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GALERKIN APPROXIMATIONS OF RICCATI OPERATORS ARISING IN THE BOUNDARY CONTROLS FOR HYPERBOLIC SYSTEMS

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1. Introduction

In [2], we have shown that the optimal boundary controls for hyperbolic systems in L^2-spaces can be attained in a feedback form via Riccati operators.

A number of authors [1, 5, 7 and 10] have investigated approximations of Riccati operators arising in distributed parameter systems. They assumed bounded controls for parabolic systems.

However, we in this paper study Galerkin approximations of Riccati operators and feedback controls for hyperbolic systems with unbounded control actions.

Let us briefly introduce some results of [2].

Let Ω be an open bounded region in R^n with smooth boundary Γ where n is a fixed positive integer. We consider a strictly hyperbolic differential operator $H(x)$ of order 1 on Ω with noncharacteristic boundary on Γ:

$$H(x)=\sum_{i=1}^{n}A_i(x)\frac{\partial}{\partial x_i}+C(x)$$

where A_i and C are $m\times m$ matrix valued smooth functions on the closure $\bar{\Omega}$.

We also assume that the normal matrix $N(x)$ at x in Γ is of following block diagonal form:

$$N(x)=A_i(x)n_i(x)=\begin{bmatrix} N^-(x) & 0 \\ 0 & N^+(x) \end{bmatrix} \tag{1.1}$$

for the unit normal vector $(n_1(x), \ldots, n_n(x))$ at x in Γ where $-N^-(x)$ and $N^+(x)$ are positive definite and smooth matrix valued functions

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on Γ with rank $(N^-)=r$ and rank $(N^+)=n-r$.

A boundary operator $\beta(x)$ on Γ is assumed as

(1. 2) $\beta(x)=[I_r : M(x)]$ where I_r is the identity matrix of order r and $M(x)$ is a $r\times(m-r)$ matrix valued smooth function on Γ.

Let us consider a hyperbolic mixed problem:

$$(1.3)\quad \begin{cases} \frac{\partial}{\partial t}y(t)=Hy(t)+f(t) & \text{on } (0,T)\times\Omega\equiv Q \\ \beta y(t)=u(t) & \text{on } (0,T)\times\Gamma\equiv\Sigma \\ y(0)=y_0 & \text{on } \Omega \end{cases}$$

for f in $L^2(Q\,;R^m)$, u in $L^2(\Sigma\,;R^r)$ and y_0 in $L^2(\Omega\,;R^m)$.

Then we have from [9],

THEOREM 1. 1. *The system* (1. 3) *is well posed in the sense of Kreiss. For simplicity, we let* $f=0$ *in* (1. 3) *and we denote that* $L^2(Q)=L^2(Q\,;R^m)$, $L^2(\Sigma)=L^2(\Sigma\,;R^r)$ *and* $L^2(\Omega)=L^2(\Omega\,;R^m)$ *etc with norms and inner products* $\|\cdot\|_Q$, $\|\cdot\|_\Sigma$, $\|\cdot\|_\Omega$ *and* $\langle .,.\rangle_Q$, $\langle .,.\rangle_\Sigma$, $\langle .,.\rangle_\Omega$ *etc respectively.*

Let us define an operator A as

(1. 4) $Ay=Hy$ for y in $\mathrm{Dom}(A)$

where $\mathrm{Dom}(A)=\{y\in L^2(\Omega)\,|\,Hy\in L^2(\Omega)$ and $\beta y=0$ on $\Gamma\}$.

Then from theorem 1. 1, it is known that A generates a strongly continuous semigroup $S(t)$ on $L^2(\Omega)$. In order to represent the system (1. 3) in an integral form we need to introduce a Dirichlet operator D on $L^2(\Gamma)$ as

$$(1.5)\quad Du=z \text{ if } \begin{cases} Hz=kz & \text{on } \Omega \\ \beta z=u & \text{on } \Gamma \end{cases}$$

for some fixed large constant k.

Then we have shown in [2],

LEMMA 1. 2. *D is a bounded linear operator of $L^2(\Gamma)$ into $L^2(\Omega)$ for a large constant k. Moreover we have*

(1. 6) $\int_0^T\|D^*(A^*-kI)S^*(t)z\|_\Gamma^2dt\le c\|z\|_\Omega^2$ *for all* $z\in L^2(\Omega)$,

(1. 7) $D^*(A^*-kI)y=N^-y^-$ *on* Γ *for* y *in* $\mathrm{Dom}(A^*)$, *where* c *is a constant depending on* T *and* $y=(y^-\,;y^+)$ *with*

$$y^-=(y_1,\dots y_r) \text{ and } y^+=(y_{r+1},\dots,y_m).$$

For simplicity, we also assume $k=0$ by translating $C(x)$ in (1. 5).

We define an operator L on $L^2(\Sigma)$: for u in $L^2(\Sigma)$,

$$(Lu)(t)=A\int_0^t S(t-s)Du(s)ds, \quad t\in[0, T].$$

In [2], we have

THEOREM 1.3. *L is a bounded linear operator of $L^2(\Sigma)$ into $C([0, T]; L^2(\Omega))$. Moreover the solution of (1.3) is given by $y(t)=S(t)y_0-(Lu)(t)$, $t\in[0, T]$.*

We consider an optimal control problem:

(OCP) Minimize the cost functional

$J(y_0, u)=\|u\|_\Sigma^2+\langle Fy, y\rangle_Q+\langle Gy(T), y(T)\rangle_\Omega$ over u in $L^2(\Sigma)$ subject to the dynamic system (1.3) where F and G are given positive self-adjoint bounded linear operators on $L^2(\Omega)$.

According to [2], (OCP) has a unique optimal control u^0 in $L^2(\Sigma)$ and the corresponding solution y^0 in $C([0, T]; L^2(\Omega))$. Furthermore, u^0 is given in the following feedback form:

(1.8) $$u^0(t)=-D^*A^*R(t)y^0(t), \quad \text{a.e. } t \text{ in } [0, T]$$

where $R(t)$ is a suitable positive self adjoint Riccati operator on $L^2(\Omega)$.

We have shown in [2] the following regularizations: Let F_n and G_n be positive self adjoint bounded operators on $L^2(\Omega)$ with ranges of them in $H_0^1(\Omega)$ such that

(1.9) $F_n\longrightarrow F$ and $G_n\longrightarrow G$ strongly on $L^2(\Omega)$ as $n\longrightarrow\infty$.

LEMMA 1.4. *The Riccati equation:*

(RDE) $$\frac{d}{dt}\langle R_n(t)x, y\rangle_\Omega=-\langle F_nx, y\rangle_\Omega-\langle A^*R_n(t)x, y\rangle_\Omega -\langle R_n(t)Ax, y\rangle_\Omega+\langle D^*A^*R_n(t)x, D^*A^*R_n(t)y\rangle_\Gamma$$

with $R_n(T)=G_n$, for x, y in $Dom(A)$ and a.e. t in $[0, T]$, has a unique positive self-adjoint bounded operator solution $R_n(t)$ on $L^2(\Omega)$ for each n.

Let $u_n(t)=-D^*A^*R_n(t)y_n(t)$, $0\le t\le T$.
Then we have from [2] the following convergences.

THEOREM 1.5.

(i) *$\|D^*A^*R_n(t)z\|_\Gamma\le c\|z\|_\Omega$ for z in $L^2(\Omega)$, $0\le t\le T$,*

(ii) *$R_n(t)\longrightarrow R(t)$ strongly in $L^2(\Omega)$ as $n\longrightarrow\infty$,*

(iii) $u_n \longrightarrow u^0$ *in* $L^2(\Sigma)$ *as* $n \longrightarrow \infty$,

(iv) $y_n(t) \longrightarrow y^0(t)$ *in* $L^2(\Omega)$ *as* $n \longrightarrow \infty$,

where $y_n(t)$ *is the solution of* (1. 3) *corresponding to the control* $u_n(t)$ *for each* n, c *is a constant depending on* T.

In section 2, we introduce semidiscrete Galerkin approximations to the open loop control systems and investigate some stability properties of the approximations.

We then in section 3 consider finite dimensional Riccati feedback controls resulted from finite element type approximations. In the main theorems, the convergences of finite dimensional feedback controls and corresponding solutions to the original optimal control and solution respectively are given.

2. Semidiscrete Galerkin approximations and stability results

We here introduce finite dimensional spline spaces W_h (of order $\geq$ 1) of $L^2(\Omega)$ where h is a discretization parameter as follows.

Let P_h be the orthogonal projection of $L^2(\Omega)$ onto W_h, $h>0$. We assume that W_h are contained in $H^1(\Omega)$ and satisfy the next:

(AP1) $P_h z \longrightarrow z$ in $L^2(\Omega)$ as $h \longrightarrow 0$ for z in $L^2(\Omega)$.

(AP2) $P_h z \longrightarrow z$ in $H^1(\Omega)$ as $h \longrightarrow 0$ for z in $H^1(\Omega)$,

(AP3) $\|z\|_\Gamma \leq c_0(\sqrt{h})^{-1}\|z\|_\Omega$ for z in $H^1(\Omega)$,

(AP4) $\|P_h z - z\|_{H^1(\Omega)} \leq c_0 h \|z\|_{H^2(\Omega)}$ for z in $H^2(\Omega)$

where c_0 is a constant independent of $h>0$.

We also assume that the domain Dom(A^*) is invariant under P_h.

Now we are in a position to define the finite dimensional operators A_h, A_h^* as follows:

(2. 1) $\langle A_h z, w\rangle_\Omega = \langle Hz, w\rangle_\Omega + \langle \beta z, N^- w^-\rangle_\Gamma$ for w, z in W_h.

That is, (2. 1) is equivalent to

(2. 2) $\langle A_h z, w\rangle_\Omega = \langle Hz, w\rangle_\Omega + \langle z^- + Mz^+, N^- w^-\rangle_\Gamma$

for $z = (z^- ; z^+)$, $w = (w^- ; w^+)$ in W_h.

By duality, we define

(2. 3) $\langle A_h^* z, w\rangle_\Omega = \langle H^* z, w\rangle_\Omega - \langle N^+ z^+, w^+\rangle_\Gamma + \langle N^- z^-, Mw^+\rangle_\Gamma$

for w, z in W_h where H^* is the formal adjoint of H.

Then A_h and A_h^* generate strongly continuous semigroups, say $S_h(t)$ and $S_h^*(t)$ respectively, on W_h.

Next we define L_h as

(2.4) $$(L_h u)(t) = \int_0^t S_h(t-s) P_h(AD) u(s) ds \text{ for } u \text{ in } L^2(\Sigma).$$

Then we obtain the following:

LEMMA 2.1. *For* $0 \le t \le T$ *and* z *in* W_h,

(i) $\|S_h(t)z\|_\Omega \le c\|z\|_\Omega$ *and*

(ii) $\|S_h^*(t)z\|_\Omega \le c\|z\|_\Omega$ *where* c *is a constant depending on* T.

Proof. Let $z(t) = S_h(t)z$ for z in W_h and let

$R = \begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix}$ be a $n \times n$ matrix where $\delta > 0$ and $\varepsilon > 0$.

We use a simple symmetrizer technique in next:

From (2.2), we have for a. e. t in $[0, T]$

(2.5) $$\langle \frac{d}{dt} z(t),\ Rz(t) \rangle_\Omega = \langle A_h z(t), Rz(t) \rangle_\Omega$$

$$= \langle Hz(t),\ Rz(t) \rangle_\Omega + \delta \langle z^-(t), N^- z^-(t) \rangle_\Gamma + \delta \langle Mz^+(t), N^- z^-(t) \rangle_\Gamma$$

As in [2] using Green's formula, we have

(2.6) $$\langle Hz(t), Rz(t) \rangle_\Omega \le -\frac{1}{2} \langle Nz(t), Rz(t) \rangle_\Gamma + d\|z(t)\|_\Omega^2$$

for some constant d depending on T.

From (2.5) and (2.6), we obtain

(2.7) $$\frac{d}{dt} \langle z(t), Rz(t) \rangle_\Omega \le -\delta \langle N^- z^-(t), z^-(t) \rangle_\Gamma - \varepsilon \langle N^+ z^+(t), z^+(t) \rangle_\Gamma + 2\delta \langle z^-(t),\ N^- z^-(t) \rangle_\Gamma + 2\delta \langle Mz^+(t), N^- z^-(t) \rangle_\Gamma + 2d\|z(t)\|_\Omega^2.$$

Since $-N^-$ and N^+ are positive definite on Γ, by taking δ and ε suitably in (2.7) we have

$$\frac{d}{dt}\|z(t)\|^2_\Omega \le -a\|z(t)\|_\Gamma^2 + b\|z(t)\|_\Omega^2 \text{ for some constants } a \text{ and } b > 0.$$

Thus by Gronwall inequality, we have

(2.8) $$\|z(t)\|_\Omega^2 + \int_0^t \|z(s)\|_\Gamma^2 ds \le k\|z\|_\Omega^2, \quad 0 \le t \le T$$

for some constant $k > 0$. This shows part (i).

By the similar way, we can show that

(2.9) $$\|S_h^*(t)z\|_\Omega^2 + \int_0^t \|S_h^*(t)z\|_\Gamma^2 dt \le k\|z\|_\Omega^2, \quad 0 \le t \le T.$$

This completes the proof.

From (2.8) and (2.9) we have

COROLLARY 2.2. *For u in $L^2(\sum)$ and $0 \leq t \leq T$,*

$\|(L_h u)(t)\|_\Omega^2 \leq \|cu\|_\Sigma$ *where c is a constant depending on T.*

Now we obtain the following convergences:

THEOREM 2.3. *As $h \longrightarrow 0$,*

(i) $S_h(t)P_h z \longrightarrow S(t)z$ *and*

$S_h^*(t)P_h z \longrightarrow S^*(t)z$ *in $C([0, T]; L^2(\Omega))$ for z in $L^2(\Omega)$,*

(ii) $(L_h u)(t) \longrightarrow (Lu)(t)$ *in $L^2(\Omega)$ for u in $L^2(\sum)$ and*

$L_h^* P_h f \longrightarrow L^* f$ *in $L^2(\sum)$ for f in $L^2(Q)$,*

(iii) $D^*A^*S_h^*(t)P_h z \longrightarrow D^*A^*S^*(t)z$ *in $L^2(\Gamma)$ for z in $H_0^1(\Omega)$.*

Proof. For (i), we prove the second part since the first part can be followed similarly.
Let $r_h(t) = S_h^*(t)P_h z - P_h S^*(t)z$ for z in $L^2(\Omega)$ and $0 \leq t \leq T$.
Then $r_h(t)$ is in $H^1(\Omega)$ for z in $H_0^1(\Omega)$ by Rauch [9].
Since $\mathrm{Dom}(A^*)$ is invariant under P_h, we obtain for z in $H_0^1(\Omega)$

$$\langle \frac{d}{dt} r_h(t), Rr_h(t)\rangle_\Omega$$
$$= \langle A_h^* r_h(t),\ Rr_h(t)\rangle_\Omega + \langle H^*(P_h - I)S^*(t)z,\ Rr_h(t)\rangle_\Omega$$
$$= \langle H^* r_h(t),\ Rr_h(t)\rangle_\Omega - \varepsilon\langle N^+ r_h^+(t),\ r_h^+(t)\rangle_\Gamma + \varepsilon\langle N^- r_h^-(t), Mr_h^+(t)\rangle_\Gamma$$
$$+ \langle H^*(P_h - I)S^*(t)z, Rr_h(t)\rangle_\Omega$$

As in (2.6) using Green's formula, we have

$$\langle H^* r_h(t), Rr_h(t)\rangle_\Omega \leq \frac{1}{2}\langle Nr_h(t), Rr_h(t)\rangle_\Gamma + d\|r_h(t)\|_\Omega^2$$

for some constant $d > 0$.
Thus by taking suitably δ large and $\varepsilon > 0$ small, and using a trace theorem, we obtain

(2.10) $\|r_h(t)\|_\Omega^2 \leq -\alpha\|r_h(t)\|_\Gamma^2 + \beta\|r_h(t)\|_\Omega^2 + \gamma\|(P_h - I)S^*(t)z\|_{H^1(\Omega)}^2$

for some constants $\alpha > 0$, $\beta > 0$ and $\gamma > 0$.
By Gronwall inequality, (2.10) becomes for $0 \leq t \leq T$ and z in $H_0(\Omega)$

$$\|r_h(t)\|_\Omega^2 + \int_0^t \|r_h(s)\|_\Gamma^2 ds \leq k\int_0^t \|(P_h - I)S^*(s)z\|^2_{H^1(\Omega)} ds \tag{2.11}$$

where k is some constant depending on T.
Using Lebesgue dominated convergence theorem in (2.11), we have

$$\|r_h(t)\|_\Omega^2 + \int_0^t \|r_h(s)\|_\Gamma^2 ds \longrightarrow 0 \text{ as } h \longrightarrow 0. \tag{2.12}$$

By density arguments, (2. 12) holds for z in $L^2(\Omega)$ which shows part (i).
From (2. 12) and lemma 1. 2, we obtain for z in $H_0^1(\Omega)$

$$(2.13)\qquad \int_0^T \|D^*A^*r_h(t)\|_\Gamma^2 dt \longrightarrow 0 \text{ as } h \longrightarrow 0.$$

By the dual inequality of corollary 2. 2 and (2. 13), part (ii) follows.
From [9], we see that for z in $H_0^2(\Omega)$, $S^*(t)z - P_hS^*(t)z$ is in $H^1(\Omega)$. Thus for z in $H_0^2(\Omega)$, from (AP4) we have, $0 \leq t \leq T$,

$$(2.14)\qquad \|S^*(t)z - P_hS^*(t)z\|_{H^1(\Omega)} \leq ch\|z\|_{H^2(\Omega)}$$

where c is some constant depending on T.
From (2. 11) and (2. 14) we obtain for z in in $H_0^2(\Omega)$,

$$(2.15)\qquad \|S_h^*(t)P_hz - P_hS^*(t)z\|_\Omega \leq \sqrt{kT}\, ch\|z\|_{H^2(\Omega)}.$$

On the other hand, from (AP3) we have for z in $H_0^2(\Omega)$,

$$(2.16)\qquad \|S_h^*(t)P_hz - S^*(t)z\|_\Gamma \leq c_0(\sqrt{h})^{-1}\|S_h^*(t)P_hz - P_hS^*(t)z\|_\Omega + c_0(\sqrt{h})^{-1}\|S^*(t)z - P_hS^*(t)z\|_\Omega.$$

Combining (2. 14), (2. 15) and (2. 16), it can be seen that for z in $H_0^2(\Omega)$.

$$\|S_h^*(t)P_hz - S^*(t)z\|_\Gamma \leq b\sqrt{h}\,\|z\|_{H^2(\Omega)}, \qquad 0 \leq t \leq T$$

where b is a constant depending on T.
By lemma 1. 2, part (iii) follows.

3. Convergences of Riccati operators and feedback controls

For simplicity, we assume that F and G are the identity operator on $L^2(\Omega)$ and $F_n = G_n$ in the assumption (1. 9).

In order to proceed our arguments, we define F_n^h on W_h and B_h of $L^2(\Gamma)$ into W_h for $n = 1, 2, 3, \ldots$, $h > 0$:

$$(3.1)\qquad \langle F_n^h z, w\rangle_\Omega = \langle F_n z, w\rangle_\Omega \text{ for } w, z \text{ in } W_h \text{ and}$$

$$(3.2)\qquad \langle B_h u, w\rangle_\Omega = \langle u, D^*A^*w\rangle_\Gamma \text{ for } u \text{ in } L^2(\Gamma),\ w \in W_h.$$

Now we consider the following finite dimensional Riccati equations on W_h: For given n and $h > 0$,

$$(\text{FRE})\qquad \begin{cases} \dfrac{d}{dt}R_n^h(t) = -F_n^h - A_h^*R_n^h(t) - R_n^h(t)A_h \\ \qquad\qquad + R_n^h(t)B_h(D^*A^*)R_n^h(t) \\ \text{with } R_n^h(T) = F_n^h. \end{cases}$$

It is well known that (FRE) has a unique positive self-adjoint matrix solution $R_n^h(t)$ on W_h for each n and h. Then by the similar method as in [2], we have the following convergence results.

THEOREM 3.1. *For each given n,*

(i) $R_n^h(t)P_h z \longrightarrow R_n(t)z$ *in* $L^2(\Omega)$ *as* $h \longrightarrow 0$ *for each* z *in* $L^2(\Omega)$ *uniformly in* $t \in [0, T]$,

(ii) $D^*A^*R_n^h(t)P_h z \longrightarrow D^*A^*R_n(t)z$ *in* $L^2(\Gamma)$ *as* $h \longrightarrow 0$ *for each* z *in* $L^2(\Omega)$ *uniformly in* $t \in [0, T]$,

where $R_n(t)$ *is the unique positive self-adjoint solution of (RDE) in section* 1.

We are in a position to introduce a boundary control system with finite dimensional feedback controls:

$$\text{(CPF)} \qquad \begin{cases} y_n^h(t) = Hy_n^h(t) & \text{on } (0, T)\times\Omega \\ \beta y_n^h(t) = u_n^h(t) & \text{on } (0, T)\times\Gamma \\ y_n^h(0) = y_0 & \text{on } \Omega \\ u_n^h(t) = -D^*A^*R_n^h(t)P_h y_n^h(t), & 0\le t\le T. \end{cases}$$

REMARK. In (CPF), we note that $y_n^h(t)$ is the real time measurement of the state of system (1.3) at time t corresponding to the finite dimensional feedback $u_n^h(t)$.

Then we have the following convergences.

THEOREM 3.2. *For each* y_0 *in* $L^2(\Omega)$, *as* $h\to 0$ *and* $n\to\infty$

(i) $y_n^h \longrightarrow y^0$ *in* $C([0, T]; L^2(\Omega))$,

(ii) $u_n^h \longrightarrow u^0$ *in* $L^2(\Sigma)$,

where u^0 *is the original optimal control and* y^0 *is the corresponding solution of system* (1.3).

Proof. Let $e_n^h(t) = y_n^h(t) - y_n(t)$, $0\le t\le T$.
Then $e_n^h(t) = (Lv_n^h)(t)$ where

$$v_n^h(t) = D^*A^*R_n(t)y_n(t) - D^*A^*R_n^h(t)P_h y_n^h(t).$$

Thus by theorem 1.3, we have

$$\|e_n^h(t)\|_\Omega \le c\|v_n^h\|_\Sigma \text{ for } 0\le t\le T \tag{3.3}$$

for a constant c depending on T.
By Lebesgue dominated convergence theorem and theorem 3.1,
(3.3) leads to

$$e_n^h(t) \longrightarrow 0 \text{ in } L^2(\Omega) \text{ as } h\to 0 \tag{3.4}$$

uniformly in $t \in [0, T]$.

On the other hand, we see that

(3.5) $$y_n^h(t) - y^0(t) = e_n^h(t) + y_n(t) - y^0(t).$$

From theorem 1.5 with (3.4) and (3.5), we deduce part (i).

Part (ii) can be easily derived from theorem 1.5 and part (i).

REMARK. Theorem 3.2 (i) can be also proved by using evolution operator type arguments as in [4].

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