Bull. Korean Math. Soc. 25(1988), No. 2, pp. 203-209

G-REGULAR SEMIGROUPS

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0. Introduction

In this paper, we define a g-regular semigroup which is a generalization of a regular semigroup. And we want to find some properties of g-regular semigroup. G-regular semigroups contains the variety of all regular semigroup and the variety of all periodic semigroup.

If a is an element of a semigroup S, the smallest left ideal containing a is $Sa \cup \{a\}$, which we may conveniently write as S^1a, and which we shall call the principal left ideal generated by a. An equivalence relation $\mathcal{L}$ on S is then defined by the rule $a\mathcal{L}b$ if and only if a and b generate the same principal left ideal, i.e. if and only if $S^1a=S^1b$. Similarly, we can define the relation $\mathcal{R}$.

The equivalence relation $\mathcal{D}$ is $\mathcal{R}\circ\mathcal{L}$ and the principal two sided ideal generated by an element a of S is S^1aS^1. We write $a\mathcal{J}b$ if $S^1aS^1=S^1bS^1$, i.e. if there exist x, y, u, v in S^1 for which $xay=b$, $ubv=a$. It is immediate that $\mathcal{D}\subset\mathcal{J}$.

A semigroup S is called periodic if all its elements are of finite order. A finite semigroup is necessarily periodic semigroup. It is well known that in a periodic semigroup, $\mathcal{D}=\mathcal{J}$.
An element a of a semigroup S is called regular if there exists x in S such that $axa=a$. The semigroup S is called regular if all its elements are regular. The following is the property of $\mathcal{D}$-classes of regular semigroup.

Lemma. *If a is a regular element of a semigroup S, then every element of D_a is regular.*

An idea of great importance in semigroup theory is that of an inverse of an element. If a is an element of semigroup S, we say

Received December 21, 1987.
This work was supported by Korea Science and Engineering Foundation 1986.

that a' is an inverse of a if $aa'a=a$, $a'aa'=a'$. Notice that an element with an inverse is necessarily regular. Less obviously, every regular element has an inverse; for if $axa=a$ we need only define $a'=xax$. An element a may well have more than one inverse. We call a semigroup S is an inverse semigroup if every a in S posesses a unique inverse, i. e. if there exists a unique element a^{-1} such that $aa^{-1}a=a$, $a^{-1}aa^{-1}=a^{-1}$. It is well known that S is an inverse semigroup if and only if S is regular and every idempotent elements commute. A semigroup S is right (left) simple if $\mathcal{R}=S\times S(\mathcal{L}=S\times S)$. S is right (left) cancellative if $ac=bc(ca=cb)$ implies $a=b$ for all a, b, c in S. A right simple left cancellative semigroup is called a right group. It is easy that a semigroup S is a right group if and only if it is isomorphic to a direct product of a group and a right zero semigroup.

1. *G*-regular semigroups

Definition 1. 1. An element a in a semigroup S is called g-regular if there exist a nonzero x in S such that $xax=x$. The semigroup S is g-regular if all its nonzero elements are g-regular.

From definition, 0 is not g-regular for any semigroup S. If S is a regular semigroup, then for any nonzero a of S, there exist x in S such that $axa=a$. Put $a=xax$, then $a'aa'=a'$ and a' is nonzero. So that every regular semigroup is g-regular. Before showing the g-regular semigroup which is not regular, we need the following lemma.

Lemma 1. 2. *If a is a g-regular element of a semigroup S, then every element of D_a is also g-regular.*

Proof. Let $x\in D_a$. Then $(x, a)\in\mathcal{D}$ and so there is an element y in S such that $(x, y)\in\mathcal{R}$, $(y, a)\in\mathcal{L}$. There are u, v, w, z in S^1 such that $xu=y$, $yv=x$, $wy=a$, $za=y$. Since a is g-regular, there exists a' in S such that $a'aa'=a'$. Hence $(ua'w)x(ua'w)=ua'aa'w=ua'w$. If $ua'w=0$, then $wx(ua'w)y=aa'a=0$. But this is a contradiction to the fact a' is nonzero. Hence x is a g-regular element.

If D is a $\mathcal{D}$-class, then either every element of D is g-regular or no elements is g-regular. this dichotomy does in general apply to

$\mathcal{J}$-classes.

LEMMA 1.3. *If a is a g-regular element of a semigroup S, then every element of J_a is also g-regular.*

Proof. Let $x \in J_a$. Then $(x, a) \in \mathcal{J}$. So there are u, v in S^1 such that $uxv = a$. Since a is g-regular, $a'aa = a'$ for some nonzero a' in S. $(va'u)x(va'u) = va'aa'u = va'u$. If $va'u = 0$, then $0 = (a'ux)va'u(xva') = a'aa'aa' = a'$. This is a contradiction. Hence every element of J_a is g-regular.

COROLLARY 1.4. *Every simple semigroup with idempotent element is g-regular semigroup.*

Above lemma 1.3 does not hold in general for regular semigroups i.e. there exists a semigroup S in which regular elements and irregular elements are contained in a same $\mathcal{J}$-class.

EXAMPLE 1.5. Let $\{1, e, 0\}$ be a semigroup with identity 1, zero 0 and $ee = 0$ and let S be the $N \times \{1, 0, e\} \times N$.
Define an operation on S by

$$(m, a, n) \cdot (p, b, q) = \begin{cases} (m, a, q-p+n) & \text{if } n > p. \\ (m-n+p, b, q) & \text{if } n < p. \\ (m, ab, q) & \text{if } n = p. \end{cases}$$

Then S becomes a simple semigroup. Since S has idempotent elements, it is a g-regular semigroup. But any elements of the $\mathcal{D}$-class $N \times \{e\} \times N$ is not regular. Hence S is a g-regular semigroup which is not a regular semigroup.

In example 1.5, we find a g-regular semigroup which is not regular and $\mathcal{D} \neq \mathcal{J}$. The following theorem shows that there are many g-regular semigroups which are not regular though $\mathcal{D} = \mathcal{J}$.

THEOREM 1.6. *If S is a periodic semigroup, then S is a g-regular semigroup.*

Proof. Let $a \in S$, m be the index of a, r be the period of a. Since $a^m = a^{m+nr}$ for all natural number n, we can choose a natural number i such that r divide $m+i$ and $0 \leqslant i \leqslant r-1$. Now we have $a^{m+i-1}aa^{m+i-1} = a^{m+i-1}$. Hence S is g-regular.

COROLLARY 1.7. *Every finite semigroup has idempotent element.*

We can prove easily that every regular semigroup with unique idempotent is a group. But a g-regular semigroup with unique idempotent is not a group. Even a g-regular semigroup with unique idempotent is not a monoid.

EXAMPLE 1.8. Let S be the subsemigroup of $\mathcal{F}(\{1, 2, \ldots 7\})$ generated by $x=\begin{pmatrix}1234567\\2345675\end{pmatrix}$. (The notation for x is an obvious generalization of the standard notation for permutations). We can prove easily that x has index 4 and period 3. The kernel K_x is $\{x^4, x^5, x^6\}$. And x^6 is the identity element of K_x. Sinc S is periodic, it is a g-regular semigroup. Also x^6 is the unique idempotent of S. But S is not a monoid.

LEMMA 1.9. *If S is a g-regular semigroup without zero, then S has unique idempotent if and only if for each a of S there exist a unique x such that $xax=x$.*

Proof. "only if" Let $xax=x$, $yay=y$. Then $x=xax=yax=yay=y$.
"if" If e, f are two idempotent elements of S, then there exists a nonzero x in S such that $xefx=x$. But since $fxe(ef)fxe=fxe$, we have $x=fxe$. From $xefxe=xe$, it follows that $f=xe$. So $x=fxe=f$ and $x=xefx=xef=xex$. Thus we have that $e=f$.

If S has a zero element the above lemma 1.9 does not hold in general.

EXAMPLE 1.10. Let $S=\{0, e_1, e_2, \ldots\}$ with the operation $e_ie_j=0$ if $i\neq j$, $e_ie_j=e_i$ if $i=j$. Then S is a g-regular semigroup. If $xe_ix=x$, then $x=e_i$ and so S satisfies the condition of the if part of lemma 1.9. But all elements of S are idempotents.

LEMMA 1.11. *If g-regular semigroup S has a unique idempotent, then it commutes with all elements of S.*

Proof. If $x\in S$, then $x'xx'=x'$ for some nonzero x' in S. Since xx' and $x'x$ are idempotent, we have $xx'=x'x$. Thus $xx'x=xx'xx'x$

$=x'x(xx')x=x'xx$.

THEOREM 1. 12. *If S is a g-regular semigroup, then the following are equivalent;*

1) *Every idempotent is a left identity of S.*
2) *S is a right simple semigroup.*
3) *S is a left cancellative semigroup.*
4) *S is a right group.*
5) *The set of all idempotents of S is a right zero semigroup and S is regular.*

Proof. 1) implies 3). If $ax=ay$, then there is a' in S such that $a'aa'=a'$. Since $a'a$ is idempotent and $a'ax=a'ay$, we have $x=a'ax=a'ay=y$. And so S is a left cancellative semigroup.

3) implies 4). For any a of S, there is a' in S such that $a'aa'=a'$. So $a'aa'a=a'a$. Since S is left cancellative, $aa'a=a$. Thus S is a regular semigroup. It is well known that any left cancellative regular semigroup is right group.

4) $\Longleftrightarrow$ 2). Since any g-regular semigroup has at least one idempotent, S is a right group. Indeed S is a right group if and only if S is right simple and it contains an idempotent.

2) implies 5). If e, f are idempotents of S, then $eS=fS$ since S is a right simple semigroup. So $e=fx$ for some x in S. From $e=fx=ffx=fe$, the set of all idempotent elements of S becomes right zero semigroup. If $a\in S$, then there is a' in S such that $a'aa'=a'$. Since S is a right simple semigroup, $aS^1=aa'S^1$. So $a=aa'x$ for some x in S. Since $a'a=a'aa'x=a'x$, we have $a=aa'x=aa'a$. Thus S is a regular semigroup.

5) implies 1). Let $x\in S$ and e be an idempotent element of S. Then there exists x' in S such that $xx'x=x$. So $ex=exx'x=xx'x=x$.

From the above theorem, we have that any right simple, g-regular semigroup is regular. But a simple g-regular semigroup is not regular in general (example 1. 5). We call a semigroup S is completely simple if it is simple and satisfies the condition $\min_L$ and $\min_R$, that is, if every non-empty set either of $\mathcal{L}$-classes or of $\mathcal{R}$-classes possesses a minimal member. Rees (1940) shows that every completely simple semigroup S is isomorphic to $M[G : I, J : P]$, the $I \times J$ Rees matrix semigroup over the group G with the regular sandwitch matrix P. Conversely, every $M[G : I, J : P]$ is a completely simple semigroup.

THEOREM 1.13. *If S is a g-regular semigroup, then every minimal right ideal of S is a completely simple semigroup.*

Proof. Let M be a minimal right ideal of S and $m \in M$. Then there is an element m' in S such that $m'mm' = m'$. Since $mm' \in M$, we have $mm'S = M$. For any $x \in M$, we have $x = mm't$, $t \in S$. So $x = mm't = mm'x$. In particular, $m = mm'm$. mM and mS are both right ideals of S we have $mM = mS = M$. Thus $mm' = ma$ for some $a \in M$. Since $a \in M$, we have $a = mm'a$ and so $m = mm'm = mam = m^2(m'a)m$. So $m = m^2(m'a)m = m(m^2m'am)m'am \in m^2Mm$. This means that M is completely regular [5] and so M is a union of disjoint of groups. Also we can prove easily that $M = MmM$ for all $m \in M$. So M is simple. Thus M is a completely simple semigroup [4, proposition 1. 1, p. 91].

COROLLARY 1.14. *If semigroup S is a right simple g-regular, then S is completely simple.*

The bicyclic semigroup S is a bisimple inverse semigroup with idempotent $e_i (i = 1, 2, \ldots)$ ordered by $e_1 < e_2 < e_3 < \cdots$. Since every bisimple semigroup S is simple, S is simple regular semigroup. But this semigroup S is not completely simple. Simple regular (g-regular) semigroup need not be completely simple. Also example 1. 5 shows that S is simple g-regular but it is not a completely simple semigroup.

References

1. Clifford, A.H. and G.B. Preston, *The algebraic theory of semigroups*, Amer. Math. Soc. Survey No. 7, 1961.
2. Clifford, A.H. and G.B. Preston, *The algebraic theory of semigroups*, Vol. II, Math. Survey No. 7, Amer. Math. Soc., Providence, R.I., 19 61.
3. K.R. Goodearl, *Ring theory*, Marcel Dekker, inc. 1967.
4. J.M. Howie, *An introduction to semigroup theory*, Academic Press, 1967.
5. S. Lajos, *Some characterizations of completely regular semigroups*, Kyung-

pook Math. J. Vol. **19**, 1979, 213-214.

6. S. Lajos, *Characterization of certain classes of semigroups I*, Math. seminar notes. Vol. **7**, 1979, 605-608.

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