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ON FINITE DIMENSIONAL C^*-SUBALGEBRAS OF AF C^*-ALGEBRA

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1. Introduction

The set of traces on a C^*-algebra is a very useful invariant of the algebra and there have been some significant recent advances concerning the relationship between traces, finiteness and comparability of elements. For example a simple C^*-algebra with a finite trace is a finite algebra [3]. An approximately finite dimensional algebra, that is AF C^*-algebra, is a C^*-algebra wich is an inductive limit of a sequence of finite of finite dimensional C^*-algebras. The study of AF C^*-algebra was begun by Bratteli [2] following earlier more specialized studies by Glimm [6] and Diximier [4]. Elliott showed that if A is an AF C^*-algebra, then A is classified up to isomorphism by $K_0(A)$, considered as apartially ordered abelian group, [5]. The relation between trace and $K_0(A)$ has been studied by J. Cuntz and G.K Pedersen. In this paper we study the finite dimensional C^*-sub algebras of AF C^*-algebra by using the trace and the partially ordered abelian group K_0.

2. Preliminaries

Let A be a C-algebra. A trace on A is a function $\phi: A_+ \rightarrow [0, \infty]$ such that

i) $\phi(\alpha x) = \phi(x)$ if $x \in A_+$ and $\alpha \in R_+$,
ii) $\phi(x+y) = \phi(x)+\phi(y)$ if x and y belong to A_+,
iii) $\phi(u^*xu) = \phi(x)$ for all X in A_+ and all unitaries u in A.

In here A is a C^*-algebra with unit containing A as a closed ideal and A_+ is the set of all positive elements in A. We say that ϕ is finite it $\phi(x) < \infty$ for $x \in A_+$ and ϕ is semi-finite if for each $x \in A_+$, $\phi(x')$ is the supremum of the numbers $\phi(y)$ for those $y \in A_+$ such that $y \leq x$ and $\phi(y) < +\infty$. Clearly ϕ may be unbounded functional on A. $y \leq x$ means that $x-y \in A_+$ for $x, y \in A$. If a trace ϕ is finite, then ϕ can be extended to A as a positive linear functional on A. ϕ is lower semi-continuous if for each $\alpha \in R_+$ the set $\{x \in A_+ | \phi(x) \leq \alpha\}$ is closed. The trace has deep relation with the type of von Neumann algebras. A cone M in the positive part of a C^*-algebra A is called hereditary if $0 \leq x \leq y$ and $y \in M$ implies $x \in M$ for each x in A. A $*$-subalgebra B of A is hereditary if B_+' is hereditary in A_+.

LEMMA 2 ([3]). Let B a hereditary C^*-subalgebra of A. Each finite trace ρ on B has an extension to a semi-finite lower semi-continuous trace $\tilde{\rho}$ on A.

3. Ablian group K_0

Let A be a *-algebra. We present the construction of $K_0(A)$, which in genera yields a pre-ordered abelian group, buil1 from the family of self adjoint projections in all matrix algebras over A. Let e, f be projections in A. e and f are *-equivalent, written $e \overset{*}{\sim} f$; if there is an element $w \in A$ such that $w=ewf$ $ww^* = e$ $w^*w = f$. We define

$$P(A) = \bigcup_{n=1}^{\infty} \{\text{projections in } M_n\ (A)\}.$$

In here $M_n(A) = \{[a_{ij}]_{n \times n} | a_{ij} \in A\}$. Given $e, f \in P(A)$ $e \overset{*}{\sim} f$ mean that $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \overset{*}{\sim} \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$ for some suitable sized zero matrices. And define $e, f \in P(A)$ to be stably *-equivalent, written $e \overset{*}{\approx} f$ provided $e \oplus g \overset{*}{\sim} f \oplus g$ for some $g \in P(A)$, i.e., $\begin{bmatrix} e & 0 \\ 0 & g \end{bmatrix} \overset{*}{\sim} \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix}$.

For $e \in P(A)$, we use $[e]$ to denote the equivalence class of e with respect to $\overset{*}{\approx}$. If $e_1, e_2, f_1, f_2 \in P(A)$ with $e_1 \overset{*}{\approx} e_2$ and $f_1 \overset{*}{\approx} f_2$, then $e_1 \oplus f_1 \overset{*}{\approx} e_2 \oplus f_2$. Hence we see that $\oplus$ inducesa well-defined binary operation $+$ on the set of equivalence classes $P(A)/\overset{*}{\approx}$, where $[e]+[f]=[e \oplus f]$ for all $e, f \in P(A)$. Then the operation is commutative and associative. Moreover the semi-group $(P(A)/\overset{*}{\approx}, +)$ satisfies cancellation law: so $(P(A)/\overset{*}{\approx} +)$ is an abelian group.

Denote

$$P(A)/\overset{*}{\approx}, +) = K_0(A).$$

For any *-algebra A, we set

$$K_0(A)_+ = \{[e] | e \in P(A)\}.$$

For any $x, y \in K_0(A)$ we define

$$x \leq y \text{ on } K_0(A) \text{ if and only if } y - x \in K_0(A)_+.$$

The relation $\leq K_0(A)$ is a pre-order. A C^*-algobra A is an AF C^*-algebra if A is the norm-closure of the union of finite demensional C^*-algebras A_n.

THEOREM 3.1([1]). If A is an AF C^*-algebra, then $K_0(A)$ is a partially ordered abelian group.

PROSITIOM 3.2. Let A be a AF C^*-algebra and p, q be projections in A. If $\phi(p) \leq \phi(q)$ for all nonzero traces ϕ on A, then $[p] \leq [q]$ in $K_0(A)$.

PROOF. We may assume that p, q lie in a finite dimensional subalgebra A_1 by replacing p and q by equivalent projections. By [2. Theorem 2.2], we can find an increasing sequence $(A_n)_{n=1}^{\infty}$ of finite dimensional subalgebras containing A_1 and A is the norm closure of $\bigcup_{n=1}^{\infty} A_n$. If $\phi(p) \leq \phi(q)$ for all trace ϕ on A, then $\phi(p) \leq \phi(q)$ for all trace ϕ on A_n for all n. If not ; let e be the unit of the finite dimensional C^*-subalgebra A_1. There exists an integer n_0 and a trace ϕ_{n_0}, on A_{n_0} such that $\phi_{n_0}(p) > \phi_{n_0}(q)$ and $\phi_{n_0}(e) = \alpha$, for some $\alpha > 0$. Let $\phi'_{n_0} = \frac{1}{\alpha} \phi_{n_0}|_{A_0} e$. Since $eA_1e = A_1 \subset eA_{n_0}e$, ϕ'_{n_0} is a trace on $eA_{n_0}e$ such that $\phi'_{n_0}(p) > \phi'_{n_0}(q)$ and $\phi'_{n_0}(e) = 1$. Then for $n > n_0$ there exists a trace ϕ_n on eA_ne such that $\phi_0|_{eA_{n_0}e} = \phi'_{n_0}$. Hence there exists a trace ϕ_n on eA_ne such that $\phi_n(p) > \phi_n(q)$ and $\phi_n(e) = 1$ for $n > n_0$. Let $eAe = B$ and $\tilde{\phi}_n$ be an extension of ϕ_n to a state on B. Since B has a unit e, $\{\tilde{\phi}_n\}$ has a weak*-limit $\tilde{\phi}$. Then $\tilde{\phi}$ is a tracial state and $\tilde{\phi}(e)=1$. Since B is a hereditary subalgebra of A and $\tilde{\phi}|_{B_+}$ is a finite trace on B, by Lemma 2.1 $\tilde{\phi}$ extended to a trace on A. Futhemore $\tilde{\phi}(p) > \tilde{\phi}(q)$ and this contradicts to the hypothesis. Hence $[p] \leq [q]$ in $K_0(A_n)$. Since $[p] \leq [q]$ in $K_0(A)$ if and only if $[p] \leq [q]$ in $K_0(A_n)$ for some n, $[p] \leq [q]$ in $K_0(A)$.

Since $K_0(A)$ is a partially ordered group for an AF C^*-algeba A, if p, q are projections and $\phi(p) = \phi(q)$ for all traces ϕ on AF C^*-algebra A, then $[p] = [q]$ in $K_0(A)$.

Proposition 3.3. Let A be an AF C^*-algebra and p, q be projections in A. Then $[p]=[q]$ in $K_0(A)$ if and only if $p \overset{*}{\sim} q$ in A.

Proof. Clearly $p \overset{*}{\sim} q$ implies $[p]=[q]$. In AF C^*-algebra by [1. Lemma 20] if $[p]=[q]$ in $K_0(A)$, then $\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \overset{*}{\sim} \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}$ for some suitable sized zero matrix. Hence there exists a $w \in M_n(A)$ such that $w^*w = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$ and $ww^* = \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$ for some n. Since e, f are in A and the zero matrices in the above is of the same sized, there exists a partial isometry $w' \in A$ such that $w = \begin{bmatrix} w' & 0 \\ 0 & 0 \end{bmatrix}$.

4. Main results.

Let A be a *-algebra. A set $n \times n$ *-matrix units in A is a set of $n \times n$ matrix units $\{e_{ij} | i, j = 1, \cdots, n\}$ of elements of A such that $e_{ij}e_{km} = \delta_{jk}e_{im}$ and $e_{ij}^* = e_{ij}$ for all i, j. In this case $e_{11}, \cdots, e_{nn}$ are orthogonal projections. A *-matricial subbasis in A is a set $\{e_{pq}^i | i = 1, \cdots, k, \ p, q = 1, \cdots, n(i)\}$ of elements of A such that

1) $\{e_{pq}^{(i)} | p, q = 1, \cdots, n(i)\}$ is a set of $n(i) \times n(i)$ *-matrix units for each $i = 1, \cdots, k$;
2) $e_{pq}^{(i)} \ e_{rs}^{(j)} = 0$ for all $i, j, \ p, q, \ r, s$ with $i \neq j$.

Then $e^{(i)} = \sum_{n=1}^{n(i)} e_{pp}^{(i)}$ are mutually orthogonal projections in A. If a *-algebra A has a *-matricial subbasis $\{e_{pq}^{(i)}\}$ that spans A, then $\{e_{pq}^{(i)}\}$ is a *-matricial basis for A. In this case $\sum_{i=1}^{k} \sum_{p=1}^{n(i)} e_{pp}^{(i)}$ is a unit of A. Thus a *-algebrais matricial if and only if it has a *-matricial basis.

THEOREM 4. Let A be an AF C^*-algebra with unit acting on a separable Hilbert space H. Suppose that $M \subset A$ and $N \subset A$ are *-isomorphic finite dimensional C^*-subalgebras of A. Then there exists a unitary element u in A such that $uMu^* = N$.

PROOF. Suppose that $\{E_{ij}^k \mid i, j = 1, \cdots, n_k, k = 1, \cdots, n\}$ and $\{F_{ij}^k \mid i, j = 1, \cdots, n_k, \ k = 1, \cdots, n\}$ are *-matricial basis of M and N respectively. We may assume that M, N have the same unit with A. We show that there exists a partial isometry $V^k \in A$ with initial projection E^k_{11} and terminal projection F^k_{11} for $k=1, \cdots, n$. Let $U=\sum_{k=1}^{n} \sum_{i=1}^{n_k} F^k_{i1} V^k E^k_{1i}$.

$$uE^r_{ij} u^*=\Big(\sum_{k}^{n} \sum_{q=1}^{n_k} F^k_{qi} V^k E^r_{iq}\Big)(E^r_{ij})\Big(\sum_{s}^{n} \sum_{p}^{n_k} F^s_{pi} V^s E^s_{ip}\Big)^*$$

$$=(\sum_{k}\sum_{q} F^k_{qi} V^k E^k_{iq})(E^r_{ij})(\sum_{s} \sum_{p} E^s_{pi} (V^s)^* F^s_{ip})$$

$$=\sum_{k,q,s,p} \delta_{kr} \delta_{qi} \delta_{rs} \delta_{jp} F^k_{qi} V^k E^k_{iq} E^r_{ij} E^s_{pi} (V^s)^* F^s_{ip}$$

$$=F^r_{i1} V^r E^r_{1i} E^r_{ij} E^r_{ji} (V^r)^* F^r_{ij}$$

$$=F^r_{i1} V^r E^r_{11} (V^r)^* F^*_{ij} = F^r_{ij}.$$

Therefore

$$uMu^*=N \text{ and } uu^*=\sum_{k,s,p,q} (F^k_{qi} V^k E^k_{iq})(F^s_{pi} V^s E^s_{ip})^*$$

$$=\sum_{k,s,p,q} \delta_{ks} \delta_{pq} F^k_{qi} V^k E^k_{iq} E^s_{pi} (V^s)^* F^s_{ip}$$

$$=\sum_{k,q} F^k_{qi} V^k E_{iq} E^k_{qi} (V^k)^* F^k_{iq}$$

$$=\sum_{k,q} F^k_{qq} = I$$

Similarly $u^*u = I$ Hence u is the unitary that we want. Let ϕ be a trace on A. Let $p^k_i = EE^k_{ii}$, $Q^k_i = EF^k_{ii}$ $1 \leq k \leq n$ for central projection $E \in A$. Then

$$\sum_{k=1}^{n} \sum_{i=1}^{n_k} p^k_i = \sum_{k=1}^{n} \sum_{i=1}^{n_k} Q^k_i = E.$$

We put

$$E^k = \sum_{i=1}^{n_k} E^k_{ii}$$

and

$$F^k = \sum_{i=1}^{n_k} F^k_{ii} .$$

Let

$$S(i) = \sum_{k=1}^{n} E^k - E^k_{11} - E^k_{ii} + E^k_{1i} + E^k_{i1}$$

and

$$V(i) = \sum_{k=1}^{n} F^k - F^k_{11} - F^k_{ii} + F^k_{1i} + F^k_{i1}$$

Then

$$\begin{aligned} S(i)\ P^k_1\ S(i)^* &= \Big(\sum_{l=1}^{n} E^l - E^l_{11} - E^l_{ii} + E^l_{1i} + E^l_{i1} \Big)(EE^k_{ll}) \\ &\quad \Big(\sum_{r=1}^{n} E^r - E^r_{11} - E^r_{ii} + E^r_{i1} + E^r_{i1} \Big)^* \\ &= \Big(\sum_{l=1}^{n} \delta_{lk}\ EE^l\ E^k_{11} - EE^k_{11} E^k_{11} - EE^l_{ii} E^k_{11} \\ &\quad + EE^l_{1i} E^k_{11} + EE^l_{i1} E^k_{11} \Big) \\ &\quad \Big(\sum_{r=1}^{n} E^r - E^r_{11} - E^r_{ii} + E^r_{i1} + E^r_{1i} \Big)^* \\ &= \sum_{r} \delta_{kr}\ EE^k_{i1} (E^r - E^r_{11} E^r_{ii} + E^r_{i1} + E^r_{1i}) \\ &= EE^k_{i1} E^k - EE^k_{i1} E^k_{11} - EE^k_{i1} E^k_{ii} + EE^k_{i1} E^r_{i1} \\ &\quad + EE^k_{i1} E^k_{1i} = EE^k_{ii} = P^k_{i1}, \end{aligned}$$

Moreover $S(i)S(i)^* = S(i)^*S(i) = \sum_{k=1}^{n} E^k = I.$ By similar computation $V_{(i)} Q^k_i V_{(i)}^* = Q^k_i$.

Since $S_{(i)}$, $V_{(i)}$ are unitary and trace is invariant under inner autorphisms, $\phi(EE_{11}^k) = \phi(EE_{jj}^k) = \phi(EE_{11}^k) = \phi(EF_{jj}^k)$ for all $k=1, \cdots, n$ $j=1, \cdots, n_k$.

Since E is a central projection, $\phi(E_{11}^k) = \phi(F_{11}^k)$ for all trace ϕ on A. By Proposition 3.2 $[E_{11}^k]=[F_{11}^k]$ in $K_0(A)$ and by Proposition 3.3 there exists a partial isometry $V^k \in A$ with initial projection E_{11}^k and terminal projection F_{11}^k.

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