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# A METHOD OF PROOF FOR THE RANGE OF CONFIDENCE INTERVALS

KWAN JOONG KANG

## 1. Introduction

Consider the one-factor nested Components-of-Variance model with equal numbers in the subclass

(1.1) $y_{ij}=\mu+A_i+B_{ij}$

where $i=1,2,\cdots,I,\ j=1,2,\cdots,J$ and $\mu$ is a constant common to all observations. $A_i, B_{ij}$ are normally and independently distributed random variables with mean zero and finite variables $\sigma_A^2$ and $\sigma_B^2$ respectively.

There is no known method for determining exact confidence limit for $\sigma_A^2$. Several approximations are available. Satterthwaite (1948) approximates the distribution of the estimate of a linear combination of variances by Chi-square distribution times a constant. Welch(1956) offers a series approximation for setting approximation is given by Moriguti(1954) are repeated by Bulmer(1957). They derive a confidence interval on $\sigma_A^2$ by assuming the confidence limit is a certain form and solving for it by forcing the confidence coefficient to be exact under certian limiting conditions.

Tukey(1951) and Williams(1962) constructed an approximate confidence interval for $\sigma_A^2$ with a guaranteed lower confidence coefficient of $1-\alpha$ by a geometrical projection. Howe(1974) derives confidence limits on $\sigma_A^2$ through a modified Cronish-Fisher expansion.

Then, in this case we don't know whether the range is quite fit for $1-\alpha$ or not. If it is not fit, we don't know how close to $1-\alpha$ the range is, either. Therefore, we need to find out the closer confidence interval to $1-\alpha$.

So in this paper the writer intend to show the new method of proof by the calculus of integration and the change of the random variables.

## 2. The range of confidence interval on $\sigma_A^2$

Let

$$\bar{y}_{i.}=\frac{\sum_{j=1}^{J} y_{ij}}{J}, \quad \bar{y}_{..}=\frac{\sum_{i=1}^{I}\sum_{j=1}^{J} y_{ij}}{IJ},$$

$$S_1^2=\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{y}_{i.}-\bar{y}_{..})^2}{I-1}, \quad S_2^2=\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\bar{y}_{i.})^2}{I(J-1)}.$$

It is well known that

$$E(S_1^2)=\theta_1=\sigma_B^2+J\sigma_A^2,$$
$$E(S_2^2)=\theta_2=\sigma_B^2,$$
$$\frac{n_1S_1^2}{\theta_1}\sim\chi_{n_1}^2 \quad \text{and} \quad \frac{n_2S_2^2}{\theta_2}\sim\chi_{n_2}^2$$

where

$$n_1=I-1, \quad \text{and} \quad n_2=I(J-1).$$

The Tukey-Williams(TW) upper confidence intervale on $\sigma_A{}^2$ is

$$(2.1) \qquad \frac{S_1{}^2-S_2{}^2\,F_{\frac{\alpha}{2},n_1,n_2}}{F_{\frac{\alpha}{2},n_1,\infty}} \leq \theta_1-\theta_2 \leq \frac{S_1{}^2-S_2{}^2F_{1-\frac{\alpha}{2},n_1,n_2}}{F_{1-\frac{\alpha}{2},n_1,\infty}}$$

THEOREM 1. The probability of TW upper confidence interval on $\sigma_A{}^2$ is

$$(2.2) \qquad P_r\left[\frac{S_1{}^2-S_2{}^2\ F_{\frac{\alpha}{2},n_1,n_2}}{F_{\frac{\alpha}{2},n_1,\infty}} \leq \theta_1-\theta_2\right] \geq 1-\frac{\alpha}{2}.$$

PROOF.

$$(S_1{}^2-S_2{}^2F_{\frac{\alpha}{2},n_1,n_2})/F_{\frac{\alpha}{2},n_1,\infty} \leq \theta_1-\theta_2$$

$$\frac{S_1{}^2}{\theta_1}-\frac{S_2{}^2}{\theta_1}F_{\frac{\alpha}{2},n_1,n_2} \leq \left(1-\frac{\theta_2}{\theta_1}\right)F_{\frac{\alpha}{2},n_1,\infty}$$

$$\frac{n_1S_1{}^2}{\theta_1} \leq n_1F_{\frac{\alpha}{2},n_1,\infty}+n_1\rho\left(\frac{S_2{}^2}{\theta_2}F_{\frac{\alpha}{2};n_1,n_2}-F_{\frac{\alpha}{2},n_1,\infty}\right).$$

Let $\rho=\frac{\theta_2}{\theta_1}$, $U_i=\frac{n_iS_i{}^2}{\theta_i}$ for $i=1,2$ and $n_1F_{\frac{\alpha}{2},n_1,\infty}=a_1$.

$$U_1 \leq a_1+n_1\rho\left(\frac{U_2}{n_2}F_{\frac{\alpha}{2},n_1,n_2}-F_{\frac{\alpha}{2};n_1,\infty}\right),$$

$$(2.3) \qquad P\left(\frac{\alpha}{2}\right)=P_r\left[U_1 \leq a_1+n_1\rho\left(\frac{U_2}{n_2}F_{\frac{\alpha}{2},n_1,n_2}-F_{\frac{\alpha}{2},n_1,\infty}\right)\right].$$

And let

$$n_1\rho\left(\frac{U_2}{n_2}F_{\frac{\alpha}{2};n_1,n_2}-F_{\frac{\alpha}{2};n_1,\infty}\right)=Y_1,$$

$$P\left(\frac{\alpha}{2}\right)=P_r[U_1 \leq a_1+Y_1].$$

$U_1$ and $U_2$ is independent Chi-square distributed with degree of freedom $n_i$ for $i=1,2$,

$$(2.4)\qquad P\left(\frac{\alpha}{2}\right)=\int_{-\infty}^{\infty}\left\{\int_0^{a_1+y_1}f_1(x)dx\right\}f_2(y_1)dy_1.$$

Let $F$ is cumulative distribution function (c. d. f.) of $f_1$.

$$\begin{aligned}\int_0^{a_1+y_1}f_1(x)dx&=\int_0^{a_1}f_1(x)dx+\int_{a_1}^{a_1+y_1}f_1(x)dx\\&=\int_0^{a_1}f_1(x)dx+[F_1(x)]_{a_1}^{a_1+y_1}\\&=\int_0^{a_1}f_1(x)dx+\{F_1(a_1+y_1)-F_1(a_1)\}.\end{aligned}$$

$$P\left(\frac{\alpha}{2}\right)=\int_{-\infty}^{\infty}\int_0^{a_1}f_1(x)f_2(y_1)dxdy_1+\int_{-\infty}^{\infty}\{F_1(a_1+y_1)-F_1(a_1)\}f_2(y_1)dy_1.$$

i) If $Y_1=0$,

$$\begin{aligned}P\left(\frac{\alpha}{2}\right)&=P\{U_1\leq a_1+Y_1]\\&=\int_{-\infty}^{\infty}\int_0^{a_1}f_1(x)f_2(y_1)dxdy_1\\&=P_r\{U_1\leq a_1]\\&=1-\frac{\alpha}{2}.\end{aligned}$$

ii) If $Y_1\neq 0$,

$$\begin{aligned}P\left(\frac{\alpha}{2}\right)&=P\{U_1\leq a_1+Y_1]\\&=\int_{-\infty}^{\infty}\int_0^{a_1}f_1(x)f_2(y_1)dxdy_1\\&\quad+\int_{-\infty}^{\infty}\frac{F_1(a_1+y_1)-F_1(a_1)}{y_1}y_1f_2(y_1)dy_1.\end{aligned}$$

There exists a constant $\xi_1$ such that

$$P\left(\frac{\alpha}{2}\right)=P\{U_1\leq a_1+Y_1]$$
$$=\int_{-\infty}^{\infty}\int_0^{a_1}f_1(x)f_2(y_1)dxdy_1$$
$$+\frac{F_1(a_1+\xi_1)-F_1(a_1)}{\xi_1}\int_{-\infty}^{\infty}y_1f_2(y)dy_1.$$

Using the mean value theorem, there exists constants $\eta_1$ for $0<|\eta_1|<|\xi_1|$ such that

$$P\left(\frac{\alpha}{2}\right)=P[U_1\leq a_1+Y_1]$$
$$=\int_{-\infty}^{\infty}\int_0^{a_1}f_1(x)\,f_2(y_1)dxdy_1$$
$$+f_1(a_1+\eta_1)\int_{-\infty}^{\infty}yf_2(y_1)dy_1$$
$$=P[U_1\leq a_1]+f_1(a_1+\eta_1)EY_1.$$

Since

$$f_1(a+\eta)EY_1\geq 0,$$
$$P\left(\frac{\alpha}{2}\right)=P\{U_1\leq a_1]+f_1(a+\eta_1)EY_1\geq P\{U_1\leq a_1]$$
$$=1-\frac{\alpha}{2}.$$

THEOREM 2. The probability of TW lower confidence interval on $\sigma_A^2$ is

$$P_r\left[\theta_1-\theta_2\leq\frac{S_1^2-S_2^2F_{1-\frac{\alpha}{2};n_1,n_2}}{F_{1-\frac{\alpha}{2},n_1,\infty}}\right]\leq 1-\frac{\alpha}{2}.$$

PROOF.

$$P\left(1-\frac{\alpha}{2}\right)=P_r\left[\theta_1-\theta_2\leq\frac{S_1^2-S_2^2F_{1-\frac{\alpha}{2};n_1,n_2}}{F_{1-\frac{\alpha}{2},n_1,\infty}}\right]$$
$$=1-P_r\left[\frac{S_1^2-S_2^2F_{1-\frac{\alpha}{2};n_1,n_2}}{F_{1-\frac{\alpha}{2},n_1,\infty}}<\theta_1-\theta_2\right]$$

$$
\begin{aligned}
&= 1 - P_r\left(1 - \frac{\alpha}{2}\right) \\
&= 1 - P_r[U_1 < n_1 F_{1-\frac{\alpha}{2}, n_1, \infty} \\
&\quad + n_1 \rho (F_{1-\frac{\alpha}{2}, n_1, n_2} - F_{1-\frac{\alpha}{2}, n_1, \infty})] \\
&= 1 - P_r[U_1 < n_1 F_{1-\frac{\alpha}{2} n_1, \infty} \\
&\quad - n_1 \rho (F_{1-\frac{\alpha}{2}; n, \infty} - F_{1-\frac{\alpha}{2}, n_1, n_2})] \\
&= 1 - P_r[U_1 < a_2 - Y_2],
\end{aligned}
$$

where

$$
\begin{aligned}
a_2 &= n_1 F_{1-\frac{\alpha}{2}, n_1, \infty} \geq 0 \\
Y_2 &= n_1 \rho (F_{1-\frac{\alpha}{2}, n_1, \infty} - F_{1-\frac{\alpha}{2}, n_1, n_2}) \geq 0 .
\end{aligned}
$$

$$
\begin{aligned}
P_r\left(1 - \frac{\alpha}{2}\right) &= 1 - P_r[U_1 < a_2] + P_r[a_2 - Y_2 < U_1 < a_2] \\
&= P_r[a_2 \leq U_1] + P_r[a_2 - Y_2 < U_1 < a_2] .
\end{aligned}
$$

i) If $Y_2 = 0$,

$$
\begin{aligned}
P\left(1 - \frac{\alpha}{2}\right) &= P_r[a_2 \leq U_1] \\
&= 1 - \frac{\alpha}{2}.
\end{aligned}
$$

ii) If $Y_2 \neq 0$,

$$
\begin{aligned}
P\left(1 - \frac{\alpha}{2}\right) &= P_r[a_2 \leq U_1] + P_r[a_2 - Y_2 < U_1 < a_2] \\
&= P_r[a_2 \leq U_1] \\
&\quad + \int_{-\infty}^{\infty} \left\{ \int_{b_1 - y_2}^{b_1} f_1(x) dx \right\} f_2(y_2) dy_2 \\
&= P_r[a_2 \leq U_1] \\
&\quad + \int_{-\infty}^{\infty} \{F(a_2) - F(a_2 - y_2)\} f_2(y_2) dy_2.
\end{aligned}
$$

And there exist constants $\xi_2$ and $\eta_2$ such that

$$P\left(1-\frac{\alpha}{2}\right)=P_r[a_2 \leq U_1]$$
$$+\frac{F(a_2)-F(a_2-\xi_2)}{\xi_2}\int_{-\infty}^{\infty} y_2 f_2(y_2)dy_2$$
$$=P_r[a_2 \leq U_1]+f_1(a_2-\xi_2)EY_2.$$

Since

$$f_1(a_2-\xi_2)EY_2 \geq 0,$$
$$P\left(1-\frac{\alpha}{2}\right)=P_r[a_2 \leq U_1]+f_1(a_2-\xi_2)EY_2$$
$$=1-\frac{\alpha}{2}+f_1(a_2-\xi_2)EY_2$$
$$\geq 1-\frac{\alpha}{2}.$$

THEOREM 3. The probability of TW confidence intervals on $\sigma_A{}^2$ is

$$P_r\left[\frac{S_1{}^2-S_2{}^2F_{\frac{\alpha}{2},n_1,n_2}}{F_{\frac{\alpha}{2},n_1,\infty}} \leq \theta_1-\theta_2 \leq \frac{S_1{}^2-S_2{}^2F_{1-\frac{\alpha}{2},n_1,n_2}}{F_{1,n_1,\infty}}\right]$$
$$\geq 1-\alpha.$$

PROOF.

$$P(\alpha)=P_r\left[\frac{S_1{}^2-S_2{}^2F_{\frac{\alpha}{2},n_1,n_2}}{F_{\frac{\alpha}{2},n_1,\infty}} \leq \theta_1-\theta_2 \leq \frac{S_1{}^2-S_2{}^2F_{1-\frac{\alpha}{2},n_1,n_2}}{F_{1-\frac{\alpha}{2},n_1,\infty}}\right]$$
$$=P_r\left[\frac{S_1{}^2-S_2{}^2F_{\frac{\alpha}{2},n_1,n_2}}{F_{\frac{\alpha}{2},n_1,\infty}} \leq \theta_1-\theta_2\right]$$
$$-P_r\left[\frac{S_1{}^2-S_2{}^2F_{1\frac{\alpha}{2}-,n_1,n_2}}{F_{1-\frac{\alpha}{2},n_1,\infty}} < \theta_1-\theta_2\right]$$
$$=P_r[U_1 \leq a_1+Y_1]-P_r[U_1 < a_2-Y_2],$$

where

$$b_1=n_1 F_{1-\frac{\alpha}{2};n_1,\infty},$$

$$Y_2 = n_1\rho\left(F_{1-\frac{\alpha}{2};n_1,\infty} - \frac{U_2}{n_2} F_{1-\frac{\alpha}{2};n_1,n_2}\right).$$

$$\begin{aligned} P(\alpha) &= P_r[U_1 \le a_1] + P_r[a_1 < U_1 \le a_1 + Y_1] \\ &\quad - P_r[U_1 \le a_2] + P_r[a_2 - Y_2 < U_1 < a_2] \\ &= P_r[a_2 \le U_1 \le a_1] \\ &\quad + P_r[a_1 < U_1 \le a_1 + Y_1] + P_r[a_2 - Y_2 < U_1 < a_2]. \end{aligned}$$

Then

i) $P_r[a_2 \le U_1 \le a_1]$

$$= P_r[n_1 F_{1-\frac{\alpha}{2},n_1,\infty} \le U_1 \le n_1 F_{\frac{\alpha}{2},n_1,\infty}]$$
$$= 1-\alpha,$$

ii) $P_r[a_1 < U_1 \le a_1 + Y_1] + P_r[a_2 - Y_2 < U_1 < a_2]$

$$= f_1(a_1+\eta_1)EY_1 + f_1(a_2-\eta_2)\ EY_2.$$

Therefore

$$\begin{aligned} P\left(1-\frac{\alpha}{2}\right) &= P_r[a_2 \le U_1 \le a_1] \\ &\quad + f_1(a_1+\eta_1)EY_1 + f(a_2-\eta_2)EY_2. \\ &\ge P_r[a_2 \le U_1 \le a_1] \\ &= 1-\alpha. \end{aligned}$$

Though we succeded in-devising the new method of proof, we don't know exactly the possibility of its application.

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Dong-A University
Pusan 604-714
Korea

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