)}{\Gamma(1+p+k)} \left\{ 1 + \frac{1}{1+p+k} \cdot \frac{(B-A)(p-\alpha)}{(1+B)} \right\}.$$

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