Bull. Korean Math. Soc. 25(1988), No. 2, pp. 167-170

AUTOMORPHISMS OF SOME C^*-ALGEBRAS

SUNG JE CHO*, SANG OG KIM** AND SA GE LEE*

Versions of Tannaka duality in operator algebraic context have been obtained in [6], [8] etc. Suppose σ is an automorphism of a von Neumann algebra M, on which there is an action α of a compact group G such that $\sigma|M^\alpha=id$, where M^τ is the fixed point algebra under the action α. Then it is shown that if there is an action τ of a group H which commutes with α, and which is ergodic in the sense that the fixed point algebra M^τ is trivial, then there exists $g\in G$ such that $\sigma=\alpha(g)$. Recently Evans and Kishimoto ([4]) showed the versions of Tannaka duality in C^*-settings under some conditions.

Our purpose of this note is to find a condition under which the Tannaka duality holds and to show that simple AF-algebras have the relative Dixmier property in the reduced crossed products if the action is discrete outer.

A unital C^*-algebra A is said to satisfy the Dixmier property if for each element x in A the closed convex hull of elements of the form u^*xu, u being a unitary in A, intersects the center of A. The von Neumann algebras and also some other classes of C^*-algebras are known to satisfy the Dixmier property (cf. [2], [7]). We prove the following theorem in a similar method as in [9] (III. 3. Theorem 3).

THEOREM 1. *Let A be a C^*-algebra acting on the Hilbert space H, which admits a unique tracial faithful state φ. If A has the Dixmier property, then any outer action α of a finite group G on A yields a simple crossed product algebra.*

Proof. Assume that τ_1 and τ_2 are tracial states of $\Pi_\varphi(A)$. Then $\tau_i\Pi_\varphi$ are tracial states of A for $i=1,2$. Since A has the unique

Received August 10, 1987.
The present studies were supported by the Basic Science Research Institute Program, Ministry of Education, 1986.

tracial state, this shows that $\Pi_{\varphi}(A)$ has unique tracial state. Since the weak-operator closure of $\Pi_{\varphi}(A)$ is a finite von Neumann algebra, it is a factor by [5] and hence the center $Z(A)$ of A is trivial. Let $\alpha: G \to \mathrm{Aut}(A)$ be the action and I be a nonzero closed two-sided ideal in $A \times_\alpha G$. We represent the element of $A \times_\alpha G$ of the form $\sum a_g u_g$ with the relation $u_g a u_g = \alpha_g(a)$ for $g \in G$ and $a \in A$. Now pick a nonzero element $x = \sum a_g u_g$ in I with the number of nonzero coefficients minimal. Considering x^*x, we may assume that $a_1 > 0$. Since A has the Dixmier property, there exists $\{u_i\}$ in the unitary group of A and $\{\lambda_i\} \subset (0,1)$ such that $\sum \lambda_i u_i a_1 u_i^* \to \mu 1$ for some $\mu \in C$. Since $\tau(a_1) = \mu$, we may assume that $\sum \lambda_i / \mu u_i a_1 u_i^*$ is invertible in A. Let $b = \sum \lambda_i / \mu u_i a_1 u_i^*$. Then

$$b^{-1}(\sum \lambda_i / \mu u_i a_1 u_i^*) + \sum_{g \neq 1} b^{-1}(\sum \lambda_i / \mu u_i a_g u_g u_i^*)$$
$$= b^{-1}(\sum \lambda_i / \mu u_i a_1 u_i^*) u_1 + \sum_{g \neq 1} b^{-1}(\sum \lambda_i / \mu u_i a_g u_g u_i^*) \in I.$$

Hence we may assume that $a_1 = 1$. Choose arbitrary $a \in A$ and fix it. Then

$$\begin{aligned} xa - ax &= (\sum a_g u_g) a - a(\sum a_g u_g) \\ &= \sum a_g \alpha_g(a) u_g - \sum a a_g u_g \\ &= \sum_{g \neq 1} (a_g \alpha_g(a) - a a_g) u_g \in I. \end{aligned}$$

From the choice of x in I, we have $a_g \alpha_g(a) = a a_g$ for $g \neq 1$. Note that $a a_g a_g^* = a_g \alpha_g(a) a_g^* = a_g (a_g \alpha_g(a)^*)^* = a_g (a^* a_g)^* = a_g a_g^* a$ Hence $a_g a_g^* \in Z(A) = CI$. Note that $a_g^* a^* = \alpha_g(a)^* a_g^*$, so replacing a by $\alpha_{g^{-1}}(a)^*$, we have $a_g^* \alpha_{g^{-1}}(a) = a a_g^*$ Hence $a a_g^* a_g = a_g^* \alpha_{g^{-1}}(a) a_g = a_g^* (a_g^* \alpha_{g^{-1}}(a^*))^* = a_g^* (a^* a_g^*)^* = a_g^* a_g a$, so $a_g^* a_g \in CI$. It follows that a_g is a nonzero scalar times a unitary or zero. But if a_g is a nonzero scalar times a unitaty, then α_g would be inner from the relation $a_g \alpha_g(a) = a a_g$, which contradicts the fact that the action is outer. Therefore $a_g = 0$ for all $g \neq 1$. Thus $x = 1$, so that $I = A \times_\alpha G$. This completes the proof.

Corollary 2. *Let A be a separable C^*-algebra. Then under the hypothesis of Theorem 1 with G abelian, A^α and $A \times_\alpha G$ are stably isomorphic.*

Proof. It follows directly from [3], since $A \times_\alpha G$ is simple.

Lemma 3 ([1]). *Let G be a compact abelian group and α be an action* *of G on a C^*-algebra A such that the fixed point algebra A^α is simple. Let τ be a $*$-automorphism of A such that*

(1) $\tau\alpha(g)=\alpha(g)\tau$ *for all* $g\in A$

(2) $\tau(x)=x$ *for all* $x\in A^\alpha$.

Then there exists a $g\in G$ *such that* $\tau=\alpha(g)$.

Hence we have the following.

COROLLARY 4. *Let A be a separable C^*-algebra and G be abelian. Then under the hypothesis of Theorem 1,* $\tau=\alpha(g)$ *for some* $g\in G$ *if* $\tau\in Aut(A)$ *and* $\tau\alpha(g)=\alpha(g)\tau$ $(g\in G)$ *and* $\tau(x)=x$ *for* $x\in A^\alpha$.

Let $D\subset B(H)$ be the algebra of diagonal operators relative to a fixed basis of a Hilbert space H, Then a long-standing open question is whether the following equivalent properties hold.

The extension property: Every pure state of D has a unique extension to a pure state of $B(H)$.

The relative Dixmier property: For every $x\in B(H)$ the set $K(x)=$ closed convex hull of $\{uxu^*|u,$ unitary in $D\}$ meets with D'.
The relative Dixmier property can be formulated in the more general setting of the embedding of two C^*-algebras $A\subset B$ and was proven to hold true by Dixmier for the case $A=B$ is a von Neumann algebra.

THEOREM 5. *Let A be a unital C^*-algebra with trivial center and let* $\alpha: G\to Aut(A)$ *be an outer action of a discrete group G. Then A has the trivial relative commutant in the reduced crossed product algebra* $A\times_{\alpha,r}G$.

Proof. We represent the element $a\in A\times_{\alpha,r}G$ as a converging series $\sum a_g u_g$, where $a_g\in A$ and $u_g\in A\times_{\alpha,r}G$ implements α_g. If $a\in A'\cap(A\times_{\alpha,r}G)$, then we have for all $x\in A$, $xa=\sum xa_g u_g=\sum a_g u_g x=\sum a_g\alpha_g(x)u_g$. Therefore for any fixed $1\neq g\in G$, we have $xa_g=a_g\alpha_g(x)$. Thus as in the proof of Theorem 1, a_g is a nonzero scalar times a unitary or zero. Since α is outer we have $a_g=0$ for $1\neq g\in G$. Hence $a\in A'\cap A=CI$. This completes the proof.

COROLLARY 6. *Let A be a unital simple AF-algebra and* $\alpha: G\to Aut(A)$ *an outer action of a discrete group G. Then A has the relative Dixmier property with respect to* $A\times_{\alpha,r}G$.

Proof. Let A be the norm-closure of $\cup A_n$ where $\{A_n\}$ is an ascending sequence of finite dimensional C^*-algebras. Take H as the inductive limit of unitary groups of each A_n and μ a mean on H. If $x \in A \times_{\alpha, r} G$, then

$\int_H uxu^* d\mu(u) \in A' \cap A \times_{\alpha, r} G = CI$. This completes the proof.

References

1. H. Araki, R. Haag, D. Kastler and M. Takesaki, *Extensions of KMS states and chemical potentials*, Comm. Math. Phys. **53**, (1977) 93–134.
2. R.J. Archbold, *On the Dixmier property of certain algebras*, Math. Proc. Camb. Phil. Soc. **86**, (1979) 251–259.
3. O. Bratteli, *Fixed point algebras versus crossed products*, Proc. Sympo. in Pure Math. **38**, (1982) Part I.
4. D.E. Evans, A. Kishimoto, *Duality for automorphisms on a compact C*-dynamical systems*, preprint.
5. R.V. Kadison, *Representations of matricial operator algebras, operator algebras and group representations*, Proc. Inter. Conf. held in Neptune (Romania), (1980) Part II, 1–20.
6. R. Longo and C. Peligrad, *Non commutative topological dynamics and compact actions on C*-algebras*, J. Funct. Analy. **58**, (1984) 157–174.
7. R.T. Powers, *Simplicity of C*-algebras associated with the free group on two generators*, Duke Math. J. **42**, (1975) 151–156.
8. M. Takesaki, *Compact automorphism groups of von Neumann algebras.*
9. A.J. Wasserman, *Automorphic actions of compact groups on operator algebras*, thesis, (1981).

* Seoul National University
Seoul 151–742, Korea

** Hallym University
Chuncheon 200–702, Korea